INTRODUCTION TO MORI THEORY

Cours de M2 – 2010/2011

Université Paris Diderot

Olivier Debarre

March 11, 2016

Contents

Chapter 1

Aim of the course

Let X be a smooth projective variety (over an algebraically closed field). Let C be a curve in X and let D be a hypersurface in X. When C and D meets transversely, we denote by $(D \cdot C)$ the number of their intersection points. This "product" can in fact be defined for any curve and any hypersurface; it is always an integer (which can be negative when C is contained in D) and does not change when one moves C and D.

Example 1.1 If C_1 and C_2 are curves in $\mathbf{P}^2_{\mathbf{k}}$, we have (this is Bézout's theorem)

$$(C_1 \cdot C_2) = \deg(C_1) \deg(C_2).$$

The intersection number is here always positive. More generally, it is possible to define the degree of a curve C in $\mathbf{P}^n$ in such a way that, for any hypersurface H, we have

$$(H \cdot C) = \deg(H) \deg(C), \tag{1.1}$$

where $\deg(H)$ is the degree of a homogeneous polynomial that defines H.

We will define intersection of curves and hypersurfaces in any smooth projective variety X. Then, we will identify two curves which have the same intersection number with each hypersurface (this defines an equivalence relation on the set of all curves). It is useful to introduce some linear algebra in the picture, as follows.

Consider finite formal linear combinations with real coefficients of irreducible curves in X (they are called *real* 1-*cycles*); these form a gigantic vector space with basis the set of all irreducible curves in X. Extend by bilinearity the intersection product between 1-cycles and hypersurfaces; it takes real values. Define

$$N_1(X) = \{\text{real vector space of all 1-cycles}\}/\{\text{1-cycles with intersection 0 with all hypersurfaces}\}.$$

The fundamental fact is that *the real vector space* $N_1(X)$ *is finite-dimensional.* In this vector space, we define the *effective (convex) cone* $NE(X)$ as the set of all linear combinations with nonnegative coefficients of classes of curves in X. It is sometimes not closed, and we consider its closure $\overline{NE}(X)$ (the geometry of closed convex cones is easier to study).

If X is a smooth variety contained in $\mathbf{P}^n$ and H is the intersection of X with a general hyperplane in $\mathbf{P}^n$, we have $(H \cdot C) > 0$ for all curves C in X (one can always choose a hyperplane which does not contain C). This means that $NE(X) - \{0\}$, and in fact also $\overline{NE}(X) - \{0\}$, is contained in an open half-space in $N_1(X)$. Equivalently, $\overline{NE}(X)$ contains no lines.

Examples 1.2 1) By (1.1), there is an isomorphism

$$N_1(\mathbf{P}^n) \quad \longrightarrow \quad \mathbf{R}$$

$$\sum \lambda_i [C_i] \quad \longmapsto \quad \sum \lambda_i \deg(C_i)$$

and $NE(\mathbf{P}^n)$ is $\mathbf{R}^+$ (not a very interesting cone).

2) If X is a smooth quadric in $\mathbf{P}^3_{\mathbf{k}}$, and C_1 and C_2 are lines in X which meet, the relations $(C_1 \cdot C_2) = 1$ and $(C_1 \cdot C_1) = (C_2 \cdot C_2) = 0$ imply that the classes $[C_1]$ and $[C_2]$ are independent in $N_1(X)$. In fact,

$$N_1(X) = \mathbf{R}[C_1] \oplus \mathbf{R}[C_2] \qquad \text{and} \qquad NE(X) = \mathbf{R}^+[C_1] \oplus \mathbf{R}^+[C_2].$$

3) If X is a smooth cubic in $\mathbf{P}^3_{\mathbf{k}}$, it contains 27 lines $C_1, \dots, C_{27}$ and one can find 6 of them which are pairwise disjoint, say $C_1, \dots, C_6$. Let C be the smooth plane cubic obtained by cutting X with a general plane. We have

$$N_1(X) = \mathbf{R}[C] \oplus \mathbf{R}[C_1] \oplus \cdots \oplus \mathbf{R}[C_6].$$

The classes of $C_7, \dots, C_{27}$ are the 15 classes $[C - C_i - C_j]$, for $1 \le i < j \le 6$, and the 6 classes $[2C - \sum_{i \ne k} C_i]$, for $1 \le k \le 6$. We have

$$NE(X) = \sum_{i=1}^{27} \mathbf{R}^+[C_i].$$

So the effective cone can be quite complicated. One can show that there exists a regular map $X \to \mathbf{P}^2_{\mathbf{k}}$ which contracts exactly $C_1, \dots, C_6$. We say that X is the *blow-up* of $\mathbf{P}^2_{\mathbf{k}}$ at 6 points.

4) Although the cone $NE(X)$ is closed in each of the examples above, this is not always the case (it is *not* closed for the surface X obtained by blowing up $\mathbf{P}^2_{\mathbf{k}}$ at 9 general points; we will come back to this in Example 5.16).

Let now $f : X \to Y$ be a regular map; we assume that fibers of f are connected, and that Y is normal. We denote by $NE(f)$ the subcone of $NE(X)$ generated by classes of curves contracted by f. The map f is determined by the curves that it contracts, and these curves are the curves whose class is in $NE(f)$.

Fundamental fact. *The regular map f is characterized (up to isomorphism) by the subcone $NE(f)$.*

The subcone $NE(f)$ also has the property that it is *extremal*: it is convex and, if c, c' are in $NE(X)$ and $c + c'$ is in $NE(f)$, then c and c' are in $NE(f)$. We are then led to the fundamental question of Mori's Minimal Model Programm (MMP):

Fundamental question. *Given a smooth projective variety X, which extremal subcones of $NE(X)$ correspond to regular maps?*

To (partially) answer this question, we need to define a canonical linear form on $N_1(X)$, called the *canonical class.*

1.3. The canonical class. Let X be a complex variety of dimension n. A meromorphic n-form is a differential form on the complex variety X which can be written, in a local holomorphic coordinate system, as

$$\omega(z_1, \dots, z_n) dz_1 \wedge \cdots \wedge dz_n,$$

where ω is a meromorphic function. This function ω has zeroes and poles along (algebraic) hypersurfaces of X, with which we build a formal linear combination $\sum_i m_i D_i$, called a *divisor*, where m_i is the order of vanishing or the order of the pole (it is an integer).

Examples 1.4 1) On $\mathbf{P}^n$, the n-form $dx_1 \wedge \cdots \wedge dx_n$ is holomorphic in the open set U_0 where $x_0 \ne 0$. In $U_1 \cap U_0$, we have

$$(x_0, 1, x_2, \dots, x_n) = (1, \frac{1}{x_0}, \frac{x_2}{x_0}, \dots, \frac{x_n}{x_0})$$

hence

$$dx_1 \wedge \cdots \wedge dx_n = d\big(\frac{1}{x_0}\big) \wedge d\big(\frac{x_2}{x_0}\big) \wedge \cdots \wedge d\big(\frac{x_n}{x_0}\big) = -\frac{1}{x_0^{n+1}} \, dx_0 \wedge dx_2 \wedge \cdots \wedge dx_n.$$

There is a pole of order $n+1$ along the hyperplane H_0 with equation $x_0 = 0$; the divisor is $-(n+1)H_0$.

2) If X is a smooth hypersurface of degree d in $\mathbf{P}^n$ defined by a homogeneous equation $P(x_0,\dots,x_n) = 0$, the $(n-1)$-form defined on $U_0 \cap X$ by

$$(-1)^i \ \frac{dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_n}{(\partial P/\partial x_i)(x)}$$

does not depend on i and does not vanish. As in 1), it can be written in $U_1 \cap U_0 \cap X$ as

$$\frac{d\big(\frac{1}{x_0}\big) \wedge d\big(\frac{x_3}{x_0}\big) \wedge \cdots \wedge d\big(\frac{x_n}{x_0}\big)}{(\partial P/\partial x_2)(1, \frac{1}{x_0}, \frac{x_2}{x_0}, \dots, \frac{x_n}{x_0})} = -\frac{1}{x_0^{n-(d-1)}} \ \frac{dx_0 \wedge dx_3 \wedge \cdots \wedge dx_n}{(\partial P/\partial x_2)(x_0, 1, x_2, \dots, x_n)},$$

so that the divisor is $-(n+1-d)(H_0 \cap X)$.

The fundamental point is that although this divisor depends on the choice of the (nonzero) n-form, the linear form that it defines on $N_1(X)$ does not. It is called the *canonical class* and is denoted by K_X.

Example 1.5 If X is a smooth hypersurface smooth of degree d in $\mathbf{P}^n$, we just saw that the canonical class is $d-n-1$ times the class of a hyperplane section: for a smooth quadric in $\mathbf{P}^3_{\mathbf{k}}$, the canonical class is $-2[C_1] - 2[C_2]$; for a smooth cubic in $\mathbf{P}^3_{\mathbf{k}}$, the canonical class is $-[C]$ (see Examples 1.2.2) and 1.2.3)).

The role of the canonical class in relation to regular maps is illustrated by the following result.

Proposition 1.6 *Let X and Y be smooth projective varieties and let $f : X \to Y$ be a birational, nonbijective, regular map. There exists a curve C in X contracted by f such that $(K_X \cdot C) < 0$.*

The curves C contained in a variety X such that $(K_X \cdot C) < 0$ therefore play an essential role. If X contains no such curves, X cannot be "simplified." Mori's Cone Theorem describes the part of $\overline{NE}(X)$ where the canonical class is negative.

Theorem 1.7 (Mori's Cone Theorem) *Let X be a smooth projective variety.*

- *There exists a countable family of curves $(C_i)_{i \in I}$ such that $(K_X \cdot C_i) < 0$ for all $i \in I$ and*

$$\overline{NE}(X) = \overline{NE}(X)_{K_X \geq 0} + \sum_{i \in I} \mathbf{R}^+[C_i].$$

- *The rays $\mathbf{R}^+[C_i]$ are extremal and, in characteristic zero, they can be contracted.*

More generally, in characteristic zero, each extremal subcone which is negative (i.e., on which the canonical class is negative) can be contracted.

Examples 1.8 1) For $\mathbf{P}^n_{\mathbf{k}}$, there is not much to say: the only extremal ray of $\overline{NE}(X)$ is the whole of $\overline{NE}(X)$ (see Example 1.2.1)), and it is negative. Its contraction is the constant morphism. Any nonconstant regular map defined on $\mathbf{P}^n$ therefore has finite fibers.

2) When X is a smooth quadric in $\mathbf{P}^3_{\mathbf{k}}$, it is isomorphic to $\mathbf{P}^1_{\mathbf{k}} \times \mathbf{P}^1_{\mathbf{k}}$ and there are two extremal rays in $\overline{NE}(X)$ (see Example 1.2.2)). They are negative and their contractions correspond to each of the two projections $X \to \mathbf{P}^1_{\mathbf{k}}$.

3) When X is a smooth cubic in $\mathbf{P}^3_{\mathbf{k}}$, the class of each of the 27 lines contained in X spans a negative extremal ray (see Example 1.2.3)). The subcone $\sum_{i=1}^6 \mathbf{R}^+[C_i]$ is negative extremal and its contraction is the blow-up $X \to \mathbf{P}^2_{\mathbf{k}}$.

4) Let X be the surface obtained by blowing up $\mathbf{P}^2_{\mathbf{k}}$ in 9 points; the vector space $N_1(X)$ has dimension 10 (each blow-up increases it by one). There exists on X a countable union of curves with self-intersection -1 and with intersection -1 with K_X (see Example 5.16), which span pairwise distinct negative extremal rays in $\overline{NE}(X)$. They accumulate on the hyperplane where K_X vanishes (it is a general fact that extremal rays are locally discrete in the open half-space where K_X is negative).

This theorem is the starting point of Mori's Minimal Model Program (MMP): starting from a smooth (complex) projective variety X, we can contract a negative extremal ray (if there are any) and obtain a regular map $c : X \to Y$. We would like to repeat this procedure with Y, until we get a variety on which the canonical class has nonnegative degree on every curve.

Several problems arise, depending on the type of the contraction $c : X \to Y$, the main problem being that Y is not, in general, smooth. There are three cases.

1) **Case** $\dim Y < \dim X$**.** This happens for example when X is a projective bundle over Y and the contracted ray is spanned by the class of a line contained in a fiber.

2) **Case** c **birational and divisorial** (c is not injective on a hypersurface of X). This happens for example when X is a blow-up of Y.

3) **Case** c **birational and "small"** (c is injective on the complement of a subvariety of X of codimension at least 2).

In the first two cases, singularities of Y are still "reasonable," but not in the third case, where they are so bad that there is no reasonable theory of intersection between curves and hypersurfaces any more. The MMP cannot be continued with Y, and we look instead for another small contraction $c' : X' \to Y$, where X' is an algebraic variety with reasonable singularities with which the program can be continued, and c' is the contraction of an extremal ray which is *positive* (recall that our aim is to make the canonical class "more and more positive"). This surgery (we replace a subvariety of X of codimension at least 2 by another) is called a *flip* and it was a central problem in Mori's theory to show their existence (which is now known by [BCHM]; see [Dr], cor. 2.5).

The second problem also comes from flips: in the first two cases, the dimension of the vector space $N_1(Y)$ is one less than the dimension of $N_1(X)$. These vector space being finite-dimensional, this ensures that the program will eventually stop. But in case of a flip $c' : X' \to Y$ of a small contraction c, the vector spaces $N_1(X')$ and $N_1(X)$ *have same dimensions*, and one needs to exclude the possibility of an infinite chain of flips (this has been done only in small dimensions).

1.9. An example of a flip. The product $P = \mathbf{P}^1_{\mathbf{k}} \times \mathbf{P}^2_{\mathbf{k}}$ can be realized as a subvariety of $\mathbf{P}^5_{\mathbf{k}}$ by the regular map

$$((x_0, x_1), (y_0, y_1, y_2)) \mapsto (x_0y_0, x_1y_0, x_0y_1, x_1y_1, x_0y_2, x_1y_2).$$

Let Y be the cone (in $\mathbf{P}^6$) over P. There exists a smooth algebraic variety X of dimension 4 and a regular map $f : X \to Y$ which replaces the vertex of the cone Y by a copy of P. There exist birational regular maps $X \to X_1$ and $X \to X_2$ (where X_1 and X_2 are smooth algebraic varieties) which coincide on P with the projections $P \to \mathbf{P}^1_{\mathbf{k}}$ and $P \to \mathbf{P}^2_{\mathbf{k}}$, which are injective on the complement of P and through which f factors. We obtain in this way regular maps $X_i \to Y$ which are small contractions of extremal rays. The ray is negative for X_2 and positive for X_1. The contraction $X_1 \to Y$ is therefore the flip of the contraction $X_2 \to Y$. We will come back to this example in more details in Example 8.21.

1.10. Conventions. (Almost) all schemes are of finite type over a field. A variety is a geometrically integral scheme (of finite type over a field). A subvariety is always closed (and integral).

Chapter 2

Divisors and line bundles

In this chapter and the rest of these notes, $\mathbf{k}$ is a field and a $\mathbf{k}$-*variety* is an integral scheme of finite type over $\mathbf{k}$.

2.1 Weil and Cartier divisors

In §1, we defined a 1-cycle on a $\mathbf{k}$-scheme X as a (finite) formal linear combination (with integral, rational, or real coefficients) of integral curves in X. Similarly, we define a (Weil) divisor as a (finite) formal linear combination with integral coefficients of integral hypersurfaces in X. We say that the divisor is *effective* if the coefficients are all nonnegative.

Assume that X is *regular in codimension 1* (for example, normal). For each integral hypersurface Y of X with generic point η, the integral local ring $\mathscr{O}_{X,\eta}$ has dimension 1 and is regular, hence is a discrete valuation ring with valuation v_Y. For any nonzero rational function f on X, the integer $v_Y(f)$ (valuation of f along Y) is the order of vanishing of f along Y if it is nonnegative, and the opposite of the order of the pole of f along Y otherwise. We define the divisor of f as

$$\operatorname{div}(f) = \sum_Y v_Y(f) Y.$$

When X is normal, a (nonzero) rational function f is regular if and only if its divisor is effective ([H1], Proposition II.6.3A).

Assume that X is *locally factorial*, i.e., that its local rings are unique factorization domains. Then one sees ([H1], Proposition II.6.11) that any hypersurface can be defined locally by 1 (regular) equation.[1] Similarly, any divisor is locally the divisor of a rational function. Such divisors are called *locally principal,* and they are the ones that we are interested in. The following formal definition is less enlightening.

Definition 2.1 (Cartier divisors.) *A Cartier divisor on a* $\mathbf{k}$*-scheme* X *is a global section of the sheaf* $\mathscr{K}_X^*/\mathscr{O}_X^*$*, where* $\mathscr{K}_X$ *is the sheaf of total quotient rings of* $\mathscr{O}_X$*.*

On an open affine subset U of X, the ring $\mathscr{K}_X(U)$ is the localization of $\mathscr{O}_X(U)$ by the multiplicative system of non zero-divisors and $\mathscr{K}_X^*(U)$ is the group of its invertible elements (if U is integral, $\mathscr{K}_X^*(U)$ is just the multiplicative group of the quotient field of $\mathscr{O}_X(U)$).

In other words, a Cartier divisor is given by a collection of pairs (U_i, f_i), where (U_i) is an open cover of X and f_i an invertible element of $\mathscr{K}_X(U_i)$, such that f_i/f_j is in $\mathscr{O}_X^*(U_i \cap U_j)$. When X is integral, we may take integral open sets U_i, and f_i is then a nonzero rational function on U_i such that f_i/f_j is a regular function on $U_i \cap U_j$ that does not vanish.

2.2. Associated Weil divisor. Assume that the $\mathbf{k}$-scheme X is *regular in codimension 1.* To a Cartier divisor D on X, given by a collection (U_i, f_i), one can associate a Weil divisor $\sum_Y n_Y Y$ on X, where the

[1] This comes from the fact that in a unique factorization domain, prime ideals of height 1 are principal.

integer n_Y is the valuation of f_i along $Y \cap U_i$ for any i such that $Y \cap U_i$ is nonempty (it does not depend on the choice of such an i).

Again, on a locally factorial variety (i.e., a variety whose local rings are unique factorization domains; for example a smooth variety), there is no distinction between Cartier divisors and Weil divisors.

2.3. Effective Cartier divisors. A Cartier divisor D is *effective* if it can be defined by a collection (U_i, f_i) where f_i is in $\mathscr{O}_X(U_i)$. We write $D \geq 0$. When D is not zero, it defines a subscheme of X of codimension 1 by the "equation" f_i on each U_i. We still denote it by D.

2.4. Principal Cartier divisors. A Cartier divisor is *principal* if it is in the image of the natural map

$$H^0(X, \mathscr{K}_X^*) \to H^0(X, \mathscr{K}_X^*/\mathscr{O}_X^*).$$

In other words, when X is integral, the divisor can be defined by a global nonzero rational function on the whole of X.

2.5. Linearly equivalent divisors. Two Cartier divisors D and D' are *linearly equivalent* if their difference is principal; we write $D \underset{\text{lin}}{\equiv} D'$. Similarly, if X is regular in codimension 1, two Weil divisors are linearly equivalent if their difference is the divisor of a nonzero rational fucntion on X.

Example 2.6 Let X be the quadric cone defined in $\mathbf{A}^3_{\mathbf{k}}$ by the equation $xy = z^2$. It is normal. The line L defined by $x = z = 0$ is contained in X hence defines a Weil divisor on X which cannot be defined near the origin by one equation (the ideal (x, z) is not principal in the local ring of X at the origin). It is therefore not a Cartier divisor. However, $2L$ is a principal Cartier divisor, defined by x.

Example 2.7 On a smooth projective curve X, a (Cartier) divisor is just a finite formal linear combination of closed points $\sum_{p \in X} n_p p$. We define its degree to be the integer $\sum n_p [k(p) : \mathbf{k}]$. One proves (see [H1], Corollary II.6.10) that the degree of the divisor of a regular function is 0, hence the degree factors through

$$\{\text{Cartier divisors on } X\}/\text{lin. equiv.} \to \mathbf{Z}.$$

This map is in general not injective.

2.2 Invertible sheaves

Definition 2.8 (Invertible sheaves) *An invertible sheaf on a scheme X is a locally free $\mathscr{O}_X$-module of rank* 1.

The terminology comes from the fact that the tensor product defines a group structure on the set of locally free sheaves of rank 1 on X, where the inverse of an invertible sheaf $\mathscr{L}$ is $\mathscr{H}om(\mathscr{L}, \mathscr{O}_X)$. This makes the set of isomorphism classes of invertible sheaves on X into an abelian group called the *Picard group* of X, and denoted by $\text{Pic}(X)$. For any $m \in \mathbf{Z}$, it is traditional to write $\mathscr{L}^m$ for the mth (tensor) power of $\mathscr{L}$ (so in particular, $\mathscr{L}^{-1}$ is the dual of $\mathscr{L}$).

Let $\mathscr{L}$ be an invertible sheaf on X. We can cover X with affine open subsets U_i on which $\mathscr{L}$ is trivial and we obtain

$$g_{ij} \in \Gamma(U_i \cap U_j, \mathscr{O}^*_{U_i \cap U_j}) \tag{2.1}$$

as changes of trivializations, or transition functions. They satisfy the cocycle condition

$$g_{ij} g_{jk} g_{ki} = 1$$

hence define a Cech 1-cocycle for $\mathscr{O}_X^*$. One checks that this induces an isomorphism

$$\text{Pic}(X) \simeq H^1(X, \mathscr{O}_X^*). \tag{2.2}$$

For any $m \in \mathbf{Z}$, the invertible sheaf $\mathscr{L}^m$ corresponds to the collection of transition functions $(g_{ij}^m)_{i,j}$.

2.9. Invertible sheaf associated with a Cartier divisor. To a Cartier divisor D on X given by a collection (U_i, f_i), one can associate an invertible subsheaf $\mathscr{O}_X(D)$ of $\mathscr{K}_X$ by taking the sub-$\mathscr{O}_X$-module of $\mathscr{K}_X$ generated by $1/f_i$ on U_i. We have

$$\mathscr{O}_X(D_1) \otimes \mathscr{O}_X(D_2) \simeq \mathscr{O}_X(D_1 + D_2).$$

Every invertible subsheaf of $\mathscr{K}_X$ is obtained in this way, and two divisors are linearly equivalent if and only if their associated invertible sheaves are isomorphic ([H1], Proposition II.6.13). When X is integral, or projective over a field, every invertible sheaf is a subsheaf of $\mathscr{K}_X$ ([H1], Remark II.6.14.1 and Proposition II.6.15), so we get an isomorphism of groups:

$$\{\text{Cartier divisors on } X\}/\text{lin. equiv.} \simeq \{\text{Invertible sheaves on } X\}/\text{isom.} = \operatorname{Pic}(X).$$

We will write $H^i(X, D)$ instead of $H^i(X, \mathscr{O}_X(D))$ and, if $\mathscr{F}$ is a coherent sheaf on X, $\mathscr{F}(D)$ instead of $\mathscr{F} \otimes_{\mathscr{O}_X} \mathscr{O}_X(D)$.

Assume that X is integral and normal. One has

$$\Gamma(X, \mathscr{O}_X(D)) \simeq \{f \in \mathscr{K}_X(X) \mid f = 0 \ \text{ or } \ \operatorname{div}(f) + D \geq 0\}. \tag{2.3}$$

Indeed, if (U_i, f_i) represents D, and f is a nonzero rational function on X such that $\operatorname{div}(f)+D$ is effective, ff_i is regular on U_i (because X is normal!), and $f|_{U_i} = (ff_i)\frac{1}{f_i}$ defines a section of $\mathscr{O}_X(D)$ over U_i. Conversely, any global section of $\mathscr{O}_X(D)$ is a rational function f on X such that, on each U_i, the product $f|_{U_i} f_i$ is regular. Hence $\operatorname{div}(f) + D$ effective.

Remark 2.10 If D is a nonzero effective Cartier divisor on X and we still denote by D the subscheme of X that it defines (see 2.3), we have an exact sequence of sheaves[2]

$$0 \to \mathscr{O}_X(-D) \to \mathscr{O}_X \to \mathscr{O}_D \to 0.$$

Remark 2.11 Going back to Definition 2.1 of Cartier divisors, one checks that the morphism

$$\begin{array}{rcl} H^0(X, \mathscr{K}_X^*/\mathscr{O}_X^*) & \to & H^1(X, \mathscr{O}_X^*) \\ D & \mapsto & [\mathscr{O}_X(D)] \end{array}$$

induced by (2.2) is the coboundary of the short exact sequence

$$0 \to \mathscr{O}_X^* \to \mathscr{K}_X^* \to \mathscr{K}_X^*/\mathscr{O}_X^* \to 0.$$

Example 2.12 An integral hypersurface Y in $\mathbf{P}^n_{\mathbf{k}}$ corresponds to a prime ideal of height 1 in $\mathbf{k}[x_0, \dots, x_n]$, which is therefore (since the ring $\mathbf{k}[x_0, \dots, x_n]$ is factorial) principal. Hence Y is defined by one (homogeneous) irreducible equation f of degree d (called the *degree of* Y). This defines a surjective morphism

$$\{\text{Cartier divisors on } \mathbf{P}^n_{\mathbf{k}}\} \to \mathbf{Z}.$$

Since f/x_0^d is a rational function on $\mathbf{P}^n_{\mathbf{k}}$ with divisor $Y - dH_0$ (where H_0 is the hyperplane defined by $x_0 = 0$), Y is linearly equivalent to dH_0. Conversely, the divisor of any rational function on $\mathbf{P}^n_{\mathbf{k}}$ has degree 0 (because it is the quotient of two homogeneous polynomials of the same degree), hence we obtain an isomorphism

$$\operatorname{Pic}(\mathbf{P}^n_{\mathbf{k}}) \simeq \mathbf{Z}.$$

We denote by $\mathscr{O}_{\mathbf{P}^n_{\mathbf{k}}}(d)$ the invertible sheaf corresponding to an integer d (it is $\mathscr{O}_{\mathbf{P}^n_{\mathbf{k}}}(D)$ for any divisor D of degree d). One checks that the space of global sections of $\mathscr{O}_{\mathbf{P}^n_{\mathbf{k}}}(d)$ is 0 for $d < 0$ and isomorphic to the vector space of homogeneous polynomials of degree d in $n+1$ variables for $d \geq 0$.

[2] Let i be the inclusion of D in X. Since this is an exact sequence of sheaves *on* X, the sheaf on the right should be $i_*\mathscr{O}_D$ (a sheaf on X with support on D). However, it is customary to drop i_*. Note that as far as cohomology calculations are concerned, this does not make any difference ([H1], Lemma III.2.10).

Exercise 2.13 Let X be an integral scheme which is regular in codimension 1. Show that

$$\mathrm{Pic}(X \times \mathbf{P}^n_{\mathbf{k}}) \simeq \mathrm{Pic}(X) \times \mathbf{Z}.$$

(*Hint:* proceed as in [H1], Proposition 6.6 and Example 6.6.1). In particular,

$$\mathrm{Pic}(\mathbf{P}^m_{\mathbf{k}} \times \mathbf{P}^n_{\mathbf{k}}) \simeq \mathbf{Z} \times \mathbf{Z}.$$

This can be seen directly as in Example 2.12 by proving first that any hypersurface in $\mathbf{P}^m_{\mathbf{k}} \times \mathbf{P}^n_{\mathbf{k}}$ is defined by a *bihomogeneous* polynomial in the variables $((x_0, \dots, x_n), (y_0, \dots, y_m))$.

Remark 2.14 In all of the examples given above, the Picard group is an abelian group of finite type. This is not always the case. For smooth projective varieties, the Picard group is in general the extension of an abelian group of finite type by a connected group (called an *abelian variety*).

2.15. Pull-back and restriction. Let $\pi : Y \to X$ be a morphism between schemes and let D be a Cartier divisor on X. The pull-back $\pi^*\mathscr{O}_X(D)$ is an invertible subsheaf of $\mathscr{K}_Y$ hence defines a linear equivalence class of divisors on Y (improperly) denoted by π^*D. Only the linear equivalence class of π^*D is well-defined in general; however, when Y is reduced and D is a divisor (U_i, f_i) whose support contains the image of none of the irreducible components of Y, the collection $(\pi^{-1}(U_i), f_i \circ \pi)$ defines a divisor π^*D in that class. In particular, it makes sense to restrict a Cartier divisor to a subvariety not contained in its support, and to restrict a Cartier divisor *class* to any subvariety.

2.3 Line bundles

A *line bundle* on a scheme X is a scheme L with a morphism $\pi : L \to X$ which is locally (on the base) "trivial", i.e., isomorphic to $\mathbf{A}^1_U \to U$, in such a way that the changes of trivializations are linear, i.e., given by $(x, t) \mapsto (x, \varphi(x)t)$, for some $\varphi \in \Gamma(U, \mathscr{O}^*_U)$. A *section* of $\pi : L \to X$ is a morphism $s : X \to L$ such that $\pi \circ s = \mathrm{Id}_X$. One checks that the sheaf of sections of $\pi : L \to X$ is an invertible sheaf on X. Conversely, to any invertible sheaf $\mathscr{L}$ on X, one can associate a line bundle on X: if $\mathscr{L}$ is trivial on an affine cover (U_i), just glue the $\mathbf{A}^1_{U_i}$ together, using the g_{ij} of (2.1). It is common to use the words "invertible sheaf" and "line bundle" interchangeably.

Assume that X is integral and normal. A nonzero section s of a line bundle $L \to X$ defines an effective Cartier divisor on X (by the equation $s = 0$ on each affine open subset of X over which L is trivial), which we denote by $\mathrm{div}(s)$. With the interpretation (2.3), if D is a Cartier divisor on X and L is the line bundle associated with $\mathscr{O}_X(D)$, we have

$$\mathrm{div}(s) = \mathrm{div}(f) + D.$$

In particular, if D is effective, the function $f = 1$ corresponds to a section of $\mathscr{O}_X(D)$ with divisor D. In general, any nonzero rational function f on X can be seen as a (regular, nowhere vanishing) section of the line bundle $\mathscr{O}_X(-\mathrm{div}(f))$.

Example 2.16 Let $\mathbf{k}$ be a field and let W be a $\mathbf{k}$-vector space. We construct a line bundle $L \to \mathbf{P}W$ whose fiber above a point x of $\mathbf{P}W$ is the line ℓ_x of W represented by x by setting

$$L = \{(x, v) \in \mathbf{P}W \times W \mid v \in \ell_x\}.$$

On the standard open set U_i (defined after choice of a basis for W), L is defined in $U_i \times W$ by the equations $v_j = v_i x_j$, for all $j \neq i$. The trivialization on U_i is given by $(x, v) \mapsto (x, v_i)$, so that $g_{ij}(x) = x_i/x_j$, for $x \in U_i \cap U_j$. One checks that this line bundle corresponds to $\mathscr{O}_{\mathbf{P}W}(-1)$ (see Example 2.12).

Example 2.17 (Canonical line bundle) Let X be a complex manifold of dimension n. Consider the line bundle ω_X on X whose fiber at a point x of X is the (one-dimensional) vector space of ($\mathbf{C}$-multilinear) differential n-forms on the (holomorphic) tangent space to X at x. It is called the *canonical (line) bundle* on X. Any associated divisor is called a *canonical divisor* and is usually denoted by K_X (note that it is not uniquely defined!).

As we saw in Examples 1.4, we have

$$\omega_{\mathbf{P}^n_{\mathbf{k}}} = \mathscr{O}_{\mathbf{P}^n_{\mathbf{k}}}(-n-1)$$

and, for any smooth hypersurface X of degree d in $\mathbf{P}^n_{\mathbf{k}}$,

$$\omega_X = \mathscr{O}_X(-n-1+d).$$

2.4 Linear systems and morphisms to projective spaces

Let $\mathscr{L}$ be an invertible sheaf on an integral normal scheme X of finite type over a field $\mathbf{k}$ and let $|\mathscr{L}|$ be the set of (effective) divisors of global nonzero sections of $\mathscr{L}$. It is called the *linear system* associated with $\mathscr{L}$. The quotient of two sections which have the same divisor is a regular function on X which does not vanish. *If X is projective*, the map $\operatorname{div} : \mathbf{P}\Gamma(X, \mathscr{L}) \to |\mathscr{L}|$ is therefore bijective.

Let D be a Cartier divisor on X. We write $|D|$ instead of $|\mathscr{O}_X(D)|$; it is the set of effective divisors on X which are linearly equivalent to D.

2.18. We now get to a very important point: the link between morphisms from X to a projective space and vector spaces of sections of invertible sheaves on X. Assume for simplicity that X is integral.

Let W be a $\mathbf{k}$-vector space of finite dimension and let $u : X \to \mathbf{P}W$ be a regular map. Consider the invertible sheaf $\mathscr{L} = u^*\mathscr{O}_{\mathbf{P}W}(1)$ and the linear map

$$\Gamma(u) : W^* \simeq \Gamma\big(\mathbf{P}W, \mathscr{O}_{PW}(1)\big) \to \Gamma(X, \mathscr{L}).$$

A section of $\mathscr{O}_{\mathbf{P}W}(1)$ vanishes on a hyperplane; its image by $\Gamma(u)$ is zero if and only if $u(X)$ is contained in this hyperplane. In particular, $\Gamma(u)$ is injective if and only if $u(X)$ is not contained in any hyperplane.

If $u : X \dashrightarrow \mathbf{P}W$ is only a rational map, it is defined on a dense open subset U of X, and we get as above a linear map $W^* \to \Gamma(U, \mathscr{L})$. If X is locally factorial, the invertible sheaf $\mathscr{L}$ is defined on U but extends to X (write $\mathscr{L} = \mathscr{O}_U(D)$ and take the closure of D in X) and, since X is normal, the restriction $\Gamma(X, \mathscr{L}) \to \Gamma(U, \mathscr{L})$ is bijective, so we get again a map $W^* \to \Gamma(X, \mathscr{L})$.

Conversely, starting from an invertible sheaf $\mathscr{L}$ on X and a finite-dimensional vector space Λ of sections of $\mathscr{L}$, we define a rational map

$$\psi_\Lambda : X \dashrightarrow \mathbf{P}\Lambda^*$$

(also denoted by $\psi_{\mathscr{L}}$ when $\Lambda = \Gamma(X, \mathscr{L})$) by associating to a point x of X the hyperplane of sections of Λ that vanish at x. This map is not defined at points where all sections in Λ vanish (they are called *base-points* of Λ). If we choose a basis $(s_0, \dots, s_r)$ for Λ, we have also

$$u(x) = \big(s_0(x), \dots, s_r(x)\big),$$

where it is understood that the $s_j(x)$ are computed via the same trivialization of $\mathscr{L}$ in a neighborhood of x; the corresponding point of $\mathbf{P}^r$ is independent of the choice of this trivialization.

These two constructions are inverse of one another. In particular, regular maps from X to a projective space, whose image is not contained in any hyperplane correspond to base-point-free linear systems on X.

Example 2.19 We saw in Example 2.12 that the vector space $\Gamma(\mathbf{P}^1_{\mathbf{k}}, \mathscr{O}_{\mathbf{P}^1_{\mathbf{k}}}(m))$ has dimension $m+1$. A basis is given by $(s^m, s^{m-1}t, \dots, t^m)$. The corresponding linear system is base-point-free and induces a morphism

$$\begin{array}{ccc} \mathbf{P}^1_{\mathbf{k}} & \to & \mathbf{P}^m_{\mathbf{k}} \\ (s,t) & \mapsto & (s^m, s^{m-1}t, \dots, t^m) \end{array}$$

whose image (the *rational normal curve*) can be defined by the vanishing of all 2×2-minors of the matrix

$$\begin{pmatrix} x_0 & \cdots & x_{m-1} \\ x_1 & \cdots & x_m \end{pmatrix}.$$

Example 2.20 (Cremona involution) The rational map

$$\begin{array}{rccccl} u: & \mathbf{P}^2_{\mathbf{k}} & \dashrightarrow & \mathbf{P}^2_{\mathbf{k}} & \\ & (x,y,z) & \longmapsto & (\frac{1}{x},\frac{1}{y},\frac{1}{z}) & = (yz,zx,xy) \end{array}$$

is defined everywhere except at the 3 points $(1,0,0)$, $(0,1,0)$, and $(0,0,1)$. It is associated with the space $\langle yz,zx,xy\rangle$ of sections of $\mathscr{O}_{\mathbf{P}^2_{\mathbf{k}}}(2)$ (which is the space of all conics passing through these 3 points).

2.5 Globally generated sheaves

Let X be a scheme of finite type over a field $\mathbf{k}$. A coherent sheaf $\mathscr{F}$ is *generated by its global sections at a point* $x \in X$ (or *globally generated at* x) if the images of the global sections of $\mathscr{F}$ (i.e., elements of $\Gamma(X,\mathscr{F})$) in the stalk $\mathscr{F}_x$ generate that stalk as a $\mathscr{O}_{X,x}$-module. The set of point at which $\mathscr{F}$ is globally generated is the complement of the support of the cokernel of the *evaluation* map

$$\mathrm{ev}: \Gamma(X,\mathscr{F}) \otimes_{\mathbf{k}} \mathscr{O}_X \to \mathscr{F}.$$

It is therefore open. The sheaf $\mathscr{F}$ is *generated by its global sections* (or *globally generated*) if it is generated by its global sections at each point $x \in X$. This is equivalent to the surjectivity of ev, and to the fact that $\mathscr{F}$ is the quotient of a free sheaf.

Since closed points are dense in X, it is enough to check global generation at every closed point x. This is equivalent, by Nakayama's lemma, to the surjectivity of

$$\mathrm{ev}_x : \Gamma(X,\mathscr{F}) \to \Gamma(X,\mathscr{F} \otimes k(x))$$

We sometimes say that $\mathscr{F}$ is *generated by finitely many global sections* (at $x \in X$) if there are $s_1,\dots,s_r \in \Gamma(X,\mathscr{F})$ such that the corresponding evaluation maps, where $\Gamma(X,\mathscr{F})$ is replaced with the vector subspace generated by $s_1,\dots,s_r$, are surjective.

Any quasi-coherent sheaf on an affine sheaf $X = \mathrm{Spec}(A)$ is generated by its global sections (such a sheaf can be written as $\widetilde{M}$, where M is an A-module, and $\Gamma(X,\widetilde{M}) = M$).

Any quotient of a globally generated sheaf has the same property. Any tensor product of globally generated sheaves has the same property. The restriction of a globally generated sheaf to a subscheme has the same property.

An invertible sheaf $\mathscr{L}$ on X is generated by its global sections if and only if for each closed point $x \in X$, there exists a global section $s \in \Gamma(X,\mathscr{L})$ that does not vanish at x (i.e., $s_x \notin \mathfrak{m}_{X,x}\mathscr{L}_x$, or $\mathrm{ev}_x(s) \neq 0$ in $\Gamma(X,\mathscr{L}\otimes k(x)) \simeq k(x)$). Another way to phrase this, using the constructions of 2.18, is to say that the invertible sheaf $\mathscr{L}$ is generated by finitely many global sections if and only if there exists a *morphism* $\psi : X \to \mathbf{P}^n_{\mathbf{k}}$ such that $\psi^*\mathscr{O}_{\mathbf{P}^n_{\mathbf{k}}}(1) \simeq \mathscr{L}$.[3]

Recall from 2.9 that Cartier divisors and invertible sheaves are more or less the same thing. For reasons that will be apparent later on (in particular when we will consider divisors with rational coefficients), we will try to use as often as possible the (additive) language of that of divisors instead of invertible sheaves. For example, if D is a Cartier divisor on X, the invertible sheaf $\mathscr{O}_X(D)$ is generated by its global sections (for brevity, we will sometimes say that D is generated by its global sections, or globally generated) if for any $x \in X$, there is a Cartier divisor on X, linearly equivalent to D, whose support does not contain x (use (2.3)).

Example 2.21 We saw in Example 2.12 that any invertible sheaf on the projective space $\mathbf{P}^n_{\mathbf{k}}$ (with $n > 0$) is of the type $\mathscr{O}_{\mathbf{P}^n_{\mathbf{k}}}(d)$ for some integer d. This sheaf is not generated by its global sections for $d \leq 0$ because any global section is constant. However, when $d > 0$, the vector space $\Gamma(\mathbf{P}^n_{\mathbf{k}}, \mathscr{O}_{\mathbf{P}^n_{\mathbf{k}}}(d))$ is isomorphic to the space of homogeneous polynomials of degree d in the homogeneous coordinates $x_0,\dots,x_n$ on $\mathbf{P}^n_{\mathbf{k}}$. At each point of $\mathbf{P}^n_{\mathbf{k}}$, one of these coordinates, say x_i, does not vanish, hence the section x_i^d does not vanish either. It follows that $\mathscr{O}_{\mathbf{P}^n_{\mathbf{k}}}(d)$ is generated by its global sections if and only if $d > 0$.

[3]If $s \in \Gamma(X,\mathscr{L})$, the subset $X_s = \{x \in X \mid \mathrm{ev}_x(s) \neq 0\}$ is open. A family $(s_i)_{i\in I}$ of sections generate $\mathscr{L}$ if and only if $X = \bigcup_{i\in I} X_{s_i}$. If X is noetherian and $\mathscr{L}$ is globally generated, it is generated by finitely many global sections.

2.6 Ample divisors

The following definition, although technical, is extremely important.

Definition 2.22 *A Cartier divisor D on a noetherian scheme X is* ample *if, for every coherent sheaf $\mathscr{F}$ on X, the sheaf $\mathscr{F}(mD)$*[4] *is generated by its global sections for all m large enough.*

Any sufficiently high multiple of an ample divisor is therefore globally generated, but an ample divisor may not be globally generated (it may have no nonzero global sections).

The restriction of an ample Cartier divisor to a closed subscheme is ample. The sum of two ample Cartier divisors is still ample. The sum of an ample Cartier divisor and a globally generated Cartier divisor is ample. Any Cartier divisor on a noetherian affine scheme is ample.

Proposition 2.23 *Let D be a Cartier divisor on a noetherian scheme. The following conditions are equivalent:*

(i) *D is ample;*

(ii) *pD is ample for all $p > 0$;*

(iii) *pD is ample for some $p > 0$.*

PROOF. We already explain that (i) implies (ii), and (ii) $\Rightarrow$ (iii) is trivial. Assume that pD is ample. Let $\mathscr{F}$ be a coherent sheaf. Then for each $j \in \{0, \dots, p-1\}$, the sheaf $\mathscr{F}(iD)(mpD) = \mathscr{F}((i+mp)D)$ is generated by its global sections for $m \gg 0$. It follows that $\mathscr{F}(mD)$ is generated by its global sections for all $m \gg 0$, hence D is ample. $\square$

Proposition 2.24 *Let D and E be Cartier divisors on a noetherian scheme. If D is ample, so is $pD + E$ for all $p \gg 0$.*

PROOF. Since D is ample, $qD + E$ is globally generated for all q large enough, and $(q+1)D + E$ is then ample. $\square$

2.25. Q-divisors. It is useful at this point to introduce $\mathbf{Q}$-*divisors* on a normal scheme X. They are simply linear combinations of integral hypersurfaces in X with rational coefficients. One says that such a divisor is $\mathbf{Q}$-*Cartier* if some multiple has integral coefficients and is a Cartier divisor; in that case, we say that it is ample if some (integral) positive multiple is ample (all further positive multiples are then ample by Proposition 2.23).

Example 2.26 Going back to the quadric cone X of Example 2.6, we see that the line L is a $\mathbf{Q}$-Cartier divisor in X.

Example 2.27 One can rephrase Proposition 2.24 by saying that if D is an ample $\mathbf{Q}$-divisor and E is any $\mathbf{Q}$-Cartier divisor, $D + tE$ is ample for all t rational small enough.

Here is the fundamental result, due to Serre, that justifies the definition of ampleness.

Theorem 2.28 (Serre) *The hyperplane divisor on $\mathbf{P}^n_{\mathbf{k}}$ is ample.*

More precisely, for any coherent sheaf $\mathscr{F}$ on $\mathbf{P}^n_{\mathbf{k}}$, the sheaf $\mathscr{F}(m)$ is generated by finitely many global sections for all $m \gg 0$.

[4] This is the traditional notation for the tensor product $\mathscr{F} \otimes \mathscr{O}_X(mD)$. Similarly, if X is a subscheme of some projective space $\mathbf{P}^n_{\mathbf{k}}$, we write $\mathscr{F}(m)$ instead of $\mathscr{F} \otimes \mathscr{O}_{\mathbf{P}^n_{\mathbf{k}}}(m)$.

PROOF. The restriction of $\mathscr{F}$ to each standard affine open subset U_i is generated by finitely many sections $s_{ik} \in \Gamma(U_i, \mathscr{F})$. We want to show that each $s_{ik}x_i^m \in \Gamma(U_i, \mathscr{F}(m))$ extends for $m \gg 0$ to a section t_{ik} of $\mathscr{F}(m)$ on $\mathbf{P}^n_{\mathbf{k}}$.

Let $s \in \Gamma(U_i, \mathscr{F})$. It follows from [H1], Lemma II.5.3.(b)) that for each j, the section

$$x_i^p s|_{U_i \cap U_j} \in \Gamma(U_i \cap U_j, \mathscr{F}(p))$$

extends to a section $t_j \in \Gamma(U_j, \mathscr{F}(p))$ for $p \gg 0$ (in other words, t_j restricts to $x_i^p s$ on $U_i \cap U_j$). We then have

$$t_j|_{U_i \cap U_j \cap U_k} = t_k|_{U_i \cap U_j \cap U_k}$$

for all j and k hence, upon multiplying again by a power of x_i,

$$x_i^q t_j|_{U_j \cap U_k} = x_i^q t_k|_{U_j \cap U_k}.$$

for $q \gg 0$ ([H1], Lemma II.5.3.(a)). This means that the $x_i^q t_j$ glue to a section t of $\mathscr{F}(p+q)$ on $\mathbf{P}^n_{\mathbf{k}}$ which extends $x_i^{p+q}s$.

We then obtain finitely many global sections t_{ik} of $\mathscr{F}(m)$ which generate $\mathscr{F}(m)$ on each U_i hence on $\mathbf{P}^n_{\mathbf{k}}$. □

Corollary 2.29 *Let X be a closed subscheme of a projective space $\mathbf{P}^n_{\mathbf{k}}$ and let $\mathscr{F}$ be a coherent sheaf on X.*

a) *The* $\mathbf{k}$*-vector spaces $H^q(X, \mathscr{F})$ all have finite dimension.*

b) *The* $\mathbf{k}$*-vector spaces $H^q(X, \mathscr{F}(m))$ all vanish for $m \gg 0$.*

PROOF. Since any coherent sheaf on X can be considered as a coherent sheaf on $\mathbf{P}^n_{\mathbf{k}}$ (with the same cohomology), we may assume $X = \mathbf{P}^n_{\mathbf{k}}$. For $q > n$, we have $H^q(X, \mathscr{F}) = 0$ and we proceed by descending induction on q.

By Theorem 2.28, there exist integers r and p and an exact sequence

$$0 \longrightarrow \mathscr{G} \longrightarrow \mathscr{O}_{\mathbf{P}^n_{\mathbf{k}}}(-p)^r \longrightarrow \mathscr{F} \longrightarrow 0$$

of coherent sheaves on $\mathbf{P}^n_{\mathbf{k}}$. The vector spaces $H^q(\mathbf{P}^n_{\mathbf{k}}, \mathscr{O}_{\mathbf{P}^n_{\mathbf{k}}}(-p))$ can be computed by hand are all finite-dimensional. The exact sequence

$$H^q(\mathbf{P}^n_{\mathbf{k}}, \mathscr{O}_X(-p))^r \longrightarrow H^q(\mathbf{P}^n_{\mathbf{k}}, \mathscr{F}) \longrightarrow H^{q+1}(\mathbf{P}^n_{\mathbf{k}}, \mathscr{G})$$

yields a).

Again, direct calculations show that $H^q(\mathbf{P}^n, \mathscr{O}_{\mathbf{P}^n_{\mathbf{k}}}(m-p))$ vanishes for all $m > p$ and all $q > 0$. The exact sequence

$$H^q(\mathbf{P}^n_{\mathbf{k}}, \mathscr{O}_X(m-p))^r \longrightarrow H^q(\mathbf{P}^n_{\mathbf{k}}, \mathscr{F}(m)) \longrightarrow H^{q+1}(\mathbf{P}^n_{\mathbf{k}}, \mathscr{G}(m))$$

yields b). □

2.7 Very ample divisors

Definition 2.30 *A Cartier divisor D on a scheme X of finite type over a field* $\mathbf{k}$ *is* very ample *if there exists an embedding $i : X \hookrightarrow \mathbf{P}^n_{\mathbf{k}}$ such that $i^*H \underset{\text{lin}}{\equiv} D$, where H is a hyperplane in $\mathbf{P}^n_{\mathbf{k}}$.*

In algebraic geometry "embedding" means that i induces an isomorphism between X and a locally closed subscheme of $\mathbf{P}^n_{\mathbf{k}}$.

In other words, a Cartier divisor is very ample if and only if its sections define a morphism from X to a projective space which induces an isomorphism between X and a locally closed subscheme of the projective space. The restriction of a very ample Cartier divisor to a locally closed subscheme is very ample. Any very ample divisor is generated by finitely many global sections.

Serre's Theorem 2.28 implies that a very ample divisor on a projective scheme over a field is also ample, but the converse is false in general (see Example 2.31.3) below). However, there exists a close relationship between the two notions (ampleness is the stabilized version of very ampleness; see Theorem 2.34).

Examples 2.31 1) A hyperplane H is by definition very ample on $\mathbf{P}^n_{\mathbf{k}}$, and so are the divisors dH for every $d > 0$, because dH is the inverse image of a hyperplane by the Veronese embedding

$$\nu_d : \mathbf{P}^n \hookrightarrow \mathbf{P}^{\binom{n+d}{d}-1}.$$

We have therefore, for any divisor $D \underset{\text{lin}}{\equiv} dH$ on $\mathbf{P}^n_{\mathbf{k}}$ (for $n > 0$),

$$D \ \text{ ample } \iff D \ \text{ very ample } \iff d > 0.$$

2) It follows from Exercise 2.13 that any divisor on $\mathbf{P}^m_{\mathbf{k}} \times \mathbf{P}^n_{\mathbf{k}}$ (with $m,\, n > 0$) is linearly equivalent to a divisor of the type $aH_1 + bH_2$, where H_1 and H_2 are the pull-backs of the hyperplanes on each factor. The divisor $H_1 + H_2$ is very ample because it is the inverse image of a hyperplane by the Segre embedding

$$\mathbf{P}^m_{\mathbf{k}} \times \mathbf{P}^n_{\mathbf{k}} \hookrightarrow \mathbf{P}^{(m+1)(n+1)-1}_{\mathbf{k}}. \tag{2.4}$$

So is the divisor $aH_1 + bH_2$, where a and b are positive: this can be seen by composing the Veronese embeddings (ν_a, ν_b) with the Segre embedding. On the other hand, since $aH_1 + bH_2$ restricts to aH_1 on $\mathbf{P}^m_{\mathbf{k}} \times \{x\}$, hence it cannot be very ample when $a \le 0$. We have therefore, for any divisor $D \underset{\text{lin}}{\equiv} aH_1 + bH_2$ on $\mathbf{P}^m_{\mathbf{k}} \times \mathbf{P}^n_{\mathbf{k}}$ (for $m, n > 0$),

$$D \ \text{ ample } \iff D \ \text{ very ample } \iff a > 0 \text{ and } b > 0.$$

3) It is a consequence of the Nakai-Moishezon criterion (Theorem 4.1) that a divisor on a smooth projective curve is ample if and only if its degree (see Example 2.7) is positive. Let $X \subset \mathbf{P}^2_{\mathbf{k}}$ be a smooth cubic curve and let $p \in X$ be a (closed) inflection point. The divisor p has degree 1, hence is ample (in this particular case, this can be seen directly: there is a line L in $\mathbf{P}^2_{\mathbf{k}}$ which has contact of order three with X at p; in other words, the divisor L on $\mathbf{P}^2_{\mathbf{k}}$ restricts to the divisor $3p$ on X, hence the latter is very ample, hence ample, on X, and by Proposition 2.23, the divisor p is ample). However, it is not very ample: if it were, p would be linearly equivalent to another point q, and there would exist a rational function f on X with divisor $p - q$. The induced map $f : X \to \mathbf{P}^1_{\mathbf{k}}$ would then be an isomorphism (because f has degree 1 by Proposition 3.16 or [H1], Proposition II.6.9, hence is an isomorphism by [H1], Corollary I.6.12), which is absurd (because X has genus 1 by Exercise 3.2).

Proposition 2.32 *Let D and E be Cartier divisors on a scheme X of finite type over a field. If D is very ample and E is globally generated, $D + E$ is very ample. In particular, the sum of two very ample divisors is very ample.*

Proof. Since D is very ample, there exists an embedding $i : X \hookrightarrow \mathbf{P}^m_{\mathbf{k}}$ such that $i^*H \underset{\text{lin}}{\equiv} D$. Since D is globally generated and X is noetherian, D is generated by finitely many global sections (footnote 3), hence there exists a morphism $j : X \to \mathbf{P}^n_{\mathbf{k}}$ such that $j^*H \underset{\text{lin}}{\equiv} E$. Consider the morphism $(i, j) : X \to \mathbf{P}^m_{\mathbf{k}} \times \mathbf{P}^n_{\mathbf{k}}$. Since its composition with the first projection is i, it is an embedding. Its composition with the Segre embedding (2.4) is again an embedding

$$k : X \hookrightarrow \mathbf{P}^{(m+1)(n+1)-1}_{\mathbf{k}}$$

such that $k^*H \underset{\text{lin}}{\equiv} D + E$. □

Corollary 2.33 *Let D and E be Cartier divisors on a scheme of finite type over a field. If D is very ample, so is $pD + E$ for all $p \gg 0$.*

PROOF. Since D is ample, $qD+E$ is globally generated for all $q \gg 0$. The divisor $(q+1)D+E$ is then very ample by Proposition 2.32. □

Theorem 2.34 *Let X be a scheme of finite type over a field and let D be a Cartier divisor on X. Then D is ample if and only if pD is very ample for some (or all) integers $p \gg 0$.*

PROOF. If pD is very ample, it is ample, hence so is D by Proposition 2.23.

Assume conversely that D is ample. Let x_0 be a point of X and let V be an affine neighborhood of x_0 in X over which $\mathscr{O}_X(D)$ is trivial (isomorphic to $\mathscr{O}_V$). Let Y be the complement of V in X and let $\mathscr{I}_Y \subset \mathscr{O}_X$ be the ideal sheaf of Y. Since D is ample, there exists a positive integer m such that the sheaf $\mathscr{I}_Y(mD)$ is globally generated. Its sections can be seen as sections of $\mathscr{O}_X(mD)$ that vanish on Y. Therefore, there exists such a section, say $s \in \Gamma(X, \mathscr{I}_Y(mD)) \subset \Gamma(X, mD)$, which does not vanish at x_0 (i.e., $\mathrm{ev}_{x_0}(s) \neq 0$). The open set

$$X_s = \{x \in X \mid \mathrm{ev}_x(s) \neq 0\}$$

is then contained in V. Since $\mathscr{L}$ is trivial on V, the section s can be seen as a regular function on V, hence X_s is an open affine subset of X containing x_0.

Since X is noetherian, we can cover X with a finite number of these open subsets. Upon replacing s with a power, we may assume that the integer m is the same for all these open subsets. We have therefore sections $s_1, \dots, s_p$ of $\mathscr{O}_X(mD)$ such that the X_{s_i} are open affine subsets that cover X. In particular, $s_1, \dots, s_p$ have no common zeroes. Let f_{ij} be (finitely many) generators of the $\mathbf{k}$-algebra $\Gamma(X_{s_i}, \mathscr{O}_{X_{s_i}})$. The same proof as that of Theorem 2.28 shows that there exists an integer r such that $s_i^r f_{ij}$ extends to a section s_{ij} of $\mathscr{O}_X(rmD)$ on X. The global sections s_i^r, s_{ij} of $\mathscr{O}_X(rmD)$ have no common zeroes hence define a morphism

$$u : X \to \mathbf{P}^N_{\mathbf{k}}.$$

Let $U_i \subset \mathbf{P}^N_{\mathbf{k}}$ be the standard open subset corresponding to the coordinate s_i^r; the open subsets $U_1, \dots, U_p$ then cover $u(X)$ and $u^{-1}(U_i) = X_{s_i}$. Moreover, the induced morphism $u_i : X_{s_i} \to U_i$ corresponds by construction to a surjection $u_i^* : \Gamma(U_i, \mathscr{O}_{U_i}) \to \Gamma(X_{s_i}, \mathscr{O}_{X_{s_i}})$, so that u_i induces an isomorphism between X_{s_i} and its image. It follows that u is an isomorphism onto its image, hence rmD is very ample. □

Corollary 2.35 *A proper scheme is projective if and only if it carries an ample divisor.*

Proposition 2.36 *Any Cartier divisor on a projective scheme is linearly equivalent to the difference of two effective Cartier divisors.*

PROOF. Assume for simplicity that the projective scheme X is integral. Let D be a Cartier divisor on X and let H be an effective very ample divisor on X. For $m \gg 0$, the invertible sheaf $\mathscr{O}_X(D+mH)$ is generated by its global sections. In particular, it has a nonzero section; let E be its (effective) divisor. We have

$$D \underset{\text{lin}}{\equiv} E - mH,$$

which proves the proposition. □

2.8 A cohomological characterization of ample divisors

Theorem 2.37 *Let X be a projective scheme over a field and let D be a Cartier divisor on X. The following properties are equivalent:*

(i) *D est ample;*

(ii) *for each coherent sheaf $\mathscr{F}$ on X, we have $H^q(X, \mathscr{F}(mD)) = 0$ for all $m \gg 0$ and all $q > 0$;*

(iii) *for each coherent sheaf $\mathscr{F}$ on X, we have $H^1(X, \mathscr{F}(mD)) = 0$ for all $m \gg 0$.*

PROOF. Assume D ample. Theorem 2.34 then implies that rD is very ample for some $r > 0$. For each $0 \le s < r$, Corollary 2.29.b) yields

$$H^q(X, (\mathscr{F}(sD))(mD)) = 0$$

for all $m \ge m_s$. For

$$m \ge r \max(m_0, \dots, m_{r-1}),$$

we have $H^q(X, \mathscr{F}(mD)) = 0$. This proves that (i) implies (ii), which trivially implies (iii).

Assume that (iii) holds. Let $\mathscr{F}$ be a coherent sheaf on X, let x be a closed point of X, and let $\mathscr{G}$ be the kernel of the surjection

$$\mathscr{F} \to \mathscr{F} \otimes k(x)$$

of $\mathscr{O}_X$-modules. Since (iii) holds, there exists an integer m_0 such that

$$H^1(X, \mathscr{G}(mD)) = 0$$

for all $m \ge m_0$ (note that the integer m_0 may depend on $\mathscr{F}$ and x). Since the sequence

$$0 \to \mathscr{G}(mD) \to \mathscr{F}(mD) \to \mathscr{F}(mD) \otimes k(x) \to 0$$

is exact, the evaluation

$$\Gamma(X, \mathscr{F}(mD)) \to \Gamma(X, \mathscr{F}(mD) \otimes k(x))$$

is surjective. This means that its global sections generate $\mathscr{F}(mD)$ in a neighborhood $U_{\mathscr{F},m}$ of x. In particular, there exists an integer m_1 such that $m_1 D$ is globally generated on $U_{\mathscr{O}_X, m_1}$. For all $m \ge m_0$, the sheaf $\mathscr{F}(mD)$ is globally generated on

$$U_x = U_{\mathscr{O}_X, m_1} \cap U_{\mathscr{F}, m_0} \cap U_{\mathscr{F}, m_0+1} \cap \cdots \cap U_{\mathscr{F}, m_0+m_1-1}$$

since it can be written as

$$(\mathscr{F}((m_0 + s)D)) \otimes \mathscr{O}_X(r(m_1 D))$$

with $r \ge 0$ and $0 \le s < m_1$. Cover X with a finite number of open subsets U_x and take the largest corresponding integer m_0. This shows that D is ample and finishes the proof of the theorem. □

Corollary 2.38 *Let X and Y be projective schemes over a field and let $u : X \to Y$ be a morphism with finite fibers. Let D be an ample* **Q**-*Cartier divisor on Y. Then the* **Q**-*Cartier divisor u^*D is ample.*

PROOF. We may assume that D Cartier divisor. Let $\mathscr{F}$ be a coherent sheaf on X. In our situation, *the sheaf $u_*\mathscr{F}$ is coherent* ([H1], Corollary II.5.20). Moreover, the morphism u is finite[5] and *the inverse image by u of any affine open subset of Y is an affine open subset of X* ([H1], Exercise II.5.17.(b)). If $\mathscr{U}$ is a covering of Y by affine open subsets, $u^{-1}(\mathscr{U})$ is then a covering of X by affine open subsets, and by definition of $u_*\mathscr{F}$, the associated cochain complexes are isomorphic. This implies

$$H^q(X, \mathscr{F}) \simeq H^q(Y, u_*\mathscr{F})$$

for all integers q. We now have (projection formula; [H1], Exercise II.5.1.(d))

$$u_*(\mathscr{F}(mu^*D)) \simeq (u_*\mathscr{F})(mD)$$

hence

$$H^1(X, \mathscr{F}(mu^*D)) \simeq H^1(Y, (u_*\mathscr{F})(mD)).$$

Since $u_*\mathscr{F}$ is coherent and D is ample, the right-hand-side vanishes for all $m \gg 0$ by Theorem 2.37, hence also the left-hand-side. By the same theorem, it follows that *the divisor u^*D est ample.* □

Exercise 2.39 In the situation of the corollary, if u is *not* finite, show that u^*D is *not* ample.

Exercise 2.40 Let X be a projective scheme over a field. Show that a Cartier divisor is ample on X if and only if it is ample on each irreducible component of X_{red}.

[5] The very important fact that a projective morphism with finite fibers is finite is deduced in [H1] from the difficult Zariski's Main Theorem. In our case, it can also be proved in an elementary fashion (see [D2], th. 3.28).

Chapter 3

Intersection of curves and divisors

3.1 Curves

A curve is a projective integral scheme X of dimension 1 over a field $\mathbf{k}$. We define its (arithmetic) genus as

$$g(X) = \dim H^1(X, \mathscr{O}_X).$$

Example 3.1 The curve $\mathbf{P}^1_{\mathbf{k}}$ has genus 0. This can be obtained by a computation in Cech cohomology: cover X with the two affine subsets U_0 and U_1. The Cech complex

$$\Gamma(U_0, \mathscr{O}_{U_0}) \oplus \Gamma(U_1, \mathscr{O}_{U_1}) \to \Gamma(U_{01}, \mathscr{O}_{U_{01}})$$

is

$$\mathbf{k}[t] \oplus \mathbf{k}[t^{-1}] \to \mathbf{k}[t, t^{-1}],$$

hence the result.

Exercise 3.2 Show that the genus of a plane curve of degree d is $(d-1)(d-2)/2$ (*Hint:* assume that $(0,0,1)$ is not on the curve, cover it with the affine subsets U_0 and U_1 and compute the Cech cohomology groups as above).

We defined in Example 2.7 the degree of a Cartier divisor (or of an invertible sheaf) on a smooth curve over a field $\mathbf{k}$ by setting

$$\deg\Big(\sum_{p \text{ closed point in } X} n_p p\Big) = \sum n_p [k(p) : \mathbf{k}].$$

In particular, when $\mathbf{k}$ is algebraically closed, this is just $\sum n_p$.

If $D = \sum_p n_p p$ is an effective divisor ($n_p \geq 0$ for all p), we can view it as a 0-dimensional subscheme of X with (affine) support at set of points p for which $n_p > 0$, where it is defined by the ideal $\mathfrak{m}_{X,p}^{n_p}$. We have

$$h^0(D, \mathscr{O}_D) = \sum_p \dim_{\mathbf{k}}(\mathscr{O}_{X,p}/\mathfrak{m}_{X,p}^{n_p}) = \sum_p n_p \dim_{\mathbf{k}}(\mathscr{O}_{X,p}/\mathfrak{m}_{X,p}) = \deg(D).$$

The central theorem in this section is the following.[1]

Theorem 3.3 (Riemann-Roch theorem) *Let X be a smooth curve. For any divisor D on X, we have*

$$\chi(X, D) = \deg(D) + \chi(X, \mathscr{O}_X) = \deg(D) + 1 - g(X).$$

[1] This should really be called the Hirzebruch-Riemann-Roch theorem (or a (very) particular case of it). The original Riemann-Roch theorem is our Theorem 3.3 with the dimension of $H^1(X, \mathscr{L})$ replaced with that of its Serre-dual $H^0(X, \omega_X \otimes \mathscr{L}^{-1})$.

PROOF. By Proposition 2.36, we can write $D \underset{\text{lin}}{\equiv} E - F$, where E and F are effective (Cartier) divisors on X. Considering them as (0-dimensional) subschemes of X, we have exact sequences (see Remark 2.10)

$$\begin{array}{ccccccc} 0 \to & \mathscr{O}_X(E-F) & \to & \mathscr{O}_X(E) & \to & \mathscr{O}_F & \to 0 \\ 0 \to & \mathscr{O}_X & \to & \mathscr{O}_X(E) & \to & \mathscr{O}_E & \to 0 \end{array}$$

(note that the sheaf $\mathscr{O}_F(E)$ is isomorphic to $\mathscr{O}_F$, because $\mathscr{O}_X(E)$ is isomorphic to $\mathscr{O}_X$ in a neighborhood of the (finite) support of F, and similarly, $\mathscr{O}_E(E) \simeq \mathscr{O}_E$). As remarked above, we have

$$\chi(F, \mathscr{O}_F) = h^0(F, \mathscr{O}_F) = \deg(F).$$

Similarly, $\chi(E, \mathscr{O}_E) = \deg(E)$. This implies

$$\begin{aligned} \chi(X, D) &= \chi(X, E) - \chi(F, \mathscr{O}_F) \\ &= \chi(X, \mathscr{O}_X) + \chi(E, \mathscr{O}_E) - \deg(F) \\ &= \chi(X, \mathscr{O}_X) + \deg(E) - \deg(F) \\ &= \chi(X, \mathscr{O}_X) + \deg(D), \end{aligned}$$

and the theorem is proved. □

Later on, we will use this theorem to *define* the degree of a Cartier divisor D on any curve X, as the leading term of (what we will prove to be) the degree-1 polynomial $\chi(X, mD)$. The Riemann-Roch theorem then becomes a tautology.

Corollary 3.4 *Let X be a smooth curve. A divisor D on X is ample if and only if* $\deg(D) > 0$.

This will be generalized later to any curve (see 4.2).

PROOF. Let p be a closed point of X. If D is ample, $mD - p$ is linearly equivalent to an effective divisor for some $m \gg 0$, in which case

$$0 \le \deg(mD - p) = m \deg(D) - \deg(p),$$

hence $\deg(D) > 0$.

Conversely, assume $\deg(D) > 0$. By Riemann-Roch, we have $H^0(X, mD) \neq 0$ for $m \gg 0$, so, upon replacing D by a positive multiple, we can assume that D is effective. As in the proof of the theorem, we then have an exact sequence

$$0 \to \mathscr{O}_X((m-1)D) \to \mathscr{O}_X(mD) \to \mathscr{O}_D \to 0,$$

from which we get a surjection[2]

$$H^1(X, (m-1)D)) \to H^1(X, mD) \to 0.$$

Since these spaces are finite-dimensional, this will be a bijection for $m \gg 0$, in which case we get a surjection

$$H^0(X, mD) \to H^0(D, \mathscr{O}_D).$$

In particular, the evaluation map ev_x (see §2.5) for the sheaf $\mathscr{O}_X(mD)$ is surjective at every point x of the support of D. Since it is trivially surjective for x outside of this support (it has a section with divisor mD), the sheaf $\mathscr{O}_X(mD)$ is globally generated.

Its global sections therefore define a morphism $u : X \to \mathbf{P}^N_{\mathbf{k}}$ such that $\mathscr{O}_X(mD) = u^*\mathscr{O}_{\mathbf{P}^N_{\mathbf{k}}}(1)$. Since $\mathscr{O}_X(mD)$ is non trivial, u is not constant, hence finite because X is a curve. But then, $\mathscr{O}_X(mD) = u^*\mathscr{O}_{\mathbf{P}^N_{\mathbf{k}}}(1)$ is ample (Corollary 2.38) hence D is ample. □

[2] Since the scheme D has dimension 0, we have $H^1(D, mD) = 0$.

3.2 Surfaces

In this section, a surface will be a smooth connected projective scheme X of dimension 2 over an algebraically closed field $\mathbf{k}$. We want to define the intersection of two curves on X. We follow [B], chap. 1.

Definition 3.5 *Let C and D be two curves on a surface X with no common component, let x be a point of $C \cap D$, and let f and g be respective generators of the ideals of C and D at x. We define the intersection multiplicity of C and D at x to be*

$$m_x(C \cap D) = \dim_{\mathbf{k}} \mathscr{O}_{X,x}/(f,g).$$

We then set

$$(C \cdot D) = \sum_{x \in C \cap D} m_x(C \cap D).$$

By the Nullstellensatz, the ideal (f,g) contains a power of the maximal ideal $\mathfrak{m}_{X,x}$, hence the number $m_x(C \cap D)$ is finite. It is 1 if and only if f and g generate $\mathfrak{m}_{X,x}$, which means that they form a system of parameters at x, i.e., that C and D meet transversally at x.

Another way to understand this definiton is to consider the scheme-theoretic intersection $C \cap D$. It is a scheme whose support is finite, and by definition, $\mathscr{O}_{C\cap D,x} = \mathscr{O}_{X,x}/(f,g)$. Hence,

$$(C \cdot D) = h^0(X, \mathscr{O}_{C\cap D}).$$

Theorem 3.6 *Under the hypotheses above, we have*

$$(C \cdot D) = \chi(X, -C-D) - \chi(X, -C) - \chi(X, -D) + \chi(X, \mathscr{O}_X). \tag{3.1}$$

Proof. Let s be a section of $\mathscr{O}_X(C)$ with divisor C and let t be a section of $\mathscr{O}_X(D)$ with divisor D. One checks that we have an exact sequence

$$0 \to \mathscr{O}_X(-C-D) \xrightarrow{(t,-s)} \mathscr{O}_X(-C) \oplus \mathscr{O}_X(-D) \xrightarrow{\binom{s}{t}} \mathscr{O}_X \to \mathscr{O}_{C\cap D} \to 0.$$

(Use the fact that the local rings of X are factorial and that local equations of C and D have no common factor.) The theorem follows. ☐

This theorem leads us to define the intersection of *any* two divisors C and D by the formula (3.1). By definition, it depends only on the linear equivalence classes of C and D. One can then prove that this defines a bilinear pairing on $\operatorname{Pic}(X)$. We refer to [B] for a direct (easy) proof, since we will do the general case in Proposition 3.15. To relate it to the degree of divisors on smooth curves defined in §3.1, we prove the following.

Lemma 3.7 *For any smooth curve C on X and any divisor D, we have*

$$(D \cdot C) = \deg(D|_C).$$

Proof. We have exact sequences

$$0 \to \mathscr{O}_X(-C) \to \mathscr{O}_X \to \mathscr{O}_C \to 0$$

and

$$0 \to \mathscr{O}_X(-C-D) \to \mathscr{O}_X(-D) \to \mathscr{O}_C(-D|_C) \to 0,$$

which give

$$(D \cdot C) = \chi(C, \mathscr{O}_C) - \chi(C, -D|_C) = \deg(D|_C)$$

by the Riemann-Roch theorem on C. ☐

Exercise 3.8 Let B be a smooth curve and let X be a smooth surface with a surjective morphism $f : X \to B$. Let x be a closed point of B and let F be the divisor f^*x on X. Prove $(F \cdot F) = 0$.

3.3 Blow-ups

We assume here that the field $\mathbf{k}$ is algebraically closed. All points are closed.

3.3.1 Blow-up of a point in $\mathbf{P}^n_{\mathbf{k}}$

Let O be a point of $\mathbf{P}^n_{\mathbf{k}}$ and let H be a hyperplane in $\mathbf{P}^n_{\mathbf{k}}$ which does not contain O. The projection $\pi : \mathbf{P}^n_{\mathbf{k}} \dashrightarrow H$ from O is a rational map defined on $\mathbf{P}^n_{\mathbf{k}} - \{O\}$.

Take coordinates such that $O = (0, \dots, 0, 1)$ and $H = V(x_n)$, so that $\pi(x_0, \dots, x_n) = (x_0, \dots, x_{n-1})$. The graph of π in $\mathbf{P}^n_{\mathbf{k}} \times H$ is the set of pairs (x, y) with $x \neq O$ and $x_i = y_i$ for $0 \le i \le n-1$. One checks that its closure $\widetilde{\mathbf{P}}^n_{\mathbf{k}}$ is defined by the homogeneous equations $x_i y_j = x_j y_i$ for $0 \le i, j \le n-1$.

The first projection $\varepsilon : \widetilde{\mathbf{P}}^n_{\mathbf{k}} \to \mathbf{P}^n_{\mathbf{k}}$ is called the *blow-up of O in $\mathbf{P}^n_{\mathbf{k}}$*, or the *blow-up of $\mathbf{P}^n_{\mathbf{k}}$ at O*. Above a point x other than O, the fiber $\varepsilon^{-1}(x)$ is the point $\pi(x)$; above O, it is $\{O\} \times H \simeq H$. The map ε induces an isomorphism from $\widetilde{\mathbf{P}}^n_{\mathbf{k}} - H$ onto $\mathbf{P}^n_{\mathbf{k}} - \{O\}$; it is therefore a birational morphism. In some sense, the point O has been "replaced" by a $\mathbf{P}^{n-1}_{\mathbf{k}}$. The construction is independent of the choice of the hyperplane H; it is in fact local and can be made completely intrinsic.

The fibers of the second projection $q : \widetilde{\mathbf{P}}^n_{\mathbf{k}} \to H$ are all isomorphic to $\mathbf{P}^1_{\mathbf{k}}$, but $\widetilde{\mathbf{P}}^n_{\mathbf{k}}$ is not isomorphic to the product $\mathbf{P}^1_{\mathbf{k}} \times H$, although it is locally a product over each standard open subset U_i of H (we say that it is a *projective bundle*): just send the point(x, y) of $\widetilde{\mathbf{P}}^n_{\mathbf{k}} \cap (\mathbf{P}^n_{\mathbf{k}} \times U_i) = q^{-1}(U_i)$ to the point $((x_i, x_n), y)$ of $\mathbf{P}^1_{\mathbf{k}} \times U_i$.

One should think of H as the set of lines in $\mathbf{P}^n_{\mathbf{k}}$ passing through O. From a more geometric point of view, we have

$$\widetilde{\mathbf{P}}^n_{\mathbf{k}} = \{(x, \ell) \in \mathbf{P}^n_{\mathbf{k}} \times H \mid x \in \ell\},$$

which gives a better understanding of the fibers of the maps $\varepsilon : \widetilde{\mathbf{P}}^n_{\mathbf{k}} \to \mathbf{P}^n_{\mathbf{k}}$ and $q : \widetilde{\mathbf{P}}^n_{\mathbf{k}} \to H$.

3.3.2 Blow-up of a point in a subvariety of $\mathbf{P}^n_{\mathbf{k}}$

When X is a subvariety of $\mathbf{P}^n_{\mathbf{k}}$ and O a point of X, we define the blow-up of X at O as the closure $\widetilde{X}$ of $\varepsilon^{-1}(X - \{O\})$ in $\varepsilon^{-1}(X)$. This yields a birational morphism $\varepsilon : \widetilde{X} \to X$ which again is independent of the embedding $X \subset \mathbf{P}^n_{\mathbf{k}}$ (this construction can be made local and intrinsic). When X is smooth at x, the inverse image $E = \varepsilon^{-1}(x)$ (called the *exceptional divisor*) is a projective space of dimension $\dim(X) - 1$; it parametrizes tangent directions to X at x, and is naturally isomorphic to $\mathbf{P}(T_{X,x})$.

Blow-ups are useful to make singularities better, or to make a rational map defined.

Examples 3.9 1) Consider the plane cubic C with equation

$$x_1^2 x_2 = x_0^2(x_2 - x_0)$$

in $\mathbf{P}^2_{\mathbf{k}}$. Blow-up $O = (0, 0, 1)$. At a point $((x_0, x_1, x_2), (y_0, y_1))$ of $\varepsilon^{-1}(C - \{O\})$ with $y_0 = 1$, we have $x_1 = x_0 y_1$, hence (as $x_0 \neq 0$)

$$x_2 y_1^2 = x_2 - x_0.$$

At a point with $y_1 = 1$, we have $x_0 = x_1 y_0$, hence (as $x_1 \neq 0$)

$$x_2 = y_0^2(x_2 - x_1 y_0).$$

These two equations define $\widetilde{C}$ in $\widetilde{\mathbf{P}}^2_{\mathbf{k}}$; one in the open set $\mathbf{P}^2_{\mathbf{k}} \times U_0$, the other in the open set $\mathbf{P}^2_{\mathbf{k}} \times U_1$. The inverse image of O consists in two points $((0, 0, 1), (1, 1))$ and $((0, 0, 1), (1, -1))$ (which are both in both open sets). We have desingularized the curve C.

2) Consider the Cremona involution $u : \mathbf{P}^2_{\mathbf{k}} \dashrightarrow \mathbf{P}^2_{\mathbf{k}}$ defined in Example 2.20 by $u(x_0, x_1, x_2) = (x_1 x_2, x_2 x_0, x_0 x_1)$, regular except at $O = (0, 0, 1)$, $(1, 0, 0)$ and $(0, 1, 0)$. Let $\varepsilon : \widetilde{\mathbf{P}}^2_{\mathbf{k}} \to \mathbf{P}^2_{\mathbf{k}}$ be the blow-up of O; on the open set $y_0 = x_2 = 1$, we have $x_1 = x_0 y_1$, where

$$u \circ \varepsilon((x_0, x_1, 1), (1, y_1)) = (x_0 y_1, x_0, x_0^2 y_1),$$

which can be extended to a regular map above O by setting

$$\tilde{u}((x_0, x_1, 1), (1, y_1)) = (y_1, 1, x_0 y_1).$$

Similarly, on the open set $y_1 = x_2 = 1$, we have $x_0 = x_1 y_0$ hence

$$u \circ \varepsilon((x_0, x_1, 1), (y_0, 1)) = (x_1, x_1 y_0, x_1^2 y_0),$$

which can be extended by $\tilde{u}((x_0, x_1, 1), (y_0, 1)) = (1, y_0, x_1 y_0)$. We see that if $\alpha : X \to \mathbf{P}^2_{\mathbf{k}}$ is the blow-up of the points O, $(1,0,0)$ and $(0,1,0)$, there exists a *regular map* $\widetilde{u} : X \to \mathbf{P}^2_{\mathbf{k}}$ such that $\widetilde{u} = u \circ \alpha$.

3.3.3 Blow-up of a point in a smooth surface

Let us now make some calculations on blow-ups on a surface X over an algebraically closed field $\mathbf{k}$.

Let $\varepsilon : \widetilde{X} \to X$ be the blow-up of a point x, with exceptional divisor E. As we saw above, it is a smooth rational curve (i.e., isomorphic to $\mathbf{P}^1_{\mathbf{k}}$).

Proposition 3.10 *Let X be a smooth projective surface over an algebraically closed field and let $\varepsilon : \widetilde{X} \to X$ be the blow-up of a point x of X, with exceptional curve E. For any divisors C and D on X, we have*

$$(\varepsilon^* C \cdot \varepsilon^* D) = (C \cdot D) \quad , \quad (\varepsilon^* C \cdot E) = 0 \quad , \quad (E \cdot E) = -1.$$

Proof. Upon replacing C and D by linearly equivalent divisors whose supports do not contain x (proceed as in Proposition 2.36), the first two equalities are obvious.

Let now C be a smooth curve in X passing through x and let $\widetilde{C} = \overline{\varepsilon^{-1}(C - x)}$ be its strict transform in $\widetilde{X}$. It meets E transversally at the point corresponding to the tangent direction to C at x. We have $\varepsilon^* C = \widetilde{C} + E$, hence

$$0 = (\varepsilon^* C \cdot E) = (\widetilde{C} \cdot E) + (E \cdot E) = 1 + (E \cdot E).$$

This finishes the proof. □

There is a very important "converse" to this proposition, due to Castelnuovo, which says that given a smooth rational curve E in a projective smooth surface $\widetilde{X}$, if $(E \cdot E) = -1$, one can "contract" E by a birational morphism $\widetilde{X} \to X$ onto a smooth surface X. We will come back to that in §5.4.

Corollary 3.11 *In the situation above, one has*

$$\operatorname{Pic}(\widetilde{X}) \simeq \operatorname{Pic}(X) \oplus \mathbf{Z}[E].$$

Proof. Let $\widetilde{C}$ be an irreducible curve on $\widetilde{X}$, distinct from E. The pull-back $\varepsilon^*(\varepsilon(\widetilde{C}))$ is the sum of $\widetilde{C}$ and a certain number of copies of E, so the map

$$\begin{array}{rcl} \operatorname{Pic}(X) \oplus \mathbf{Z} & \longrightarrow & \operatorname{Pic}(\widetilde{X}) \\ (D, m) & \longmapsto & \varepsilon^* D + mE \end{array}$$

is surjective. If $\varepsilon^* D + mE \underset{\text{lin}}{\equiv} 0$, we get $-m = 0$ by taking intersection numbers with E. We then have

$$\mathscr{O}_X \simeq \varepsilon_* \mathscr{O}_{\widetilde{X}} \simeq \varepsilon_*(\mathscr{O}_{\widetilde{X}}(\varepsilon^* D)) \simeq \mathscr{O}_X(D),$$

hence $D \underset{\text{lin}}{\equiv} 0$ (here we used Zariski's main theorem (the first isomorphism is easy to check directly (see for example the proof of [H1], Corollary III.11.4) and the last one uses the projection formula ([H1], Exercise II.5.1.(d))). □

3.4 General intersection numbers

If X is a closed subscheme of $\mathbf{P}^N_{\mathbf{k}}$ of dimension n, it is proved in [H1], Theorem I.7.5, that the function

$$m \mapsto \chi(X, \mathscr{O}_X(m))$$

is *polynomial of degree* n, i.e., takes the same values on the integers as a (uniquely determined) polynomial of degree n with rational coefficients, called the *Hilbert polynomial* of X. The degree of X in $\mathbf{P}^N_{\mathbf{k}}$ is then defined as $n!$ times the coefficient of m^n. It generalizes the degree of a hypersurface defined in Example 2.12.

If X is reduced and $H_1, \ldots, H_n$ are general hyperplanes, and if $\mathbf{k}$ is algebraically closed, the degree of X is also the number of points of the intersection $X \cap H_1 \cap \cdots \cap H_n$. If H_i^X is the Cartier divisor on X defined by H_i, the degree of X is therefore the number of points in the intersection $H_1^X \cap \cdots \cap H_n^X$. Our aim in this section is to generalize this and to define an intersection number

$$(D_1 \cdot \ldots \cdot D_n)$$

for any Cartier divisors $D_1, \ldots, D_n$ on a projective n-dimensional scheme, which only depends on the linear equivalence class of the D_i.

Instead of trying to define, as in Definition 3.5, the multiplicity of intersection at a point, which can be difficult on a general X, we give a definition based on Euler characteristics, as in Theorem 3.6 (compare with (3.3)). It has the advantage of being quick and efficient, but has very little geometric feeling to it.

Theorem 3.12 *Let $D_1, \ldots, D_r$ be Cartier divisors on a projective scheme X over a field. The function*

$$(m_1, \ldots, m_r) \longmapsto \chi(X, m_1 D_1 + \cdots + m_r D_r)$$

takes the same values on $\mathbf{Z}^r$ as a polynomial with rational coefficients of total degree at most the dimension of X.

Proof. We prove the theorem first in the case $r = 1$ by induction on the dimension of X. If X has dimension 0, we have

$$\chi(X, D) = h^0(X, \mathscr{O}_X)$$

for any D and the conclusion holds trivially.

Write $D_1 = D \underset{\text{lin}}{\equiv} E_1 - E_2$ with E_1 and E_2 effective (Proposition 2.36). There are exact sequences

$$\begin{array}{ccccccc} 0 \to & \mathscr{O}_X(mD - E_1) & \to & \mathscr{O}_X(mD) & \to & \mathscr{O}_{E_1}(mD) & \to 0 \\ & \| & & & & & \\ 0 \to & \mathscr{O}_X((m-1)D - E_2) & \to & \mathscr{O}_X((m-1)D) & \to & \mathscr{O}_{E_2}((m-1)D) & \to 0 \end{array} \qquad (3.2)$$

which yield

$$\chi(X, mD) - \chi(X, (m-1)D) = \chi(E_1, mD) - \chi(E_2, (m-1)D).$$

By induction, the right-hand side of this equality is a rational polynomial function in m of degree $d < \dim(X)$. But if a function $f : \mathbf{Z} \to \mathbf{Z}$ is such that $m \mapsto f(m) - f(m-1)$ is rational polynomial of degree δ, the function f itself is rational polynomial of degree $\delta + 1$ ([H1], Proposition I.7.3.(b)); therefore, $\chi(X, mD)$ is a rational polynomial function in m of degree $\leq d + 1 \leq \dim(X)$.

Note that for any divisor D_0 on X, the function $m \mapsto \chi(X, D_0 + mD)$ is a rational polynomial function of degree $\leq \dim(X)$ (the same proof applies upon tensoring the diagram (3.2) by $\mathscr{O}_X(D_0)$). We now treat the general case.

Lemma 3.13 *Let d be a positive integer and let $f : \mathbf{Z}^r \to \mathbf{Z}$ be a map such that for each $(n_1, \ldots, n_{i-1}, n_{i+1}, \ldots, n_r)$ in $\mathbf{Z}^{r-1}$, the map*

$$m \longmapsto f(n_1, \ldots, n_{i-1}, m, n_{i+1}, \ldots, n_r)$$

is rational polynomial of degree at most d. The function f takes the same values as a rational polynomial in r indeterminates.

PROOF. We proceed by induction on r, the case $r = 1$ being trivial. Assume $r > 1$; there exist functions $f_0, \dots, f_d : \mathbf{Z}^{r-1} \to \mathbf{Q}$ such that

$$f(m_1, \dots, m_r) = \sum_{j=0}^{d} f_j(m_1, \dots, m_{r-1}) m_r^j.$$

Pick distinct integers $c_0, \dots, c_d$; for each $i \in \{0, \dots, d\}$, there exists by the induction hypothesis a polynomial P_i with rational coefficients such that

$$f(m_1, \dots, m_{r-1}, c_i) = \sum_{j=0}^{d} f_j(m_1, \dots, m_{r-1}) c_i^j = P_i(m_1, \dots, m_{r-1}).$$

The matrix (c_i^j) is invertible and its inverse has rational coefficients. This proves that each f_j is a linear combination of $P_0, \dots, P_d$ with rational coefficients hence the lemma. □

From the remark before Lemma 3.13 and the lemma itself, we deduce that there exists a polynomial $P \in \mathbf{Q}[T_1, \dots, T_r]$ such that

$$\chi(X, m_1 D_1 + \cdots + m_r D_r) = P(m_1, \dots, m_r)$$

for all integers $m_1, \dots, m_r$. Let d be its total degree, and let $n_1, \dots, n_r$ be integers such that the degree of the polynomial

$$Q(T) = P(n_1 T, \dots, n_r T)$$

is still d. Since

$$Q(m) = \chi(X, m(n_1 D_1 + \cdots + n_r D_r)),$$

it follows from the case $r = 1$ that d is at most the dimension of X. □

Definition 3.14 *Let $D_1, \dots, D_r$ be Cartier divisors on a projective scheme X over a field, with $r \geq \dim(X)$. We define the intersection number*

$$(D_1 \cdot \ldots \cdot D_r)$$

as the coefficient of $m_1 \cdots m_r$ in the rational polynomial

$$\chi(X, m_1 D_1 + \cdots + m_r D_r).$$

Of course, this number only depends on the linear equivalence classes of the divisors D_i, since it is defined from the invertible sheaves $\mathscr{O}_X(D_i)$.

For any polynomial $P(T_1, \dots, T_r)$ of total degree at most r, the coefficient of $T_1 \cdots T_r$ in P is

$$\sum_{I \subset \{1, \dots, r\}} \varepsilon_I P(-m^I),$$

where $\varepsilon_I = (-1)^{\mathrm{Card}(I)}$ and $m_i^I = 1$ if $i \in I$ and 0 otherwise (this quantity vanishes for all other monomials of degree $\leq r$). It follows that we have

$$(D_1 \cdot \ldots \cdot D_r) = \sum_{I \subset \{1, \dots, r\}} \varepsilon_I \ \chi(X, -\sum_{i \in I} D_i). \tag{3.3}$$

This number is therefore an integer and it vanishes for $r > \dim(X)$ (Theorem 3.12).

In case X is a subscheme of $\mathbf{P}_{\mathbf{k}}^N$ of dimension n, and if H^X is a hyperplane section of X, the intersection number $((H^X)^n)$ is the degree of X as defined in [H1], §I.7.

More generally, if $D_1, \dots, D_n$ are effective and meet properly in a finite number of points, and if $\mathbf{k}$ is algebraically closed, the intersection number does have a geometric interpretation as the number of points in $D_1 \cap \cdots \cap D_n$, counted with multiplicity. This is the length of the 0-dimensional scheme-theoretic intersection $D_1 \cap \cdots \cap D_n$ (the proof is analogous to that of Theorem 3.6; see [Ko1], Theorem VI.2.8).

Of course, it coincides with our previous definition on surfaces (compare (3.3) with (3.1)). On a curve X, we can use it to define the degree of a Cartier divisor D by setting $\deg(D) = (D)$ (by the Rieman-Rch theoreme 3.3, it coincides with our previous definition of the degree of a divisor on a smooth projective curve (Example 2.7)). Given a morphism $f : C \to X$ from a projective curve to a quasi-projective scheme X, and a Cartier divisor D on X, we define

$$(D \cdot C) = \deg(f^* D). \tag{3.4}$$

Finally, if D is a Cartier divisor on the projective n-dimensional scheme X, the function $m \mapsto \chi(X, mD)$ is a polynomial $P(T) = \sum_{i=0}^{n} a_i T^i$, and

$$\chi(X, m_1 D + \cdots + m_n D) = P(m_1 + \cdots + m_n) = \sum_{i=0}^{n} a_i (m_1 + \cdots + m_n)^i.$$

The coefficient of $m_1 \cdots m_n$ in this polynomial is $a_n n!$, hence

$$\chi(X, mD) = m^n \frac{(D^n)}{n!} + O(m^{n-1}). \tag{3.5}$$

We now prove multilinearity.

Proposition 3.15 *Let $D_1, \ldots, D_n$ be Cartier divisors on a projective scheme X of dimension n over a field.*

a) The map

$$(D_1, \ldots, D_n) \longmapsto (D_1 \cdot \ldots \cdot D_n)$$

*is **Z**-multilinear, symmetric and takes integral values.*

b) If D_n is effective,

$$(D_1 \cdot \ldots \cdot D_n) = (D_1|_{D_n} \cdot \ldots \cdot D_{n-1}|_{D_n}).$$

Proof. The map in a) is symmetric by construction, but its multilinearity is not obvious. The right-hand side of (3.3) vanishes for $r > n$, hence, for any divisors $D_1, D'_1, D_2, \ldots, D_n$, the sum

$$\sum_{I \subset \{2,\ldots,n\}} \varepsilon_I \Big(\chi(X, -\sum_{i \in I} D_i) - \chi(X, -D_1 - \sum_{i \in I} D_i) - \chi(X, -D'_1 - \sum_{i \in I} D_i) + \chi(X, -D_1 - D'_1 - \sum_{i \in I} D_i) \Big)$$

vanishes. On the other hand, $((D_1 + D'_1) \cdot D_2 \cdot \ldots \cdot D_n)$ is equal to

$$\sum_{I \subset \{2,\ldots,n\}} \varepsilon_I \Big(\chi(X, -\sum_{i \in I} D_i) - \chi(X, -D_1 - D'_1 - \sum_{i \in I} D_i) \Big)$$

and $(D_1 \cdot D_2 \cdot \ldots \cdot D_n) + (D'_1 \cdot D_2 \cdot \ldots \cdot D_n)$ to

$$\sum_{I \subset \{2,\ldots,n\}} \varepsilon_I \Big(2\chi(X, -\sum_{i \in I} D_i) - \chi(X, -D_1 - \sum_{i \in I} D_i) - \chi(X, -D'_1 - \sum_{i \in I} D_i) \Big).$$

Putting all these identities together gives the desired equality

$$((D_1 + D'_1) \cdot D_2 \cdot \ldots \cdot D_n) = (D_1 \cdot D_2 \cdot \ldots \cdot D_n) + (D'_1 \cdot D_2 \cdot \ldots \cdot D_n)$$

and proves a).

In the situation of b), we have

$$(D_1 \cdot \ldots \cdot D_n) = \sum_{I \subset \{1,\ldots,n-1\}} \varepsilon_I \Big(\chi(X, -\sum_{i \in I} D_i) - \chi(X, -D_n - \sum_{i \in I} D_i) \Big).$$

From the exact sequence

$$0 \to \mathscr{O}_X(-D_n - \sum_{i\in I} D_i) \to \mathscr{O}_X(-\sum_{i\in I} D_i) \to \mathscr{O}_{D_n}(-\sum_{i\in I} D_i) \to 0$$

we get

$$(D_1 \cdot \ldots \cdot D_n) = \sum_{I\subset\{1,\ldots,n-1\}} \varepsilon_I\ \chi(D_n, -\sum_{i\in I} D_i) = (D_1|_{D_n} \cdot \ldots \cdot D_{n-1}|_{D_n}),$$

which proves b). □

Recall that the degree of a dominant morphism $\pi : Y \to X$ between varieties is the degree of the field extension $\pi^* : K(X) \hookrightarrow K(Y)$ if this extension is finite, and 0 otherwise.

Proposition 3.16 (Pull-back formula) *Let $\pi : Y \to X$ be a surjective morphism between projective varieties. Let $D_1, \ldots, D_r$ be Cartier divisors on X with $r \geq \dim(Y)$. We have*

$$(\pi^* D_1 \cdot \ldots \cdot \pi^* D_r) = \deg(\pi)(D_1 \cdot \ldots \cdot D_r).$$

Sketch of proof. For any coherent sheaf $\mathscr{F}$ on Y, the sheaves $R^q\pi_*\mathscr{F}$ are coherent ([G1], th. 3.2.1) and there is a spectral sequence

$$H^p(X, R^q\pi_*\mathscr{F}) \Longrightarrow H^{p+q}(Y, \mathscr{F}).$$

It follows that we have

$$\chi(Y, \mathscr{F}) = \sum_{q\geq 0} (-1)^q \chi(X, R^q\pi_*\mathscr{F}).$$

Applying it to $\mathscr{F} = \mathscr{O}_Y(m_1\pi^*D_1 + \cdots + m_r\pi^*D_r)$ and using the projection formula

$$R^q\pi_*\mathscr{F} \simeq R^q\pi_*\mathscr{O}_Y \otimes \mathscr{O}_Y(m_1D_1 + \cdots + m_rD_r)$$

([G1], prop. 12.2.3), we get that $(\pi^* D_1 \cdot \ldots \cdot \pi^* D_r)$ is equal to the coefficient of $m_1 \cdots m_r$ in

$$\sum_{q\geq 0} (-1)^q \chi(X, R^q\pi_*\mathscr{O}_Y \otimes \mathscr{O}_X(m_1D_1 + \cdots + m_rD_r)).$$

(Here we need an extension of Theorem 3.12 which says that for any coherent sheaf $\mathscr{F}$ on X, the function

$$(m_1, \ldots, m_r) \longmapsto \chi(X, \mathscr{F}(m_1D_1 + \cdots + m_rD_r))$$

is still polynomial of degree $\leq \dim(\operatorname{Supp} \mathscr{F})$. The proof is exactly the same.)

If π is not generically finite, we have $r > \dim(X)$ and the coefficient of $m_1 \cdots m_r$ in each term of the sum vanishes by Theorem 3.12.

Otherwise, π is finite of degree d over a dense open subset U of Y, the sheaves $R^q\pi_*\mathscr{O}_Y$ have support outside of U for $q > 0$ ([H1], Corollary III.11.2) hence the coefficient of $m_1 \cdots m_r$ in the corresponding term vanishes for the same reason. Finally, $\pi_*\mathscr{O}_Y$ is free of rank d on some dense open subset of U and it is not too hard to conclude that the coefficients of $m_1 \cdots m_r$ in $\chi(X, \pi_*\mathscr{O}_Y \otimes \mathscr{O}_X(m_1D_1 + \cdots + m_rD_r))$ and $\chi(X, \mathscr{O}_X^{\oplus d} \otimes \mathscr{O}_X(m_1D_1 + \cdots + m_rD_r))$ are the same. □

3.17. Projection formula. Let $\pi : X \to Y$ be a morphism between projective varieties and let C be a curve on X. We define the 1-cycle π_*C as follows: if C is contracted to a point by π, set $\pi_*C = 0$; if $\pi(C)$ is a curve on Y, set $\pi_*C = d\,\pi(C)$, where d is the degree of the morphism $C \to \pi(C)$ induced by π. If D is a Cartier divisor on Y, we obtain from the pull-back formula for curves the so-called *projection formula*

$$(\pi^* D \cdot C) = (D \cdot \pi_*C). \tag{3.6}$$

Corollary 3.18 *Let X be a curve of genus 0 over a field $\mathbf{k}$. If X has a $\mathbf{k}$-point, X is isomorphic to $\mathbf{P}^1_{\mathbf{k}}$.*

Any plane conic with no rational point (such as the real conic with equation $x_0^2 + x_1^2 + x_2^2 = 0$) has genus 0 (see Exercise 3.2), but is of course not isomorphic to the projective line.

Proof. Let p be a $\mathbf{k}$-point of X. Since $H^1(X, \mathscr{O}_X) = 0$, the long exact sequence in cohomology associated with the exact sequence

$$0 \to \mathscr{O}_X \to \mathscr{O}_X(p) \to k(p) \to 0$$

reads

$$0 \to H^0(X, \mathscr{O}_X) \to H^0(X, \mathscr{O}_X(p)) \to \mathbf{k}_p \to 0.$$

In particular, $h^0(X, \mathscr{O}_X(p)) = 2$ and the invertible sheaf $\mathscr{O}_X(p)$ is generated by two global sections which define a finite morphism $u : X \to \mathbf{P}^1_{\mathbf{k}}$ such that $u^*\mathscr{O}_{\mathbf{P}^1_{\mathbf{k}}}(1) = \mathscr{O}_X(p)$. By the pull-back formula for curves,

$$1 = \deg(\mathscr{O}_X(p)) = \deg(u),$$

and u is an isomorphism. □

Exercise 3.19 Let E be the exceptional divisor of the blow-up of a smooth point on an n-dimensional projective scheme (see §3.3.2). Compute (E^n).

3.20. Intersection of Q-divisors. Of course, we may define, by linearity, intersection of $\mathbf{Q}$-Cartier $\mathbf{Q}$-divisors. For example, let X be the cone in $\mathbf{P}^3_{\mathbf{k}}$ with equation $x_0x_1 = x_2^2$ (its vertex is $(0, 0, 0, 1)$) and let L be the line defined by $x_0 = x_2 = 0$ (compare with Example 2.6). Then $2L$ is a hyperplane section of X, hence $(2L)^2 = \deg(X) = 2$. So we have $(L^2) = 1/2$.

3.5 Intersection of divisors over the complex numbers

Let X be a smooth projective complex manifold of dimension n. There is a short exact sequence of sheaves

$$0 \to \mathbf{Z} \xrightarrow{\cdot 2i\pi} \mathscr{O}_{X,\mathrm{an}} \xrightarrow{\exp} \mathscr{O}^*_{X,\mathrm{an}} \to 0$$

which induces a morphism

$$c_1 : H^1(X, \mathscr{O}^*_{X,\mathrm{an}}) \to H^2(X, \mathbf{Z})$$

called the *first Chern class.* So we can in particular define the first Chern class of an algebraic line bundle on X. Given divisors $D_1, \ldots, D_n$ on X, the intersection product $(D_1 \cdot \ldots \cdot D_n)$ defined above is the cup product

$$c_1(\mathscr{O}_X(D_1)) \smile \cdots \smile c_1(\mathscr{O}_X(D_n)) \in H^{2n}(X, \mathbf{Z}) \simeq \mathbf{Z}.$$

In particular, the degree of a divisor D on a curve $C \subset X$ is

$$c_1(\nu^*\mathscr{O}_X(D)) \in H^2(\widetilde{C}, \mathbf{Z}) \simeq \mathbf{Z}.$$

where $\nu : \widetilde{C} \to C$ is the normalization of C.

Remark 3.21 A theorem of Serre says that the canonical map $H^1(X, \mathscr{O}^*_X) \to H^1(X, \mathscr{O}^*_{X,\mathrm{an}})$ is bijective. In other words, isomorphism classes of holomorphic and algebraic line bundles on X are the same.

3.6 Exercises

1) Let X be a curve and let p be a closed point. Show that $X - \{p\}$ is affine (*Hint:* apply Corollary 3.4).

Chapter 4

Ampleness criteria and cones of curves

In this chapter, we prove two ampleness criteria for a divisor on a projective variety X: the Nakai-Moishezon ampleness criterion, which involves intersection numbers on all integral subschemes of X, and (a weak form of) the Kleiman criterion, which involves only intersection numbers with 1-cycles.

We also define nef divisors, which should be thought of as limits of ample divisors, and introduce a fundamental object, the cone of effective 1-cycles on X.

4.1 The Nakai-Moishezon ampleness criterion

This is an ampleness criterion for Cartier divisors that involves only intersection numbers with curves, but with all integral subschemes. Recall that our aim is to prove eventually that ampleness is a *numerical property* in the sense that it depends only on intersection numbers with 1-cycles. This we will prove in Proposition 4.10.

Theorem 4.1 (Nakai-Moishezon criterion) *A Cartier divisor D on a projective scheme X over a field is ample if and only if, for every integral subscheme Y of X, of dimension r,*

$$((D|_Y)^r) > 0.$$

The same result of course holds when D is a $\mathbf{Q}$-Cartier $\mathbf{Q}$-divisor.

Having $(D \cdot C) > 0$ for every curve C on X does not in general imply that D is ample (see Example 5.16 for an example) although there are some cases where it does (e.g., when $\mathrm{NE}(X)$ is closed, by Proposition 4.10.a)).

Proof. One direction is easy: if D is ample, some positive multiple mD is very ample hence defines an embedding $f : X \hookrightarrow \mathbf{P}^N_{\mathbf{k}}$ such that $f^*\mathscr{O}_{\mathbf{P}^N_{\mathbf{k}}}(1) \simeq \mathscr{O}_X(mD)$. In particular, for every (closed) subscheme Y of X of dimension r,

$$((mD|_Y)^r) = \deg(f(Y)) > 0,$$

by [H1], Proposition I.7.6.(a).

The converse is more subtle. Let D be a Cartier divisor such that $(D^r \cdot Y) > 0$ for every integral subscheme Y of X of dimension r. We show by induction on the dimension of X that D is ample on X. By Exercise 2.40, we may assume that X is integral. The proof follows the ideas of Corollary 3.4.

Write $D \underset{\mathrm{lin}}{\equiv} E_1 - E_2$, with E_1 and E_2 effective. Consider the exact sequences (3.2). By induction, D is ample on E_1 and E_2, hence $H^i(E_j, mD)$ vanishes for $i > 0$ and all $m \gg 0$. It follows that for $i \geq 2$,

$$h^i(X, mD) = h^i(X, mD - E_1) = h^i(X, (m-1)D - E_2) = h^i(X, (m-1)D)$$

for all $m \gg 0$. Since $(D^{\dim(X)})$ is positive, $\chi(X, mD)$ goes to infinity with m by (3.5); it follows that

$$h^0(X, mD) - h^1(X, mD)$$

hence also $h^0(X, mD)$, go to infinity with m. To prove that D is ample, we may replace it with any positive multiple. So we may assume that D is effective; the exact sequence

$$0 \to \mathscr{O}_X((m-1)D) \to \mathscr{O}_X(mD) \to \mathscr{O}_D(mD) \to 0$$

and the vanishing of $H^1(D, mD)$ for all $m \gg 0$ (Theorem 2.37) yield a surjection

$$\rho_m : H^1(X, (m-1)D) \to H^1(X, mD).$$

The dimensions $h^1(X, mD)$ form a nonincreasing sequence of numbers which must eventually become stationary, in which case ρ_m is bijective and the restriction

$$H^0(X, mD) \to H^0(D, mD)$$

is surjective. By induction, D is ample on D, hence $\mathscr{O}_D(mD)$ is generated by its global sections for all m sufficiently large. As in the proof of Corollary 3.4, it follows that the sheaf $\mathscr{O}_X(mD)$ is also generated by its global sections for m sufficiently large, hence defines a proper morphism f from X to a projective space $\mathbf{P}^N_{\mathbf{k}}$. Since D has positive degree on every curve, f has finite fibers hence, being projective, is finite (see footnote 5). Since $\mathscr{O}_X(D) = f^*\mathscr{O}_{\mathbf{P}^N_{\mathbf{k}}}(1)$, the conclusion follows from Corollary 2.38. □

4.2. On a curve, the Nakai-Moishezon criterion just says that a divisor is ample if and only if its degree is positive. This generalizes Corollary 3.4.

4.2 Nef divisors

It is natural to make the following definition: a Cartier divisor D on a projective scheme X is *nef* [1] if it satisfies, for every subscheme Y of X of dimension r,

$$((D|_Y)^r) \geq 0. \tag{4.1}$$

The restriction of a nef divisor to a subscheme is again nef. A divisor on a curve is nef if and only if its degree is nonnegative.

This definition still makes sense for **Q**-Cartier divisors, and even, on a normal variety, for **Q**-Cartier **Q**-divisors. As for ample divisors, whenever we say "nef **Q**-divisor", or "nef divisor", it will always be understood that the divisor is **Q**-Cartier, and that the variety is normal if it is a **Q**-divisor.

Note that by the pull-back formula (Proposition 3.16), the pull-back of a nef divisor by any morphism between projective schemes is still nef.

4.3. Sum of ample and nef divisors. Let us begin with a lemma that will be used repeatedly in what follows.

Lemma 4.4 *Let X be a projective scheme of dimension n over a field, let D be a Cartier divisor and let H be an ample divisor on X. If $((D|_Y)^r) \geq 0$ for every subscheme Y of X of dimension r, we have*

$$(D^r \cdot H^{n-r}) \geq 0.$$

PROOF. We proceed by induction on n. Let m be an integer such that mH is very ample. The linear system $|mH|$ contains an effective divisor E. If $r = n$, there is nothing to prove. If $r < n$, using Proposition 3.15.b), we get

$$\begin{aligned}(D^r \cdot H^{n-r}) &= \frac{1}{m}(D^r \cdot H^{n-r-1} \cdot (mH)) \\ &= \frac{1}{m}((D|_E)^r \cdot (H|_E)^{n-r-1})\end{aligned}$$

[1] This acronym comes from "numerically effective," or "numerically eventually free" (according to [R], D.1.3).

and this is nonnegative by the induction hypothesis. □

Let now X be a projective variety, let D be a nef divisor on X, let H be an ample divisor, and let Y be an r-dimensional subscheme of X. Since $D|_Y$ is nef, the lemma implies

$$((D|_Y)^s \cdot (H|_Y)^{r-s}) \geq 0 \tag{4.2}$$

for $0 \leq s \leq r$, hence

$$((D|_Y + H|_Y)^r) = ((H|_Y)^r) + \sum_{s=1}^{r} \binom{r}{s} ((D|_Y)^s \cdot (H|_Y)^{r-s}) \geq ((H|_Y)^r) > 0$$

because $H|_Y$ is ample. By the Nakai-Moishezon criterion, $D + H$ is ample: *on a projective scheme, the sum of a nef divisor and an ample divisor is ample.* This still holds for **Q**-Cartier **Q**-divisors.

4.5. Sum of nef divisors. Let D and E be nef divisors on a projective scheme X of dimension n, and let H be an ample divisor on X. We just saw that for all positive rationals t, the divisor $E + tH$ is ample, and so is $D + (E + tH)$. For every subscheme Y of X of dimension r, we have, by the easy direction of the Nakai-Moishezon criterion (Theorem 4.1),

$$((D|_Y + E|_Y + tH|_Y)^r) > 0.$$

By letting t go to 0, we get, using multilinearity,

$$((D|_Y + E|_Y)^r) \geq 0.$$

It follows that $D + E$ is nef: on a projective scheme, *a sum of nef divisors is nef.*

Exercise 4.6 Let X be a projective scheme over a field. Show that a Cartier divisor is nef on X if and only if it is nef on each irreducible component of X_{red}.

Theorem 4.7 *Let X be a projective scheme over a field. A Cartier divisor on X is nef if and only if it has nonnegative intersection with every curve on X.*

Recall that for us, a curve is always projective integral. The same result of course holds when D is a **Q**-Cartier **Q**-divisor.

Proof. We may assume by Exercise 4.6, we may assume that X is integral. Let D be a Cartier divisor on X with nonnegative degree on every curve. Proceeding by induction on $n = \dim(X)$, it is enough to prove $(D^n) \geq 0$. Let H be an ample divisor on X and set $D_t = D + tH$. Consider the degree n polynomial

$$P(t) = (D_t^n) = (D^n) + \binom{n}{1}(D^{n-1} \cdot H)t + \cdots + (H^n)t^n.$$

We need to show $P(0) \geq 0$. Assume the contrary; since the leading coefficient of P is positive, it has a largest positive real root t_0 and $P(t) > 0$ for $t > t_0$.

For every subscheme Y of X of positive dimension $r < n$, the divisor $D|_Y$ is nef by induction. By (4.2), we have

$$((D|_Y)^s \cdot (H|_Y)^{r-s}) \geq 0$$

for $0 \leq s \leq r$. Also, $((H|_Y)^r) > 0$ because $H|_Y$ is ample. This implies, for $t > 0$,

$$((D_t|_Y)^r) = ((D|_Y)^r) + \binom{r}{1}((D|_Y)^{r-1} \cdot H|_Y)t + \cdots + ((H|_Y)^r)t^r > 0.$$

Since $(D_t^n) = P(t) > 0$ for $t > t_0$, the Nakai-Moishezon criterion implies that D_t is ample for t rational and $t > t_0$.

Note that P is the sum of the polynomials

$$Q(t) = (D_t^{n-1} \cdot D) \qquad \text{and} \qquad R(t) = t(D_t^{n-1} \cdot H).$$

Since D_t is ample for t rational $> t_0$ and D has nonnegative degree on curves, we have $Q(t) \geq 0$ for *all* $t \geq t_0$ by Lemma 4.4.[2] By the same lemma, the induction hypothesis implies

$$(D^r \cdot H^{n-r}) \geq 0$$

for $0 \leq r < n$, hence

$$R(t_0) = (D^{n-1} \cdot H)t_0 + \binom{n-1}{1}(D^{n-2} \cdot H^2)t_0^2 + \cdots + (H^n)t_0^n \geq (H^n)t_0^n > 0.$$

We get the contradiction

$$0 = P(t_0) = Q(t_0) + R(t_0) \geq R(t_0) > 0.$$

This proves that $P(t)$ does not vanish for $t > 0$ hence

$$0 \leq P(0) = (D^n).$$

This proves the theorem. □

4.3 The cone of curves and the effective cone

Let X be a projective scheme over a field. We say that two Cartier divisors D and D' on X are *numerically equivalent* if they have same degree on every curve C on X. In other words (see (3.4),

$$(D \cdot C) = (D' \cdot C).$$

We write $D \underset{\text{num}}{\equiv} D'$. The quotient of the group of Cartier divisors by this equivalence relation is denoted by $N^1(X)_{\mathbf{Z}}$. We set

$$N^1(X)_{\mathbf{Q}} = N^1(X)_{\mathbf{Z}} \otimes \mathbf{Q} \qquad , \qquad N^1(X)_{\mathbf{R}} = N^1(X)_{\mathbf{Z}} \otimes \mathbf{R}.$$

These spaces are finite-dimensional vector spaces[3] and their dimension is called the *Picard number* of X, which we denote by ρ_X.

We say that a property of a divisor is *numerical* if it depends only on its numerical equivalence class, in other words, if it depends only of its intersection numbers with real 1-cycles. For example, we will see in §4.4 that ampleness is a numerical property.

Two 1-cycles C and C' on X are *numerically equivalent* if they have the same intersection number with every Cartier divisor; we write $C \underset{\text{num}}{\equiv} C'$. Call $N_1(X)_{\mathbf{Z}}$ the quotient group, and set

$$N_1(X)_{\mathbf{Q}} = N_1(X)_{\mathbf{Z}} \otimes \mathbf{Q} \qquad , \qquad N_1(X)_{\mathbf{R}} = N_1(X)_{\mathbf{Z}} \otimes \mathbf{R}.$$

The intersection pairing

$$N^1(X)_{\mathbf{R}} \times N_1(X)_{\mathbf{R}} \to \mathbf{R}$$

is by definition nondegenerate. In particular, $N_1(X)_{\mathbf{R}}$ is a finite-dimensional real vector space. We now make a very important definition.

Definition 4.8 *The* cone of curves $\mathrm{NE}(X)$ *is the set of classes of effective* 1*-cycles in* $N_1(X)_{\mathbf{R}}$.

[2] Here I am cheating a bit: to apply the lemma, one needs to know that D has nonnegative degree on all 1-dimensional subschemes C of X. One can show that if $C_1, \ldots, C_s$ are the irreducible components of C_{red}, with generic points $\eta_1, \ldots, \eta_s$, one has

$$(D \cdot C) = \sum_{i=1}^{s} [\mathscr{O}_{C,\eta_i} : \mathscr{O}_{C_i,\eta_i}](D \cdot C_i) \geq 0$$

(see [Ko1], Proposition VI.(2.7.3)).

[3] Over the complex numbers, we saw in §3.5, $N^1(X)_{\mathbf{Q}}$ is a subspace of $H^2(X, \mathbf{Q})$. For the general case, see [K], p. 334.

Note that since X is projective, no class of curve is 0 in $N_1(X)_{\mathbf{R}}$.

We can make an analogous definition for divisors and define similarly the *effective cone* $\mathrm{NE}^1(X)$ as the set of classes of effective (Cartier) divisors in $N^1(X)_{\mathbf{R}}$. These convex cones are not necessarily closed. We denote their closures by $\overline{\mathrm{NE}}(X)$ and $\overline{\mathrm{NE}}^1(X)$ respectively; we call them *the closed cone of curves* and *the pseudo-effective cone*, respectively.

Exercise 4.9 Let X a projective scheme of dimension n over a field and let D be a Cartier divisor on X. Show that the following properties are equivalent:

(i) the divisor D is numerically equivalent to 0;

(ii) for any coherent sheaf $\mathscr{F}$ on X, we have $\chi(X, \mathscr{F}(D)) = \chi(X, \mathscr{F})$;

(iii) for all Cartier divisors $D_1, \ldots, D_{n-1}$ on X, we have $(D \cdot D_1 \cdot \ldots \cdot D_{n-1}) = 0$;

(iv) for any Cartier divisor E on X, we have $(D \cdot E^{n-1}) = 0$.

(*Hint:* you might want to look up the difficult implication (i) $\Rightarrow$ (ii) in [K], §2, Theorem 1. The other implications are more elementary.)

4.4 A numerical characterization of ampleness

We have now gathered enough material to prove our main characterization of ample divisors, which is due to Kleiman ([K]). It has numerous implications, the most obvious being that ampleness is a numerical property, so we can talk about ample classes in $N^1(X)_{\mathbf{Q}}$. These classes generate an open (convex) cone (by 2.25) in $N^1(X)_{\mathbf{R}}$, called the ample cone, whose closure is the nef cone (by Theorem 4.7 and 4.3).

The criterion also implies that the closed cone of curves of a *projective* variety contains no lines: by Lemma 4.24.a), a closed convex cone contains no lines if and only if it is contained in an open half-space plus the origin.

Theorem 4.10 (Kleiman's criterion) *Let X be a projective variety.*

a) A Cartier divisor D on X is ample if and only if $D \cdot z > 0$ for all nonzero z in $\overline{\mathrm{NE}}(X)$.

b) For any ample divisor H and any integer k, the set $\{z \in \overline{\mathrm{NE}}(X) \mid H \cdot z \leq k\}$ is compact hence contains only finitely many classes of curves.

Item a) of course still holds when D is a $\mathbf{Q}$-Cartier $\mathbf{Q}$-divisor.

Proof. Assume D is ample and let z be in $\overline{\mathrm{NE}}(X)$. Since D is nef, one has $D \cdot z \geq 0$. Assume $D \cdot z = 0$ and $z \neq 0$; since the intersection pairing is nondegenerate, there exists a divisor E such that $E \cdot z < 0$, hence $(D + tE) \cdot z < 0$ for all positive t. In particular, $D + tE$ cannot be ample, which contradicts Example 2.27.

Assume for the converse that D is positive on $\overline{\mathrm{NE}}(X) - \{0\}$. Choose a norm $\|\cdot\|$ on $N_1(X)_{\mathbf{R}}$. The set

$$K = \{z \in \overline{\mathrm{NE}}(X) \mid \|z\| = 1\}$$

is compact. The linear form $z \mapsto D \cdot z$ is positive on K hence is bounded from below by a positive rational number a. Let H be an ample divisor on X; the linear form $z \mapsto H \cdot z$ is bounded from above on K by a positive rational number b. It follows that $D - \frac{a}{b}H$ is nonnegative on K hence on the cone $\overline{\mathrm{NE}}(X)$; this is exactly saying that $D - \frac{a}{b}H$ is nef, and by 4.3,

$$D = (D - \frac{a}{b}H) + \frac{a}{b}H$$

is ample. This proves a).

Let $D_1, \dots, D_r$ be Cartier divisors on X such that $\mathscr{B} := ([D_1], \dots, [D_r])$ is a basis for $N^1(X)_{\mathbf{R}}$. There exists an integer m such that $mH \pm D_i$ is ample for each i in $\{1, \dots, r\}$. For any z in $\overline{\mathrm{NE}}(X)$, we then have $(mH \pm D_i) \cdot z \geq 0$ hence $|D_i \cdot z| \leq mH \cdot z$. If $H \cdot z \leq k$, this bounds the coordinates of z in the dual basis $\mathscr{B}^*$ and defines a closed bounded set. It contains at most finitely many classes of curves, because the set of this classes is discrete in $N_1(X)_{\mathbf{R}}$ (they have integral coordinates in the basis $\mathscr{B}^*$). □

We can express Kleiman's criterion in the language of duality for closed convex cones (see §4.7).

Corollary 4.11 *Let X be a projective scheme over a field.*

The dual of the closed cone of curves on X is the cone of classes of nef divisors, called the nef cone.

The interior of the nef cone is the ample cone.

4.5 Around the Riemann-Roch theorem

We know from (3.5) that the growth of the Euler characteristic $\chi(X, mD)$ of successive multiples of a divisor D on a projective scheme X of dimension n is polynomial in m with leading coefficient $(D^n)/n!$. The full Riemann-Roch theorem identifies the coefficients of that polynomial (see §5.1.4 for surfaces).

We study here the dimensions $h^0(X, mD)$ and show that they grow in general not faster than some multiple of m^n and exactly like $\chi(X, mD)$ when D is nef (this is obvious when D is ample because $h^i(X, mD)$ vanishes for $i > 0$ and all $m \gg 0$ by Theorem 2.37). Item b) in the proposition is particularly useful when D is in addition big.

Proposition 4.12 *Let D be a Cartier divisor on a projective scheme X of dimension n over a field.*

a) For all i, we have
$$h^i(X, mD) = O(m^n).$$

b) If D is nef, we have
$$h^i(X, mD) = O(m^{n-1})$$
for all $i > 0$, hence
$$h^0(X, mD) = m^n \frac{(D^n)}{n!} + O(m^{n-1}).$$

Proof. We write $D \underset{\text{lin}}{\equiv} E_1 - E_2$, with E_1 and E_2 effective, and we use again the exact sequences (3.2). The long exact sequences in cohomology give

$$\begin{aligned} h^i(X, mD) &\leq h^i(X, mD - E_1) + h^i(E_1, mD) \\ &= h^i(X, (m-1)D - E_2) + h^i(E_1, mD) \\ &\leq h^i(X, (m-1)D) + h^{i-1}(E_2, (m-1)D) + h^i(E_1, mD). \end{aligned}$$

To prove a) and b), we proceed by induction on n. These inequalities imply, with the induction hypothesis,

$$h^i(X, mD) \leq h^i(X, (m-1)D) + O(m^{n-1})$$

and a) follows by summing up these inequalities over m. If D is nef, so are $D|_{E_1}$ and $D|_{E_2}$, and we get in the same way, for $i \geq 2$,

$$h^i(X, mD) \leq h^i(X, (m-1)D) + O(m^{n-2})$$

hence $h^i(X, mD) = O(m^{n-1})$. This implies in turn, by the very definition of (D^n),

$$\begin{aligned} h^0(X, mD) - h^1(X, mD) &= \chi(X, mD) + O(m^{n-1}) \\ &= m^n \frac{(D^n)}{n!} + O(m^{n-1}). \end{aligned}$$

If $h^0(X, mD) = 0$ for all $m > 0$, the left-hand side of this equality is nonpositive. Since (D^n) is nonnegative, it must be 0 and $h^1(X, mD) = O(m^{n-1})$.

Otherwise, there exists an effective divisor E in some linear system $|m_0 D|$ and the exact sequence

$$0 \to \mathscr{O}_X((m - m_0)D) \to \mathscr{O}_X(mD) \to \mathscr{O}_E(mD) \to 0$$

yields

$$\begin{aligned} h^1(X, mD) &\leq h^1(X, (m - m_0)D) + h^1(E, mD) \\ &= h^1(X, (m - m_0)D) + O(m^{n-2}) \end{aligned}$$

by induction. Again, $h^1(X, mD) = O(m^{n-1})$ and b) is proved. □

4.13. Big divisors. A Cartier divisor D on a projective scheme X over a field is *big* if

$$\limsup_{m \to +\infty} \frac{h^0(X, mD)}{m^n} > 0.$$

It follows from the theorem that a *nef* Cartier divisor D on a projective scheme of dimension n is *big* if and only if $(D^n) > 0$.

Ample divisors are nef and big, but not conversely. Nef and big divisors share many of the properties of ample divisors: for example, Proposition 4.12 shows that the dimensions of the spaces of sections of their successive multiples grow in the same fashion. They are however much more tractable; for instance, the pull-back of a nef and big divisor by a generically finite morphism is still nef and big.

Corollary 4.14 *Let D be a nef and big $\mathbf{Q}$-divisor on a projective variety X. There exists an effective $\mathbf{Q}$-Cartier $\mathbf{Q}$-divisor E on X such that $D - tE$ is ample for all rationals t in $(0, 1]$.*

PROOF. We may assume that D has integral coefficients. Let n be the dimension of X and let H be an effective ample divisor on X. Since $h^0(H, mD) = O(m^{n-1})$, we have $H^0(X, mD - H) \neq 0$ for all m sufficiently large by Proposition 4.12.b). Writing $mD \underset{\text{lin}}{\equiv} H + E'$, with E' effective, we get

$$D = \Big(\frac{t}{m} H + (1 - t)D\Big) + \frac{t}{m} E'$$

where $\frac{t}{m} H + (1 - t)D$ is ample for all rationals t in $(0, 1]$ by 4.3. This proves the corollary with $E = \frac{1}{m} E'$. □

4.6 Relative cone of curves

Let X and Y be projective varieties, and let $\pi : X \to Y$ be a morphism. There are induced morphisms

$$\pi^* : N^1(Y)_{\mathbf{Z}} \to N^1(X)_{\mathbf{Z}} \qquad \text{and} \qquad \pi_* : N_1(X)_{\mathbf{Z}} \to N_1(Y)_{\mathbf{Z}}$$

defined by (see 3.17)

$$\pi^*([D]) = [\pi^*(D)] \qquad \text{and} \qquad \pi_*([C]) = [\pi_*(C)] = \deg\big(C \xrightarrow{\pi} \pi(C)\big)\ [\pi(C)]$$

which can be extended to $\mathbf{R}$-linear maps

$$\pi^* : N^1(Y)_{\mathbf{R}} \to N^1(X)_{\mathbf{R}} \qquad \text{and} \qquad \pi_* : N_1(X)_{\mathbf{R}} \to N_1(Y)_{\mathbf{R}}$$

which satisfy the projection formula (see (3.6))

$$\pi^*(d) \cdot c = d \cdot \pi_*(c).$$

This formula implies for example that when π is surjective, $\pi^* : N^1(Y)_{\mathbf{R}} \to N^1(X)_{\mathbf{R}}$ is injective and $\pi_* : N_1(X)_{\mathbf{R}} \to N_1(Y)_{\mathbf{R}}$ is surjective. Indeed, for any curve $C \subset Y$, there is then a curve $C' \subset X$ such that $\pi(C') = C$, so that $\pi_*([C']) = m[C]$ for some positive integer m and π_* is surjective. By the projection formula, the kernel of π^* is orthogonal to the image of π_*, hence is 0.

Definition 4.15 *The* relative cone of curves *is the convex subcone* $\mathrm{NE}(\pi)$ *of* $\mathrm{NE}(X)$ *generated by the classes of curves contracted by* π.

Since Y is projective, an irreducible curve C on X is contracted by π if and only if $\pi_*[C] = 0$: being contracted is a numerical property. Equivalently, if H is an ample divisor on Y, the curve C is contracted if and only if $(\pi^* H \cdot C) = 0$.

The cone $\mathrm{NE}(\pi)$ is the intersection of $\mathrm{NE}(X)$ with the hyperplane $(\pi^* H)^\perp$. It is therefore closed in $\mathrm{NE}(X)$ and

$$\overline{\mathrm{NE}}(\pi) \subset \overline{\mathrm{NE}}(X) \cap (\pi^* H)^\perp. \tag{4.3}$$

Example 4.16 The vector space $N_1(\mathbf{P}^n_\mathbf{k})_\mathbf{R}$ has dimension 1; it is generated by the class of a line ℓ.The cone of curves is

$$\mathrm{NE}(\mathbf{P}^n_\mathbf{k}) = \mathbf{R}^+ \ell.$$

Consider the following morphisms starting from $\mathbf{P}^n_\mathbf{k}$: the identity and the map to a point. The corresponding relative subcones of $\mathrm{NE}(X)$ are $\{0\}$ and $\mathrm{NE}(X)$.

Example 4.17 Let X be a product $\mathbf{P} \times \mathbf{P}'$ of two projective spaces over a field. It easily follows from Exercise 2.13 that $N^1(X)_\mathbf{R}$ has dimension 2. Hence, $N_1(X)_\mathbf{R}$ has dimension 2 as well, and is generated by the class ℓ of a line in $\mathbf{P}$ and the class ℓ' of a line in $\mathbf{P}'$. The cone of curves of X is

$$\mathrm{NE}(X) = \mathbf{R}^+ \ell + \mathbf{R}^+ \ell'.$$

Consider the following morphisms starting from X: the identity, the map to a point, and the two projections. The corresponding relative subcones of $\mathrm{NE}(X)$ are $\{0\}$, $\mathrm{NE}(X)$, and $\mathbf{R}^+ \ell$ and $\mathbf{R}^+ \ell'$.

Exercise 4.18 Let $\pi : X \to Y$ a projective morphism of schemes over a field. We say that a Cartier divisor D on X is π*-ample* if the restriction of D to every fiber of π is ample. Show the relative version of Kleiman's criterion: *D is π-ample if and only if it is positive on* $\overline{\mathrm{NE}}(\pi) - \{0\}$. Deduce from this criterion that if D is π-ample and H is ample on Y, the divisors $m\pi^* H + D$ are ample for all $m \gg 0$.

We are interested in projective surjective morphisms $\pi : X \to Y$ which are characterized by the curves they contract. A moment of thinking will convince the reader that this kind of information can only detect the connected components of the fibers, so we want to require at least connectedness of the fibers. When the characteristic of the base field is positive, this is not quite enough because of inseparability phenomena. The actual condition is

$$\pi_* \mathscr{O}_X \simeq \mathscr{O}_Y. \tag{4.4}$$

Exercise 4.19 Show that condition (4.4) for a projective surjective morphism $\pi : X \to Y$ between integral schemes, with Y normal, is equivalent to each of the following properties (see [G1], III, Corollaire (4.3.12)):

(i) the field $K(Y)$ is algebraically closed in $K(X)$;

(ii) the generic fiber of π is geometrically integral.

If condition (4.4) holds (and π is projective), π is surjective[4] and its fibers are indeed connected ([H1], Corollary III.11.3), and even geometrically connected ([G1], III, Corollaire (4.3.12)).

4.20. Recall that any projective morphism $\pi : X \to Y$ has a *Stein factorization* ([H1], Corollary III.11.5)

$$\pi : X \xrightarrow{\pi'} Y' \xrightarrow{g} Y,$$

[4] It is a general fact that (the closure of) the image of a morphism $\pi : X \to Y$ is defined by the ideal sheaf kernel of the canonical map $\mathscr{O}_Y \to \pi_* \mathscr{O}_X$.

where Y' is the scheme $\mathbf{Spec}(\pi_* \mathscr{O}_X)$ (for a definition, see [H1], Exercise II.5.17), so that $\pi'_* \mathscr{O}_X \simeq \mathscr{O}_{Y'}$ (the morphism π' has connected fibers) and g is finite. When X is integral and normal, another way to construct Y' is as the normalization of $\pi(X)$ in the field $K(X)$.[5]

If the fibers of π are connected, the morphism g is bijective, but may not be an isomorphism. However, *if the characteristic is zero and Y is normal*, g is an isomorphism and $\pi_* \mathscr{O}_X \simeq \mathscr{O}_Y$.[6] In positive characteristic, g might very well be a bijection without being an isomorphism (even if Y is normal: think of the Frobenius morphism).

For any projective morphism $\pi : X \to Y$ with Stein factorization $\pi : X \xrightarrow{\pi'} Y' \to Y$, the curves contracted by π and the curves contracted by π' are the same, hence the relative cones of π and π' are the same, so the condition (4.4) is really not too restrictive.

Our next result shows that morphisms π defined on a projective variety X which satisfy (4.4) are characterized by their relative closed cone $\overline{\mathrm{NE}}(\pi)$. Moreover, this closed convex subcone of $\overline{\mathrm{NE}}(X)$ has a simple geometric property: it is *extremal*, meaning that if a and b are in $\overline{\mathrm{NE}}(X)$ and $a+b$ is in $\overline{\mathrm{NE}}(\pi)$, both a and b are in $\overline{\mathrm{NE}}(\pi)$ (geometrically, this means that $\overline{\mathrm{NE}}(X)$ lies on one side of some hyperplane containing $\overline{\mathrm{NE}}(\pi)$; we will prove this in Lemma 4.24 below, together with other elementary results on closed convex cones and their extremal subcones).

It is one of the aims of Mori's Minimal Model Program to give sufficient conditions on an extremal subcone of $\overline{\mathrm{NE}}(X)$ for it to be associated with an actual morphism, thereby converting geometric data on the (relatively) simple object $\overline{\mathrm{NE}}(X)$ into information about the variety X.

Proposition 4.21 *Let X, Y, and Y' be projective varieties and let $\pi : X \to Y$ be a morphism.*

a) The subcone $\overline{\mathrm{NE}}(\pi)$ of $\overline{\mathrm{NE}}(X)$ is extremal *and, if H is an ample divisor on Y, it is equal to the intersection of $\overline{\mathrm{NE}}(X)$ with the supporting hyperplane $(\pi^* H)^\perp$.*

b) Assume $\pi_* \mathscr{O}_X \simeq \mathscr{O}_Y$ and let $\pi' : X \to Y'$ be another morphism.

- *If $\overline{\mathrm{NE}}(\pi)$ is contained in $\overline{\mathrm{NE}}(\pi')$, there is a* unique *morphism $f : Y \to Y'$ such that $\pi' = f \circ \pi$.*
- *The morphism π is uniquely determined by $\overline{\mathrm{NE}}(\pi)$ up to isomorphism.*

Proof. The divisor $\pi^* H$ is nonnegative on the cone $\overline{\mathrm{NE}}(X)$, hence it defines a supporting hyperplane of this cone and it is enough to show that there is equality in (4.3). Proceeding by contradiction, if the inclusion is strict, there exists by Lemma 4.24.a), a linear form ℓ which is positive on $\overline{\mathrm{NE}}(\pi) - \{0\}$ but is such that $\ell(z) < 0$ for some $z \in \overline{\mathrm{NE}}(X) \cap (\pi^* H)^\perp$. We can choose ℓ to be rational, and we can even assume that it is given by intersecting with a Cartier divisor D. By the relative version of Kleiman's criterion (Exercise 4.18), D is π-ample, and by the same exercise, $mH + D$ is ample for $m \gg 0$. But $(mH + D) \cdot z = D \cdot z < 0$, which contradicts Kleiman's criterion. This proves a).

To prove b), we first note that if $\overline{\mathrm{NE}}(\pi) \subset \overline{\mathrm{NE}}(\pi')$, any curve contained in a fiber of π is contracted by π', hence π' contracts (to a point) each (closed) fiber of π. We use the following rigidity result.

Lemma 4.22 *Let X, Y and Y' be integral schemes and let $\pi : X \to Y$ and $\pi' : X \to Y'$ be projective morphisms. Assume $\pi_* \mathscr{O}_X \simeq \mathscr{O}_Y$.*

a) If π' contracts one fiber $\pi^{-1}(y_0)$ of π, there is an open neighborhood Y_0 of y_0 in Y and a factorization

$$\pi'|_{\pi^{-1}(Y_0)} : \pi^{-1}(Y_0) \xrightarrow{\pi} Y_0 \longrightarrow Y'.$$

b) If π' contracts each *fiber of π, it factors through π.*

[5] This is constructed exactly as the standard normalization (see [H1], Exercise II.3.8) by patching up the spectra of the integral closures *in* $K(X)$ of the coordinate rings of affine open subsets of $\pi(X)$. The fact that g is finite follows from the finiteness of integral closure ([H1], Theorem I.3.9A).

[6] By generic smoothness ([H1], Corollary III.10.7), g is birational. If U is an affine open subset of Y, the ring $H^0(g^{-1}(U), \mathscr{O}_{Y'})$ is finite over the integrally closed ring $H^0(U, \mathscr{O}_Y)$, with the same quotient field, hence they are equal and g is an isomorphism.

Proof. Note that π is surjective. Let Z be the image of

$$g : X \xrightarrow{(\pi,\pi')} Y \times Y'$$

and let $p : Z \to Y$ and $p' : Z \to Y'$ be the two projections. Then $\pi^{-1}(y_0) = g^{-1}(p^{-1}(y_0))$ is contracted by π', hence by g. It follows that the fiber $p^{-1}(y_0) = g(g^{-1}(p^{-1}(y_0)))$ is a point hence the proper surjective morphism p is finite over an open affine neighborhood Y_0 of y_0 in Y. Set $X_0 = \pi^{-1}(Y_0)$ and $Z_0 = p^{-1}(Y_0)$, and let $p_0 : Z_0 \to Y_0$ be the (finite) restriction of p; we have $\mathscr{O}_{Z_0} \subset g_*\mathscr{O}_{X_0}$ and

$$\mathscr{O}_{Y_0} \subset p_{0*}\mathscr{O}_{Z_0} \subset p_{0*}g_*\mathscr{O}_{X_0} = \pi_*\mathscr{O}_{X_0} = \mathscr{O}_{Y_0}$$

hence $p_{0*}\mathscr{O}_{Z_0} \simeq \mathscr{O}_{Y_0}$. But the morphism p_0, being finite, is affine, hence Z_0 is affine and the isomorphism $p_{0*}\mathscr{O}_{Z_0} \simeq \mathscr{O}_{Y_0}$ says that p_0 induces an isomorphism between the coordinate rings of Z_0 and Y_0. Therefore, p_0 is an isomorphism, and $\pi' = p' \circ p_0^{-1} \circ \pi|_{X_0}$. This proves a).

If π' contracts *each* fiber of π, the morphism p above is finite, one can take $Y_0 = Y$ and π' factors through π. This proves b). □

Going back to the proof of item b) in the proposition, we assume now $\pi_*\mathscr{O}_X \simeq \mathscr{O}_Y$ and $\overline{\mathrm{NE}}(\pi) \subset \overline{\mathrm{NE}}(\pi')$. This means that every irreducible curve contracted by π is contracted by π', hence every (connected) fiber of π is contracted by π'. The existence of f follows from item b) of the lemma. If $f' : Y \to Y'$ satisfies $\pi' = f' \circ \pi$, the composition $Z \xrightarrow{p} Y \xrightarrow{f'} Y'$ must be the second projection, hence $f' \circ p = p'$ and $f' = p' \circ p^{-1}$.

The second item in b) follows from the first. □

Example 4.23 Refering to Example 4.16, the (closed) cone of curves for $\mathbf{P}^n_{\mathbf{k}}$ has two extremal subcones: $\{0\}$ and $\mathrm{NE}(\mathbf{P}^n_{\mathbf{k}})$. By the Proposition 4.21 (and the existence of the Stein factorization), this means that any proper morphism $\mathbf{P}^n_{\mathbf{k}} \to Y$ is either finite or constant (prove that directly: it is not too difficult).

Refering to Example 4.17, the cone of curves of the product $X = \mathbf{P} \times \mathbf{P}'$ of two projective spaces has four extremal subcones. By the Proposition 4.21, this means that any proper morphism $\pi : X \to Y$ satisfying (4.4) is, up to isomorphism, either the identity, the map to a point, or one of the two projections.

4.7 Elementary properties of cones

We gather in this section some elementary results on closed convex cones that we have been using.

Let V be a cone in $\mathbf{R}^m$; we define its dual cone by

$$V^* = \{\ell \in (\mathbf{R}^m)^* \mid \ell \geq 0 \text{ on } V\}$$

Recall that a subcone W of V is *extremal* if it is closed and convex and if any two elements of V whose sum is in W are both in W. An extremal subcone of dimension 1 is called an *extremal ray.* A nonzero linear form ℓ in V^* is a *supporting function* of the extremal subcone W if it vanishes on W.

Lemma 4.24 *Let V be a closed convex cone in $\mathbf{R}^m$.*

a) We have $V = V^{**}$ and

$$V \text{ contains no lines} \iff V^* \text{ spans } (\mathbf{R}^m)^*.$$

The interior of V^* is

$$\{\ell \in (\mathbf{R}^m)^* \mid \ell > 0 \text{ on } V - \{0\}\}.$$

b) If V contains no lines, it is the convex hull of its extremal rays.

c) Any proper extremal subcone of V has a supporting function.

d) If V contains no lines[7] *and W is a proper closed subcone of V, there exists a linear form in V^* which is positive on $W - \{0\}$ and vanishes on some extremal ray of V.*

PROOF. Obviously, V is contained in V^{**}. Choose a scalar product on $\mathbf{R}^m$. If $z \notin V$, let $p_V(z)$ be the projection of z on the closed convex set V; since V is a cone, $z - p_V(z)$ is orthogonal to $p_V(z)$. The linear form $\langle p_V(z) - z, \cdot\rangle$ is nonnegative on V and negative at z, hence $z \notin V^{**}$.

If V contains a line L, any element of V^* must be nonnegative, hence must vanish, on L: the cone V^* is contained in $L^\perp$. Conversely, if V^* is contained in a hyperplane H, its dual V contains the line by $H^\perp$ in $\mathbf{R}^m$.

Let ℓ be an interior point of V^*; for any nonzero z in V, there exists a linear form ℓ' with $\ell'(z) > 0$ and small enough so that $\ell - \ell'$ is still in V^*. This implies $(\ell - \ell')(z) \geq 0$, hence $\ell(z) > 0$. Since the set $\{\ell \in (\mathbf{R}^m)^* \mid \ell > 0 \text{ on } V - \{0\}\}$ is open, this proves a).

Assume that V contains no lines; we will prove by induction on m that any point of V is in the linear span of m extremal rays.

4.25. Note that for any point v of ∂V, there exists by a) a nonzero element ℓ in V^* that vanishes at v. An extremal ray $\mathbf{R}^+ r$ in $\mathrm{Ker}(\ell) \cap V$ (which exists thanks to the induction hypothesis) is still extremal in V: if $r = x_1 + x_2$ with x_1 and x_2 in V, since $\ell(x_i) \geq 0$ and $\ell(r) = 0$, we get $x_i \in \mathrm{Ker}(\ell) \cap V$ hence they are both proportional to r.

Given $v \in V$, the set $\{\lambda \in \mathbf{R}^+ \mid v - \lambda r \in V\}$ is a closed nonempty interval which is bounded above (otherwise $-r = \lim_{\lambda \to +\infty} \frac{1}{\lambda}(v - \lambda r)$ would be in V). If λ_0 is its maximum, $v - \lambda_0 r$ is in ∂V, hence there exists by a) an element ℓ' of V^* that vanishes at $v - \lambda_0 r$. Since

$$v = \lambda_0 r + (v - \lambda_0 r)$$

item b) follows from the induction hypothesis applied to the closed convex cone $\mathrm{Ker}(\ell') \cap V$ and the fact that any extremal ray in $\mathrm{Ker}(\ell') \cap V$ is still extremal for V.

Let us prove c). We may assume that V spans $\mathbf{R}^m$. Note that an extremal subcone W of V distinct from V is contained in ∂V: if W contains an interior point v, then for any small x, we have $v \pm x \in V$ and $2v = (v + x) + (v - x)$ implies $v \pm x \in W$. Hence W is open in the interior of V; since it is closed, it contains it. In particular, the interior of W is empty, hence its span $\langle W\rangle$ is not $\mathbf{R}^m$. Let w be a point of its interior in $\langle W\rangle$; by a), there exists a nonzero element ℓ of V^* that vanishes at w. By a) again (applied to W^* in its span), ℓ must vanish on $\langle W\rangle$ hence is a supporting function of W.

Let us prove d). Since W contains no lines, there exists by a) a point in the interior of W^* which is not in V^*. The segment connecting it to a point in the interior of V^* crosses the boundary of V^* at a point in the interior of W^*. This point corresponds to a linear form ℓ that is positive on $W - \{0\}$ and vanishes at a nonzero point of V. By b), the closed cone $\mathrm{Ker}(\ell) \cap V$ has an extremal ray, which is still extremal in V by 4.25. This proves d). □

4.8 Exercises

1) Let X be a smooth projective variety and let $\varepsilon : \widetilde{X} \to X$ be the blow-up of a point, with exceptional divisor E.

a) Prove

$$\mathrm{Pic}(\widetilde{X}) \simeq \mathrm{Pic}(X) \oplus \mathbf{Z}[\mathscr{O}_{\widetilde{X}}(E)]$$

(see Corollary 3.11) and

$$N^1(\widetilde{X})_{\mathbf{R}} \simeq N^1(X)_{\mathbf{R}} \oplus \mathbf{Z}[E].$$

[7] This assumption is necessary, as shown by the example $V = \{(x, y) \in \mathbf{R}^2 \mid y \geq 0\}$ and $W = \{(x, y) \in \mathbf{R}^2 \mid x, y \geq 0\}$.

b) If ℓ is a line contained in E, prove

$$N_1(\widetilde{X})_{\mathbf{R}} \simeq N_1(X)_{\mathbf{R}} \oplus \mathbf{Z}[\ell].$$

c) If $X = \mathbf{P}^n$, compute the cone of curves $\mathrm{NE}(\widetilde{\mathbf{P}}^n)$.

2) Let X be a projective scheme, let $\mathscr{F}$ be a coherent sheaf on X, and let $H_1, \dots, H_r$ be ample divisors on X. Show that for each $i > 0$, the set

$$\{(m_1, \dots, m_r) \in \mathbf{N}^r \mid H^i(X, \mathscr{F}(m_1 H_1 + \cdots + m_r H_r)) \neq 0\}$$

is finite.

3) Let $D_1, \dots, D_n$ be Cartier divisors on an n-dimensional projective scheme. Prove the following:

a) If $D_1, \dots, D_n$ are ample, $(D_1 \cdot \ldots \cdot D_n) > 0$;

b) If $D_1, \dots, D_n$ are nef, $(D_1 \cdot \ldots \cdot D_n) \geq 0$.

4) Let D be a Cartier divisor on a projective scheme X (see 4.13).

a) Show that the following properties are equivalent:

(i) D is big;

(ii) D is the sum of an ample $\mathbf{Q}$-divisor and of an effective $\mathbf{Q}$-divisor;

(iii) D is numerically equivalent to the sum of an ample $\mathbf{Q}$-divisor and of an effective $\mathbf{Q}$-divisor;

(iv) there exists a positive integer m such that the rational map

$$X \dashrightarrow \mathbf{P}H^0(X, mD)$$

associated with the linear system $|mD|$ is birational onto its image.

b) It follows from (iii) above that being big is a numerical property. Show that the set of classes of big Cartier divisors on X generate a cone which is the interior of the pseudo-effective cone (i.e., of the closure of the effective cone).

5) Let X be a projective variety. Show that any surjective morphism $X \to X$ is finite.

Chapter 5

Surfaces

In this chapter, all surfaces are 2-dimensional integral schemes over an algebraically closed field $\mathbf{k}$.

5.1 Preliminary results

5.1.1 The adjunction formula

Let X be a smooth projective variety. We "defined" in Example 2.17 (at least over $\mathbf{C}$), "the" canonical class K_X. Let $Y \subset X$ be a smooth hypersurface. We have ([H1], Proposition 8.20)

$$K_Y = (K_X + Y)|_Y.$$

We saw an instance of this formula in Examples 1.4 and 2.17.

We will explain the reason for this formula using the (locally free) sheaf of differentials $\Omega_{X/\mathbf{k}}$ (see [H1], II.8 for more details); over $\mathbf{C}$, this is just the dual of the sheaf of local sections of the tangent bundle T_X of X. If f_i is a local equation for Y in X on an open set U_i, the sheaf $\Omega_{Y/\mathbf{k}}$ is just the quotient of the restriction of $\Omega_{X/\mathbf{k}}$ to Y by the ideal generated by df_i. Dually, over $\mathbf{C}$, this is just saying that in local analytic coordinates $x_1, \dots, x_n$ on X, the tangent space $T_{Y,p} \subset T_{X,p}$ at a point p of Y is defined by the equation

$$df_i(p)(t) = \frac{\partial f_i}{\partial x_1}(p)t_1 + \cdots + \frac{\partial f_i}{\partial x_n}(p)t_n = 0.$$

If we write as usual, on the intersection of two such open sets, $f_i = g_{ij} f_j$, we have $df_i = dg_{ij} f_j + g_{ij} df_j$, hence $df_i = g_{ij} df_j$ on $Y \cap U_{ij}$. Since the collection (g_{ij}) defines the invertible sheaf $\mathscr{O}_X(-Y)$ (which is also the ideal sheaf of Y in X), we obtain an exact sequence of locally free sheaves (see also [H1], Proposition II.8.20)

$$0 \to \mathscr{O}_Y(-Y) \to \Omega_{X/\mathbf{k}} \otimes \mathscr{O}_Y \to \Omega_{Y/\mathbf{k}} \to 0.$$

In other words, the normal bundle of Y in X is $\mathscr{O}_Y(Y)$. Since $\mathscr{O}_X(K_X) = \det(\Omega_{X/\mathbf{k}})$, we obtain the adjunction formula by taking determinants.

5.1.2 Serre duality

Let X be a smooth projective variety of dimension n, with canonical class K_X. Serre duality says that for any divisor D on X, the natural pairing

$$H^i(X, D) \otimes H^{n-i}(X, K_X - D) \to H^n(X, K_X) \simeq \mathbf{k},$$

given by cup-product, is non-degenerate. In particular,

$$h^i(X, D) = h^{n-i}(X, K_X - D).$$

5.1.3 The Riemann-Roch theorem for curves

Let X be a smooth projective curve and let D be a divisor on X. Serre duality gives $h^0(X, K_X) = g(X)$ and the Riemann-Roch theorem (Theorem 3.3) gives

$$h^0(X, D) - h^0(X, K_X - D) = \deg(D) + 1 - g(X).$$

Taking $D = K_X$, we obtain $\deg(K_X) = 2g(X) - 2$.

5.1.4 The Riemann-Roch theorem for surfaces

Let X be a smooth projective surface and let D be a divisor on X. We know from (3.5) that there is a rational number a such that for all m,

$$\chi(X, mD) = \frac{m^2}{2}(D^2) + am + \chi(X, \mathscr{O}_X).$$

The Riemann-Roch theorem for surfaces identifies this number a in terms of the canonical class of X and states

$$\chi(X, D) = \frac{1}{2}((D^2) - (K_X \cdot D)) + \chi(X, \mathscr{O}_X).$$

The proof is not really difficult (see [H1], Theorem V.1.6) but it uses an ingredient that we haven't proved yet: the fact that any divisor D on X is linearly equivalent to the difference of two smooth curves C and C'. We then have (Theorem 3.6)

$$\begin{aligned} \chi(X, D) &= -(C \cdot C') + \chi(X, C) + \chi(X, -C') - \chi(X, \mathscr{O}_X) \\ &= -(C \cdot C') + \chi(X, \mathscr{O}_X) + \chi(C, C|_C) - \chi(C', \mathscr{O}_{C'}) \\ &= -(C \cdot C') + \chi(X, \mathscr{O}_X) + (C^2) + 1 - g(C) - (1 - g(C')), \end{aligned}$$

using the exact sequences

$$0 \to \mathscr{O}_X(-C') \to \mathscr{O}_X \to \mathscr{O}_{C'} \to 0$$

and

$$0 \to \mathscr{O}_X \to \mathscr{O}_X(C) \to \mathscr{O}_C(C) \to 0.$$

and Riemann-Roch on C and C'.

We then use

$$2g(C) - 2 = \deg(K_C) = \deg(K_X + C)|_C = ((K_X + C) \cdot C)$$

and similarly for C' and obtain

$$\begin{aligned} \chi(X, D) - \chi(X, \mathscr{O}_X) &= -(C \cdot C') + (C^2) - \frac{1}{2}((K_X + C) \cdot C) \\ &\qquad + \frac{1}{2}((K_X + C') \cdot C') \\ &= \frac{1}{2}((D^2) - (K_X \cdot D)). \end{aligned}$$

It is traditional to write

$$p_g(X) = h^0(X, K_X) = h^2(X, \mathscr{O}_X),$$

the *geometric genus* of X, and

$$q(X) = h^1(X, K_X) = h^1(X, \mathscr{O}_X),$$

the *irregularity* of X, so we have

$$\chi(X, \mathscr{O}_X) = p_g - q + 1.$$

Note that for any irreducible curve C in X, we have

$$\begin{aligned} g(C) &= h^1(C, \mathscr{O}_C) = 1 - \chi(C, \mathscr{O}_C) \\ &= 1 + \chi(C, \mathscr{O}_X(-C)) - \chi(X, \mathscr{O}_X) \\ &= 1 + \frac{1}{2}((C^2) + (K_X \cdot C)). \end{aligned} \tag{5.1}$$

In particular, we deduce from Corollary 3.18 that

$$(C^2) + (K_X \cdot C) = -2$$

if and only if the curve C is smooth and rational.

Example 5.1 (Self-product of a curve) Let C be a smooth curve of genus g and let X be the surface $C \times C$, with p_1 and p_2 the two projections to C. We consider the classes x_1 of $\{\star\} \times C$, x_2 of $C \times \{\star\}$, and Δ of the diagonal. The canonical class of X is

$$K_X = p_1^* K_C + p_2^* K_C \underset{\text{num}}{\equiv} (2g-2)(x_1 + x_2).$$

Since we have $(\Delta \cdot x_j) = 1$, we compute $(K_X \cdot \Delta) = 4(g-1)$. Since Δ has genus g, the genus formula (5.1) yields

$$(\Delta^2) = 2g - 2 - (K_X \cdot \Delta) = -2(g-1).$$

5.2 Ruled surfaces

We begin with a result that illustrates the use of the Riemann-Roch theorem for curves over a non-algebraically closed field.

Theorem 5.2 (Tsen's theorem) *Let X be a projective surface with a morphism $\pi : X \to B$ onto a smooth curve B, over an algebraically closed field $\mathbf{k}$. Assume that the generic fiber is a geometrically integral curve of genus 0. Then X is birational over B to $B \times \mathbf{P}^1_{\mathbf{k}}$.*

Proof. We will use the fact that any geometrically integral curve C of genus 0 over any field $\mathbf{K}$ is isomorphic to a nondegenerate conic in $\mathbf{P}^2_{\mathbf{K}}$ (this comes from the fact that the anticanonical class $-K_C$ is defined over $\mathbf{K}$, is very ample, and has degree 2 by Riemann-Roch).

We must show that when $\mathbf{K} = K(B)$, any such conic has a $\mathbf{K}$-point. Let

$$q(x_0, x_1, x_2) = \sum_{0 \le i,j \le 2} a_{ij} x_i x_j = 0$$

be an equation for this conic. All the elements a_{ij} of $K(B)$ can be viewed as sections of $\mathscr{O}_B(E)$ for some effective nonzero divisor E on B. We consider, for any positive integer m, the map

$$\begin{aligned} f_m : H^0(B, mE)^3 &\longrightarrow H^0(B, 2mE + E) \\ (x_0, x_1, x_2) &\longmapsto \sum_{0 \le i,j \le 2} a_{ij} x_i x_j. \end{aligned}$$

Since E is ample, by Riemann-Roch and Serre's theorems, the dimension of the vector space on the left-hand-side is, for $m \gg 0$,

$$a_m = 3(m \deg(E) + 1 - g(B)),$$

whereas the dimension of the vector space on the right-hand-side is

$$b_m = (2m+1)\deg(E) + 1 - g(B).$$

We are looking for a nonzero $(x_0, x_1, x_2) \in H^0(B, mE)^3$ such that $q(x_0, x_1, x_2) = 0$. In other words, (x_0, x_1, x_2) should be an element in the intersection of b_m quadrics in a projective space (over $\mathbf{k}$) of dimension $a_m - 1$. For $m \gg 0$, we have $a_m - 1 \ge b_m$, and such a (x_0, x_1, x_2) exists because $\mathbf{k}$ is algebraically closed. It is a $\mathbf{K}$-point of the conic. ☐

Theorem 5.3 *Let X be a projective surface with a morphism $\pi : X \to B$ onto a smooth curve B, over an algebraically closed field $\mathbf{k}$. Assume that fibers over closed points are all isomorphic to $\mathbf{P}^1_{\mathbf{k}}$. Then there exists a locally free rank-2 sheaf $\mathscr{E}$ on B such that X is isomorphic over B to $\mathbf{P}(\mathscr{E})$.*

PROOF. We need to use some theorems far beyond this course. The sheaf $\pi_*\mathscr{O}_X$ is a locally free on B. Since π is flat, and $H^0(X_b, \mathscr{O}_{X_b}) = 1$ for all closed points $b \in B$, the base change theorem ([H1], Theorem III.12.11) implies that it has rank 1 hence is isomorphic to $\mathscr{O}_B$. In particular (Exercise 4.19), the generic fiber of π is geometrically integral.

Similarly, since $H^1(X_b, \mathscr{O}_{X_b}) = 0$ for all closed points $b \in B$, the base change theorem again implies that the sheaf $R^1\pi_*\mathscr{O}_X$ is zero and that the generic fiber also has genus 0.

It follows from Tsen's theorem that π has a rational section which, since B is smooth, extends to a section $\sigma : B \to X$ whose image we denote by C. We then have $(C \cdot X_b) = 1$ for all $b \in B$, hence, by the base change theorem again, $\mathscr{E} = \pi_*(\mathscr{O}_X(C))$ is a locally free rank-2 sheaf on B. Furthermore, the canonical morphism

$$\pi^*(\pi_*(\mathscr{O}_X(C))) \to \mathscr{O}_X(C)$$

is surjective, hence there exists, by the universal property of $\mathbf{P}(\mathscr{E})$ ([H1], Proposition II.7.12), a morphism $f : X \to \mathbf{P}(\mathscr{E})$ over B with the property $f^*\mathscr{O}_{\mathbf{P}(\mathscr{E})}(1) = \mathscr{O}_X(C)$. Since $\mathscr{O}_X(C)$ is very ample on each fiber, f is an isomorphism. □

Keeping the notation of the proof, note that since $\pi_*\mathscr{O}_X = \mathscr{O}_B$ and $R^1\pi_*\mathscr{O}_X = 0$, the direct image by π_* of the exact sequence

$$0 \to \mathscr{O}_X \to \mathscr{O}_X(C) \to \mathscr{O}_C(C) \to 0$$

is

$$0 \to \mathscr{O}_B \to \mathscr{E} \to \sigma^*\mathscr{O}_C(C) \to 0.$$

In particular,

$$(C^2) = \deg(\det \mathscr{E})). \tag{5.2}$$

Moreover, the invertible sheaf $\mathscr{O}_{\mathbf{P}(\mathscr{E})}(1)$ is $\mathscr{O}_X(C)$, so that $\sigma^*\mathscr{O}_C(C) \simeq \sigma^*\mathscr{O}_{\mathbf{P}(\mathscr{E})}(1)$.

Definition 5.4 *A ruled surface is a projective surface X with a surjective morphism $\pi : X \to B$ onto a smooth projective curve B, such that the fiber of every closed point is isomorphic to $\mathbf{P}^1_{\mathbf{k}}$.*

The terminology is not constant in the literature: for some, a ruled surface is just a surjective morphism $\pi : X \to B$ whose generic fiber is rational, and our ruled surfaces are called *geometrically ruled* surfaces.

By Theorem 5.3, the ruled surfaces over B are the $\mathbf{P}(\mathscr{E})$, for some locally free rank-2 sheaf $\mathscr{E}$ on B. In particular, they are *smooth*. Such a surface comes with an invertible sheaf $\mathscr{O}_{\mathbf{P}(\mathscr{E})}(1)$ such that $\pi_*\mathscr{O}_{\mathbf{P}(\mathscr{E})}(1) \simeq \mathscr{E}$. For any invertible sheaf $\mathscr{M}$ on B, there is an isomorphism $f : \mathbf{P}(\mathscr{E}) \xrightarrow{\sim} \mathbf{P}(\mathscr{E} \otimes \mathscr{M})$ over B, and $f^*\mathscr{O}_{\mathbf{P}(\mathscr{E}\otimes\mathscr{M})}(1) \xrightarrow{\sim} \mathscr{O}_{\mathbf{P}(\mathscr{E})}(1) \otimes \pi^*\mathscr{M}$.

Proposition 5.5 *Let $\pi : X \to B$ be a ruled surface. Let $B \to C$ be a section and let F be a fiber. The map*

$$\begin{array}{rcl} \mathbf{Z} \times \operatorname{Pic}(B) & \longrightarrow & \operatorname{Pic}(X) \\ (n, [D]) & \longmapsto & [nC + \pi^*D] \end{array}$$

is a group isomorphism, and

$$N^1(X) \simeq \mathbf{Z}[C] \oplus \mathbf{Z}[F].$$

Moreover, $(C \cdot F) = 1$ and $(F^2) = 0$.

Note that the *numerical* equivalence class of F does not depend on the fiber F (this follows for example from the projection formula (3.6)), whereas its *linear* equivalence class does (except when $B = \mathbf{P}^1_{\mathbf{k}}$).

PROOF. Let E be a divisor on X and let $n = (E \cdot F)$. As above, by the base change theorem, $\pi_*(\mathscr{O}_X(E - nC))$ is an invertible sheaf $\mathscr{M}$ on B, and the canonical morphism $\pi^*(\pi_*(\mathscr{O}_X(E-nC))) \to \mathscr{O}_X(E-nC)$ is bijective. Hence

$$\mathscr{O}_X(E) \simeq \mathscr{O}_X(nC) \otimes \pi^*\mathscr{M},$$

so that the map is surjective.

To prove injectivity, note first that if $nC+\pi^*D \underset{\text{lin}}{\equiv} 0$, we have $0=((nC+\pi^*D)\cdot F)=n$, hence $n=0$ and $\pi^*D \underset{\text{lin}}{\equiv} 0$. Then,

$$\mathscr{O}_B \simeq \pi_*\mathscr{O}_X \simeq \pi_*\mathscr{O}_X(\pi^*D) \simeq \pi_*\pi^*\mathscr{O}_B(D) \simeq \mathscr{O}_B(D)\otimes\pi_*\mathscr{O}_X \simeq \mathscr{O}_B(D)$$

by the projection formula ([H1], Exercise II.5.1.(d)), hence $D \underset{\text{lin}}{\equiv} 0$. □

In particular, if $\mathscr{E}$ and $\mathscr{E}'$ are locally free rank-2 sheaves on B such that there is an isomorphism $f:\mathbf{P}(\mathscr{E})\xrightarrow{\sim}\mathbf{P}(\mathscr{E}')$ over B, since $\mathscr{O}_{\mathbf{P}(\mathscr{E})}(1)$ and $f^*\mathscr{O}_{\mathbf{P}(\mathscr{E}')}(1)$ both have intersection number 1 with a fiber, there is by the proposition an invertible sheaf $\mathscr{M}$ on B such that $f^*\mathscr{O}_{\mathbf{P}(\mathscr{E}')}(1)\simeq\mathscr{O}_{\mathbf{P}(\mathscr{E})}(1)\otimes\pi^*\mathscr{M}$. By taking direct images, we get $\mathscr{E}'\simeq\mathscr{E}\otimes\mathscr{M}$.

Let us prove the following formula:

$$((\mathscr{O}_{\mathbf{P}(\mathscr{E})}(1))^2)=\deg(\det\mathscr{E}). \tag{5.3}$$

If C is any section, this formula holds for $\mathscr{E}'=\pi_*\mathscr{O}_X(C)$ by (5.2). By what we just saw, there exists an invertible sheaf $\mathscr{M}$ on B such that $\mathscr{E}\simeq\mathscr{E}'\otimes\mathscr{M}$, hence $\mathscr{O}_{\mathbf{P}(\mathscr{E})}(1)\simeq\mathscr{O}_{\mathbf{P}(\mathscr{E}')}(1)\otimes\pi^*\mathscr{M}$. But then,

$$\deg(\det\mathscr{E})=\deg((\det\mathscr{E}')\otimes\mathscr{M}^2)=\deg(\det\mathscr{E}')+2\deg(\mathscr{M})=(C^2)+2\deg(\mathscr{M}),$$

whereas

$$((\mathscr{O}_{\mathbf{P}(\mathscr{E})}(1))^2)=((\mathscr{O}_{\mathbf{P}(\mathscr{E}')}(1)\otimes\pi^*\mathscr{M})^2)=((C+2\deg(\mathscr{M})F)^2)=(C^2)+2\deg(\mathscr{M}),$$

and the formula is proved.

5.6. Sections. Sections of $\mathbf{P}(\mathscr{E})\to B$ correspond to invertible quotients $\mathscr{E}\twoheadrightarrow\mathscr{L}$ ([H1], §V.2) by taking a section σ to $\mathscr{L}=\sigma^*\mathscr{O}_{\mathbf{P}(\mathscr{E})}(1)$. If $\mathscr{L}$ is such a quotient, the corresponding section σ is such that

$$(\sigma(B))^2=2\deg(\mathscr{L})-\deg(\det\mathscr{E}). \tag{5.4}$$

Indeed, setting $C=\sigma(B)$ and $\mathscr{E}'=\pi_*\mathscr{O}_X(C)$, we have as above $\mathscr{E}'\simeq\mathscr{E}\otimes\mathscr{M}$ for some invertible sheaf $\mathscr{M}$ on B, and

$$\mathscr{O}_X(C)\simeq\mathscr{O}_{\mathbf{P}(\mathscr{E}')}(1)\simeq\mathscr{O}_{\mathbf{P}(\mathscr{E})}(1)\otimes\pi^*\mathscr{M}.$$

Applying σ^*, we obtain

$$\sigma^*\mathscr{O}_X(C)\simeq\mathscr{L}\otimes\mathscr{M},$$

hence $(C^2)=\deg(\mathscr{L})+\deg(\mathscr{M})$. This implies

$$(C^2)=\deg(\det\mathscr{E}')=\deg(\det\mathscr{E})+2\deg(\mathscr{M})=\deg(\det\mathscr{E})+2((C^2)-\deg(\mathscr{L})),$$

which is the desired formula.

Example 5.7 It can be shown that any locally free rank-2 sheaf on $\mathbf{P}^1_{\mathbf{k}}$ is isomorphic to $\mathscr{O}_{\mathbf{P}^1_{\mathbf{k}}}(a)\oplus\mathscr{O}_{\mathbf{P}^1_{\mathbf{k}}}(b)$, for some integers a and b. It follows that any ruled surface over $\mathbf{P}^1_{\mathbf{k}}$ is isomorphic to one of the *Hirzebruch surfaces*

$$\mathbf{F}_n=\mathbf{P}(\mathscr{O}_{\mathbf{P}^1_{\mathbf{k}}}\oplus\mathscr{O}_{\mathbf{P}^1_{\mathbf{k}}}(n)),$$

for $n\in\mathbf{N}$ (note that $\mathbf{F}_0$ is $\mathbf{P}^1_{\mathbf{k}}\times\mathbf{P}^1_{\mathbf{k}}$; what is $\mathbf{F}_1$?). The quotient $\mathscr{O}_{\mathbf{P}^1_{\mathbf{k}}}\oplus\mathscr{O}_{\mathbf{P}^1_{\mathbf{k}}}\twoheadrightarrow\mathscr{O}_{\mathbf{P}^1_{\mathbf{k}}}$ gives a section $C_n\subset\mathbf{F}_n$ such that $(C_n^2)=-n$.

Exercise 5.8 When $n<0$, show that C_n is the only (integral) curve on $\mathbf{F}_n$ with negative self-intersection.

5.3 Extremal rays

Our first result will help us locate extremal curves on the closed cone of curves of a smooth projective surface.

Proposition 5.9 *Let X be a smooth projective surface.*

a) The class of an irreducible curve C with $(C^2) \leq 0$ is in $\partial \overline{\mathrm{NE}}(X)$.

b) The class of an irreducible curve C with $(C^2) < 0$ spans an extremal ray of $\overline{\mathrm{NE}}(X)$.

c) If the class of an irreducible curve C with $(C^2) = 0$ and $(K_X \cdot C) < 0$ spans an extremal ray of $\overline{\mathrm{NE}}(X)$, the surface X is ruled over a smooth curve, C is a fiber and X has Picard number 2.

d) If r spans an extremal ray of $\overline{\mathrm{NE}}(X)$, either $r^2 \leq 0$ or X has Picard number 1.

e) If r spans an extremal ray of $\overline{\mathrm{NE}}(X)$ and $r^2 < 0$, the extremal ray is spanned by the class of an irreducible curve.

Proof. Assume $(C^2) = 0$; then $[C]$ has nonnegative intersection with the class of any effective divisor, hence with any element of $\overline{\mathrm{NE}}(X)$. Let H be an ample divisor on X. If $[C]$ is in the interior of $\overline{\mathrm{NE}}(X)$, so is $[C] + t[H]$ for all t small enough; this implies

$$0 \leq (C \cdot (C + tH)) = t(C \cdot H)$$

for all t small enough, which is absurd since $(C \cdot H) > 0$.

Assume now $(C^2) < 0$ and $[C] = z_1 + z_2$, where z_i is the limit of a sequence of classes of effective $\mathbf{Q}$-divisors $D_{i,m}$. Write

$$D_{i,m} = a_{i,m} C + D'_{i,m}$$

with $a_{i,m} \geq 0$ and $D'_{i,m}$ effective with $(C \cdot D'_{i,m}) \geq 0$. Taking intersections with H, we see that the upper limit of the sequence $(a_{i,m})_m$ is at most 1, so we may assume that it has a limit a_i. In that case, $([D'_{i,m}])_m$ also has a limit $z'_i = z_i - a_i[C]$ in $\overline{\mathrm{NE}}(X)$ which satisfies $C \cdot z'_i \geq 0$. We have then $[C] = (a_1 + a_2)[C] + z'_1 + z'_2$, and by taking intersections with C, we get $a_1 + a_2 \geq 1$. But

$$0 = (a_1 + a_2 - 1)[C] + z'_1 + z'_2$$

and since X is projective, this implies $z'_1 = z'_2 = 0$ and proves b) and a).

Let us prove c). By the adjunction formula (§5.1.1), $(K_X \cdot C) = -2$ and C is smooth rational.

For any divisor D on X such that $(D \cdot H) > 0$, the divisor $K_X - mD$ has negative intersection with H for $m > \frac{(K_X \cdot H)}{(D \cdot H)}$, hence cannot be equivalent to an effective divisor. It follows that $H^0(X, K_X - mD)$ vanishes for $m \gg 0$, hence

$$H^2(X, mD) = 0 \tag{5.5}$$

by Serre duality. In particular, $H^2(X, mC)$ vanishes for $m \gg 0$, and the Riemann-Roch theorem yields, since $(C^2) = 0$ and $(K_X \cdot C) = -2$,

$$h^0(X, mC) - h^1(X, mC) = m + \chi(X, \mathscr{O}_X).$$

In particular, there is an integer $m > 0$ such that $h^0(X, (m-1)C) < h^0(X, mC)$. Since $\mathscr{O}_C(C) \simeq \mathscr{O}_C$, we have an exact sequence

$$0 \to H^0(X, (m-1)C) \to H^0(X, mC) \xrightarrow{\rho} H^0(C, mC) \simeq H^0(C, \mathscr{O}_C) \simeq \mathbf{k},$$

and the restriction map ρ is surjective. It follows that the linear system $|mC|$ has no base-points: the only possible base-points are on C, but a section $s \in H^0(C, mC)$ such that $\rho(s) = 1$ vanishes at no point of C. It defines a morphism from X to a projective space whose image is a curve. Its Stein factorization yields a morphism from X onto a smooth curve whose general fiber F is numerically equivalent to some positive rational multiple of C. Since $(K_X \cdot C) = -2$, we have $(K_X \cdot F) < 0$, and since $(F^2) = 0$, we obtain

$(K_X \cdot F) = -2 = (K_X \cdot C)$, hence F is rational and $F \underset{\text{num}}{\equiv} C$. All fibers are integral since $\mathbf{R}^+[C]$ is extremal and $[C]$ is not divisible in $N^1(X)$. This proves c).

Let us prove d). Let D be a divisor on X with $(D^2) > 0$ and $(D \cdot H) > 0$. For m sufficiently large, $H^2(X, mD)$ vanishes by (5.5), and the Riemann-Roch theorem yields

$$h^0(X, mD) \geq \frac{1}{2}m^2(D^2) + O(m).$$

Since (D^2) is positive, this proves that mD is linearly equivalent to an effective divisor for m sufficiently large, hence D is in $\mathrm{NE}(X)$. Therefore,

$$\{z \in N_1(X)_{\mathbf{R}} \mid z^2 > 0\ ,\ H \cdot z > 0\} \tag{5.6}$$

is contained in $\mathrm{NE}(X)$; since it is open, it is contained in its interior hence does not contain any extremal ray of $\overline{\mathrm{NE}}(X)$, except if X has Picard number 1. This proves d).

Let us prove e). Express r as the limit of a sequence of classes of effective $\mathbf{Q}$-divisors D_m. There exists an integer m_0 such that $r \cdot [D_{m_0}] < 0$, hence there exists an irreducible curve C such that $r \cdot C < 0$. Write

$$D_m = a_m C + D'_m$$

with $a_m \geq 0$ and D'_m effective with $(C \cdot D'_m) \geq 0$. Taking intersections with an ample divisor, we see that the upper limit of the sequence (a_m) is finite, so we may assume that it has a nonnegative limit a. In that case, $([D'_m])$ also has a limit $r' = r - a[C]$ in $\overline{\mathrm{NE}}(X)$ which satisfies

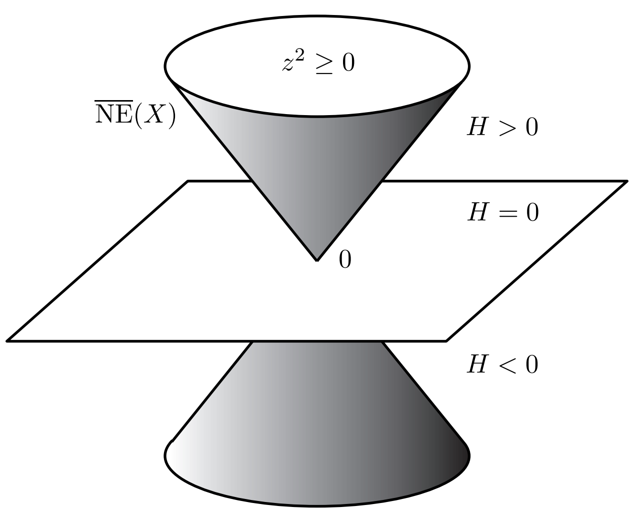
$$0 \leq r' \cdot C = r \cdot C - a(C^2) < -a(C^2)$$

It follows that a is positive and (C^2) is negative; since $\mathbf{R}^+ r$ is extremal and $r = a[C] + r'$, the class r must be a multiple of $[C]$. ☐

Example 5.10 (Abelian surfaces) An abelian surface is a smooth projective surface X which is an (abelian) algebraic group (the structure morphisms are regular maps). This implies that any curve on X has nonnegative self-intersection (because $(C^2) = (C \cdot (g + C)) \geq 0$ for any $g \in X$). Fixing an ample divisor H on X, we have

$$\overline{\mathrm{NE}}(X) = \{z \in N_1(X)_{\mathbf{R}} \mid z^2 \geq 0\ ,\ H \cdot z \geq 0\}$$

Indeed, one inclusion follows from the fact that any curve on X has nonnegative self-intersection, and the other from (5.6). By the Hodge index theorem (Exercise 5.7.2)), the intersection form on $N_1(X)_{\mathbf{R}}$ has exactly one positive eigenvalue, so that when this vector space has dimension 3, the closed cone of curves of X looks like this.



The effective cone of an abelian surface X

In particular, it is not finitely generated. Every boundary point generates an extremal ray, hence there are extremal rays whose only rational point is 0: they cannot be generated by the class of a curve on X.

Example 5.11 (Ruled surfaces) Let X be a $\mathbf{P}^1_{\mathbf{k}}$-bundle over a smooth curve B of genus g. By Proposition 5.5, $\overline{\mathrm{NE}}(X)$ is a closed convex cone in $\mathbf{R}^2$ hence has two extremal rays.

Let F be a fiber; since $F^2 = 0$, its class lies in the boundary of $\overline{\mathrm{NE}}(X)$ by Proposition 5.9.a) hence spans an extremal ray. Let ξ be the other extremal ray. Proposition 5.9.d) implies $\xi^2 \le 0$.

- If $\xi^2 < 0$, we may, by Proposition 5.9.d), take for ξ the class of an irreducible curve C on X, and $\mathrm{NE}(X) = \mathbf{R}^+[C] + \mathbf{R}^+[F]$ is closed.
- If $\xi^2 = 0$, decompose ξ in a basis $([F], z)$ for $N_1(X)_{\mathbf{Q}}$ as $\xi = az + b[F]$. Then $\xi^2 = 0$ implies that a/b is rational, so that we may take ξ rational. However, it may happen that no multiple of ξ can be represented by an effective divisor, in which case $\mathrm{NE}(X)$ is *not* closed.

For example, when $g(B) \ge 2$ and the base field is $\mathbf{C}$, there exists a rank-2 locally free sheaf $\mathscr{E}$ of degree 0 on B, with a nonzero section, all of whose symmetric powers are *stable*.[1] For the associated ruled surface $X = \mathbf{P}(\mathscr{E})$, let E be a divisor class representing $\mathscr{O}_X(1)$. We have $(E^2) = 0$ by (5.3). We first remark that $H^0(X, \mathscr{O}_X(m)(\pi^*D))$ *vanishes for any* $m > 0$ *and any divisor* D *on* B *of degree* ≤ 0. Indeed, this vector space is isomorphic to $H^0(B, (\mathrm{Sym}^m \mathscr{E})(D))$, and, by stability of $\mathscr{E}$, there are no nonzero morphisms from $\mathscr{O}_B(-D)$ to $\mathrm{Sym}^m \mathscr{E}$.

The cone $\mathrm{NE}(X)$ is therefore contained in $\mathbf{R}^+[E]+\mathbf{R}^{+*}[F]$, a cone over which the intersection product is nonnegative. It follows from the discussion above that the extremal ray of $\overline{\mathrm{NE}}(X)$ other than $\mathbf{R}^+[F]$ is generated by a class ξ with $\xi^2 = 0$, which must be proportional to E. Hence we have

$$\mathrm{NE}(X) = \mathbf{R}^+[E] + \mathbf{R}^{+*}[F]$$

and this cone is not closed. In particular, the divisor E is not ample, although it has positive intersection with every curve on X.

5.4 The cone theorem for surfaces

Without proving it (although this can be done quite elementarily for surfaces; see [R]), we will examine the consequences of the cone theorem for surfaces. This theorem states the following.

Let X be a smooth projective surface. There exists a countable family of irreducible rational curves C_i such that $-3 \le (K_X \cdot C_i) < 0$ and

$$\overline{NE}(X) = \overline{NE}(X)_{K_X \ge 0} + \sum_i \mathbf{R}^+[C_i].$$

The rays $\mathbf{R}^+[C_i]$ are extremal and can be contracted. They can only accumulate on the hyperplane $K_X^\perp$.

We will now explain directly how the rays $\mathbf{R}^+[C_i]$ can be contracted. There are several cases.

- Either $(C_i^2) > 0$ for some i, in which case it follows from Proposition 5.9.d) that X has Picard number 1 and $-K_X$ is ample. The contraction of the ray $\mathbf{R}^+[C_i]$ is the map to a point. In fact, X is isomorphic to $\mathbf{P}^2_{\mathbf{k}}$.[2]
- Or $(C_i^2) = 0$ for some i, in which case it follows from Proposition 5.9.c) that X has the structure of a ruled surface $X \to B$ for which C_i is a fiber. The contraction of the ray $\mathbf{R}^+[C_i]$ is the map $X \to B$ (see Example 5.11).
- Or $(C_i^2) < 0$ for all i, in which case it follows from the adjunction formula that C_i is smooth and $(K_X \cdot C_i) = (C_i^2) = -1$.

In the last case, the contraction of the ray $\mathbf{R}^+[C_i]$ must contract only the curve C_i. Its existence is a famous and classical theorem of Castelnuovo.

[1]For the definition of stability and the construction of $\mathscr{E}$, see [H2], §I.10.
[2]This is proved in [Ko1], Theorem III.3.7.

Theorem 5.12 (Castelnuovo) *Let X be a smooth projective surface and let C be a smooth rational curve on X such that $(C^2) = -1$. There exist a smooth projective surface Y, a point $p \in Y$, and a morphism $\varepsilon : X \to Y$ such that $\varepsilon(C) = \{p\}$ and ε is isomorphic to the blow-up of Y at p.*

PROOF. We will only prove the existence of a morphism $\varepsilon : X \to Y$ that contracts C and refer the reader, for the delicate proof of the smoothness of Y, to [H1], Theorem V.5.7.

Let H be a very ample divisor on X. Upon replacing H with mH with $m \gg 0$, we may assume $H^1(X, H) = 0$. Let $k = (H \cdot C) > 0$ and set $D = H + kC$, so that $(D \cdot C) = 0$. We will prove that $\mathscr{O}_X(D)$ is generated by its global sections. Since $(D \cdot C) = 0$, the associated morphism to the projective space will contract C to a point, and no other curve.

Using the exact sequences

$$0 \to \mathscr{O}_X(H + (i-1)C) \to \mathscr{O}_X(H + iC) \to \mathscr{O}_C(k - i) \to 0,$$

we easily see by induction on $i \in \{0, \dots, k\}$ that $H^1(X, H + iC)$ vanishes. In particular, we get for $i = k$ a surjection

$$H^0(X, D) \to H^0(C, \mathscr{O}_C) \simeq \mathbf{k}.$$

As in the proof of Proposition 5.9.c), it follows that the sheaf $\mathscr{O}_X(D)$ is generated by its global sections hence defines a morphism $f : X \to \mathbf{P}^r_{\mathbf{k}}$ which contracts the curve C to a point p. Since H is very ample, f also induces an isomorphism between $X - C$ and $f(X) - \{p\}$. □

Exercise 5.13 Let X be a smooth projective surface and let C be a smooth rational curve on X such that $(C^2) < 0$. Show that there exist a (possibly singular) projective surface Y, a point $p \in Y$, and a morphism $\varepsilon : X \to Y$ such that $\varepsilon(C) = \{p\}$ and ε induces an isomorphism between $X - C$ and $Y - \{p\}$.

Exercise 5.14 Let C be a smooth curve in $\mathbf{P}^n_{\mathbf{k}}$ and let $X \subset \mathbf{P}^{n+1}_{\mathbf{k}}$ be the cone over C with vertex O. Let $\varepsilon : \widetilde{X} \to X$ be the blow-up of O and let E be the exceptional divisor. Show that:

a) the surface $\widetilde{X}$ is isomorphic to the ruled surface $\mathbf{P}(\mathscr{O}_C \oplus \mathscr{O}_C(1))$ (see §5.2);

b) the divisor E is the image of the section of $\mathbf{P}(\mathscr{O}_C \oplus \mathscr{O}_C(1)) \to C$ that corresponds to the quotient $\mathscr{O}_C \oplus \mathscr{O}_C(1) \to \mathscr{O}_C$;

c) compute (E^2) in terms of the degree of C in $\mathbf{P}^n_{\mathbf{k}}$ (use (5.4)).

What is the surface $\widetilde{X}$ obtained by starting from the *rational normal curve* $C \subset \mathbf{P}^n_{\mathbf{k}}$, i.e., the image of the morphism $\mathbf{P}^1_{\mathbf{k}} \to \mathbf{P}^n_{\mathbf{k}}$ corresponding to vector space of all sections of $\mathscr{O}_{\mathbf{P}^1_{\mathbf{k}}}(n)$?

Example 5.15 (Del Pezzo surfaces) A del Pezzo surface X is a smooth projective surface such that $-K_X$ is ample (the projective plane is an example; a smooth cubic hypersurface in $\mathbf{P}^3_{\mathbf{k}}$ is another example). The cone $\overline{\mathrm{NE}}(X) - \{0\}$ is contained in the half-space $N_1(X)_{K_X<0}$ (Theorem 4.10.a)). By the cone theorem stated at the beginning of this section, the set of extremal rays is discrete and compact, hence finite. Furthermore,

$$\overline{\mathrm{NE}}(X) = \mathrm{NE}(X) = \sum_{i=1}^{m} \mathbf{R}^+[C_i].$$

According to the discussion following the statement of the cone theorem, either X is isomorphic to $\mathbf{P}^2_{\mathbf{k}}$, or X is a ruled surface (one checks that the only possible cases are $\mathbf{F}_0 = \mathbf{P}^1_{\mathbf{k}} \times \mathbf{P}^1_{\mathbf{k}}$ and $\mathbf{F}_1$, which is $\mathbf{P}^2_{\mathbf{k}}$ blown-up at a point), or the C_i are all exceptional curves.

For example, when X is a smooth cubic surface,

$$\mathrm{NE}(X) = \sum_{i=1}^{27} \mathbf{R}^+[C_i] \subset \mathbf{R}^7,$$

where the C_i are the 27 lines on X.

Example 5.16 (A cone of curves with infinitely many negative extremal rays) Let $X \to \mathbf{P}^2_{\mathbf{k}}$ be the blow-up of the nine base-points of a general pencil of cubics, let $\pi : X \to \mathbf{P}^1_{\mathbf{k}}$ be the morphism given by the pencil of cubics. The exceptional divisors $E_0, \dots, E_8$ are sections of π. Smooth fibers of π are elliptic curves, hence become abelian groups by choosing E_0 as the origin; translations by elements of E_i then generate a subgroup G of $\mathrm{Aut}(X)$ which can be shown to be isomorphic to $\mathbf{Z}^8$.

For each $\sigma \in G$, the curve $E_\sigma = \sigma(E_0)$ is rational with self-intersection -1 and $(K_X \cdot E_\sigma) = -1$. It follows from Proposition 5.9.b) that $\overline{\mathrm{NE}}(X)$ has infinitely many extremal rays contained in the open half-space $N_1(X)_{K_X<0}$, which are *not* locally finite in a neighborhood of $K_X^\perp$, because $(K_X \cdot E_\sigma) = -1$ but $(E_\sigma)_{\sigma \in G}$ is unbounded since the set of classes of irreducible curves is discrete in $N_1(X)_{\mathbf{R}}$.

5.5 Rational maps between smooth surfaces

5.17. Domain of definition of a rational map. Let X and Y be integral schemes and let $\pi : X \dashrightarrow Y$ be a rational map. There exists a largest open subset $U \subset X$ over which π is defined. If X is normal and Y is proper, *$X - U$ has codimension at least 2 in X*. Indeed, if x is a point of codimension 1 in X, the ring $\mathscr{O}_{X,x}$ is an integrally closed noetherian local domain of dimension 1, hence is a discrete valuation ring; by the local valuative criterion for properness, the generic point $\mathrm{Spec}(K(X)) \to Y$ extends to $\mathrm{Spec}(\mathscr{O}_{X,x}) \to Y$.

In particular, a rational map from a smooth curve is actually a morphism (a fact that we have already used several times), and a rational map from a smooth surface is defined on the complement of a finite set.

Let X' be the closure in $X \times Y$ of the graph of $\pi|_U : U \to X$; we will call it the graph of π. The first projection $p : X' \to X$ is birational and U is the largest open subset over which p is an isomorphism.

If X is normal and Y is proper, p is proper and its fibers are connected by Zariski's Main Theorem ([H1], Corollary III.11.4). If a fiber $p^{-1}(x)$ is a single point, x has a neighborhood V in X such that the map $p^{-1}(V) \to V$ induced by p is finite; since it is birational and X is normal, it is an isomorphism by Zariski's Theorem. It follows that $X - U$ is exactly the set of points of X where p has positive-dimensional fibers (we recover the fact that $X - U$ has codimension at least 2 in X).

We now study rational maps from a smooth projective surface.

Theorem 5.18 (Elimination of indeterminacies) *Let $\pi : X \dashrightarrow Y$ be a rational map, where X is a smooth projective surface and Y is projective. There exists a birational morphism $\varepsilon : \widetilde{X} \to X$ which is a composition of blow-ups of points, such that $\pi \circ \varepsilon : \widetilde{X} \to Y$ is a* morphism.

This elementary theorem was vastly generalized by Hironaka to the case where X is any smooth projective variety over an algebraically closed field *of characteristic 0;* the morphism ε is then a composition of blow-ups of *smooth* subvarieties.

Corollary 5.19 *Under the hypotheses of the theorem, if Y contains no rational curves, π is a morphism.*

This corollary holds in all dimensions (see Corollary 8.24).

Proof. Let $\varepsilon : \widetilde{X} \to X$ be a *minimal* composition of blow-ups such that $\tilde{\pi} = \pi \circ \varepsilon : \widetilde{X} \to Y$ is a morphism. If ε is not an isomorphism, let $E \subset \widetilde{X}$ be the last exceptional curve. Then $\tilde{\pi}(E)$ must be a curve, and it must be rational, which contradicts the hypothesis. Hence ε is an isomorphism. □

Proof of the Theorem. We can replace Y with a projective space $\mathbf{P}^N_{\mathbf{k}}$, so that π can be written as

$$\pi(x) = (s_0(x), \dots, s_N(x)),$$

where $s_0, \dots, s_N$ are sections of the invertible sheaf $\pi^* \mathscr{O}_{\mathbf{P}^N_{\mathbf{k}}}(1)$ (see 2.18). Since $\mathscr{O}_{\mathbf{P}^N_{\mathbf{k}}}(1)$ is globally generated, so is $\pi^* \mathscr{O}_{\mathbf{P}^N_{\mathbf{k}}}(1)$ on the largest open subset $U \subset X$ where π is defined. In particular, we can find two effective divisors D and D' in the linear system $\pi^*|\mathscr{O}_{\mathbf{P}^N_{\mathbf{k}}}(1)|$ with no common component in U. Since, by 5.17, $X - U$ is just a finite set of points, D and D' have no common component, hence

$$(D^2) = (D \cdot D') \geq 0.$$

If π is an morphism, there is nothing to prove. Otherwise, let x be a point of X where $s_0, \dots, s_N$ all vanish and let $\varepsilon : \widetilde{X} \to X$ be the blow-up of this point, with exceptional curve E. The sections $s_0 \circ \varepsilon, \dots, s_N \circ \varepsilon \in H^0(\widetilde{X}, \varepsilon^* D)$ all vanish identically on E. Let $m > 0$ be the largest integer such that they all vanish there at order m. If $s_E \in H^0(\widetilde{X}, E)$ has divisor E, we can write $s_i \circ \varepsilon = \tilde{s}_i s_E^m$, where $\tilde{s}_0, \dots, \tilde{s}_N$ do no all vanish identically on E. These sections define $\tilde{\pi} := \pi \circ \varepsilon : \widetilde{X} \to \mathbf{P}^N_{\mathbf{k}}$ and $\tilde{\pi}^* \mathscr{O}_{\mathbf{P}^N_{\mathbf{k}}}(1)$ is $\mathscr{O}_{\widetilde{X}}(\widetilde{D})$, with $\widetilde{D} = \varepsilon^* D - mE$. We have $(\widetilde{D}^2) = (D^2) - m^2 < (D^2)$; since $(\widetilde{D}^2)$ must remain nonnegative for the same reason that (D^2) was, this process must stop after at most (D^2) steps. □

Theorem 5.20 (Factorization of birational morphisms) *Let X and Y be smooth projective surfaces. Any birational morphism $\pi : X \to Y$ is a composition of blow-ups of points and an isomorphism.*

Corollary 5.21 *Let X and Y be smooth projective surfaces. Any birational map $\pi : X \dashrightarrow Y$ can be factored as the inverse of a composition of blow-ups of points, followed by a composition of blow-ups of points, and an isomorphism.*

Proof. By Theorem 5.18, there is a composition of blow-ups $\varepsilon : \widetilde{X} \to X$ such that $\pi \circ \varepsilon$ is a (birational) morphism, to which Theorem 5.20 applies. □

The corollary was generalized in higher dimensions in 2002 by Abramovich, Karu, Matsuki, Wlodarczyk, and Morelli: they prove that any birational map between smooth projective varieties over an algebraically closed field *of characteristic 0* can be factored as a composition of blow-ups of smooth subvarieties or inverses of such blow-ups, and an isomorphism (*weak factorization*).

It is conjectured that a birational morphism between smooth projective varieties can be factored as the inverse of a composition of blow-ups of smooth subvarieties, followed by a composition of blow-ups of smooth subvarieties and an isomorphism (*strong factorization*).

However, the analog of Theorem 5.20 is in general false in dimensions ≥ 3: a birational morphism between smooth projective varieties cannot always be factored as a composition of blow-ups of smooth subvarieties (recall that any birational projective morphism is a blow-up; but this is mostly useless since arbitrary blow-ups are untractable).

Proof of the Theorem. If π is an isomorphism, there is nothing to prove. Otherwise, let y be a point of Y where π^{-1} is not defined and let $\varepsilon : \widetilde{Y} \to Y$ be the blow-up of y, with exceptional curve E. Let $f = \varepsilon^{-1} \circ \pi : X \dashrightarrow \widetilde{Y}$ and $g = f^{-1} : \widetilde{Y} \dashrightarrow X$.

We want to show that f is a morphism. If f is not defined at a point x of X, there is a curve in $\widetilde{Y}$ that g maps to x. This curve must be E. Let $\tilde{y}$ be a point of E where g is defined. Since π^{-1} is not defined at y and $\pi(x) = y$, there is a curve $C \subset X$ such that $x \in C$ and $\pi(C) = \{y\}$.

We consider the local inclusions of local rings

$$\mathscr{O}_{Y,y} \stackrel{\pi^*}{\hookrightarrow} \mathscr{O}_{X,x} \stackrel{g^*}{\hookrightarrow} \mathscr{O}_{\widetilde{Y},\tilde{y}} \subset K(X).$$

We may choose a system of parameters (t, v) on $\widetilde{Y}$ at $\tilde{y}$ (i.e., elements of $\mathfrak{m}_{\widetilde{Y},\tilde{y}}$ whose classes in $\mathfrak{m}_{\widetilde{Y},\tilde{y}} / \mathfrak{m}^2_{\widetilde{Y},\tilde{y}}$ generate this $\mathbf{k}$-vector space) such that E is defined locally by v and (u, v) is a system of parameters on Y at y, with $u = tv$. Let $w \in \mathfrak{m}_{X,x}$ be a local defining equation for C at x.

Since $\pi(C) = y$, we have $w \mid u$ and $w \mid v$, so we can write $u = wa$ and $v = wb$, with $a, b \in \mathscr{O}_{X,x}$. Since $v \notin \mathfrak{m}^2_{\widetilde{Y},\tilde{y}}$, we have $b \notin \mathfrak{m}_{X,x}$ hence b is invertible and $t = u/v = a/b \in \mathscr{O}_{X,x}$. Since $t \in \mathfrak{m}_{\widetilde{Y},\tilde{y}}$, we have $t \in \mathfrak{m}_{X,x}$. On the other hand, since $g(E) = x$, any element of $g^* \mathfrak{m}_{X,x}$ must be divisible in $\mathscr{O}_{\widetilde{Y},\tilde{y}}$ by the equation v of E. This implies $v \mid t$, which is absurd since (t, v) is a system of parameters.

Each time π^{-1} is not defined at a point of the image, we can therefore factor π through the blow-up of that point. But for each factorization of π as $X \xrightarrow{f'} Y' \to Y$, we must have an injection (see §4.6)

$$f'^* : N^1(Y')_{\mathbf{R}} \hookrightarrow N^1(X)_{\mathbf{R}}.$$

In other words, the Picard numbers of the Y' must remain bounded (by the (finite) Picard number of X). Since these Picard numbers increase by 1 at each blow-up, the process must stop after finitely many blow-ups of Y, in which case we end up with an isomorphism. □

5.6 The minimal model program for surfaces

Let X be a smooth projective surface. It follows from Castelnuovo's criterion (Theorem 5.12) that by contracting exceptional curves on X one arrives eventually (the process must stop because the Picard number decreases by 1 at each step by Exercise 4.8.1)) at a surface X_0 with no exceptional curves. Such a surface is called a *minimal surface.* According to the cone theorem (§5.4),

- either K_{X_0} is nef,
- or there exists a rational curve C_i as in the theorem. This curve cannot be exceptional, hence X_0 is either $\mathbf{P}^2_{\mathbf{k}}$ or a ruled surface, and the original surface X has a morphism to a smooth curve whose generic fiber is $\mathbf{P}^1_{\mathbf{k}}$. Starting from a given surface X of this type, there are several possible different end products X_0 (see Exercise 5.7.1)b)).

In particular, *if X is not birational to a ruled surface, it has a minimal model X_0 with K_{X_0} nef.* We prove that this model is unique. In dimension at least 3, the proposition below is not true anymore: there are smooth varieties with nef canonical classes which are birationally isomorphic but not isomorphic.

Proposition 5.22 *Let X and Y be smooth projective surfaces and let $\pi : X \dashrightarrow Y$ be a birational map. If K_Y is nef, π is a morphism. If both K_X and K_Y are nef, π is an isomorphism.*

Proof. Let $f : Z \to Y$ be the blow-up of a point and let $C \subset Z$ be an integral curve other than the exceptional curve E, with image $f(C) \subset Y$. We have $f^*f(C) \underset{\text{lin}}{\equiv} C+mE$ for some $m \geq 0$ and $K_Z = f^*K_Y+E$. Therefore,

$$(K_Z \cdot C) = (K_Z \cdot C) + m \geq (K_Z \cdot C) \geq 0.$$

If now $f : Z \to Y$ is any birational morphism, it decomposes by Theorem 5.20 as a composition of blow-ups, and we obtain again, by induction on the number of blow-ups, $(K_Z \cdot C) \geq 0$ for any integral curve $C \subset Z$ not contracted by f.

There is by Theorem 5.18 a (minimal) composition of blow-ups $\varepsilon : \tilde{X} \to X$ such that $\tilde{\pi} = \pi \circ \varepsilon$ is a morphism, itself a composition of blow-ups by Theorem 5.20. If ε is not an isomorphism, its last exceptional curve E is not contracted by $\tilde{\pi}$ hence must satisfy, by what we just saw, $(K_{\tilde{X}} \cdot E) \geq 0$. But this is absurd since this integer is -1. hence π is a morphism. □

5.7 Exercises

1) Let $\pi : X \to B$ be a ruled surface.

a) Let $\tilde{X} \to X$ be the blow-up a point x. Describe the fiber of the composition $\tilde{X} \to X \to B$ over $\pi(x)$.

b) Show that the strict transform in $\tilde{X}$ of the fiber $\pi^{-1}(\pi(x))$ can be contracted to give another ruled surface $X(x) \to B$.

c) Let $\mathbf{F}_n$ be a Hirzebruch surface (with $n \in \mathbf{N}$; see Example 5.7). Describe the surface $\mathbf{F}_n(x)$ (*Hint:* distinguish two cases according to whether x is on the curve C_n of Example 5.7).

2) Let X be a projective surface and let D and H be Cartier divisors on X.

a) Assume H is ample, $(D \cdot H) = 0$, and $D \not\equiv_{\text{num}} 0$. Prove $(D^2) < 0$.

b) Assume $(H^2) > 0$. Prove the inequality (Hodge Index Theorem)

$$(D \cdot H)^2 \geq (D^2)(H^2).$$

When is there equality?

c) Assume $(H^2) > 0$. If $D_1, \ldots, D_r$ are divisors on X, setting $D_0 = H$, prove

$$(-1)^r \det((D_i \cdot D_j))_{0 \leq i,j \leq r} \geq 0.$$

3) Let $D_1, \ldots, D_n$ be nef Cartier divisors on a projective variety X of dimension n. Prove

$$(D_1 \cdot \ldots \cdot D_n)^n \geq (D_1^n) \cdot \ldots \cdot (D_n^n).$$

(*Hint:* first do the case when the divisors are ample by induction on n, using Exercise 2)b) when $n = 2$).

4) Let $\mathbf{K}$ be the function field of a curve over an algebraically closed field, and let X be a subscheme of $\mathbf{P}^N_{\mathbf{K}}$ defined by homogeneous equations $f_1, \ldots, f_r$ of respective degrees $d_1, \ldots, d_r$. If $d_1 + \cdots + d_r \leq N$, show that X has a $\mathbf{K}$-point (*Hint:* proceed as in the proof of Theorem 5.2).

5) (Weil) Let C be a smooth projective curve over a finite field $\mathbf{F}_q$, and let $F : C \to C$ be the Frobenius morphism obtained by taking qth powers (it is indeed an endomorphism of C because C is defined over $\mathbf{F}_q$). Let $X = C \times C$, let $\Delta \subset X$ be the diagonal (see Example 5.1), and let $\Gamma \subset X$ be the graph of F.

a) Compute (Γ^2) (*Hint:* proceed as in Example 5.1).

b) Let x_1 and x_2 be the respective classes of $\{\star\} \times C$ and $C \times \{\star\}$. For any divisor D on X, prove

$$(D^2) \leq 2(D \cdot x_1)(D \cdot x_2)$$

(*Hint:* apply Exercise 2)c) above).

c) Set $N = \Gamma \cdot \Delta$. Prove

$$|N - q - 1| \leq 2g\sqrt{q}$$

(*Hint:* apply b) to $r\Gamma + s\Delta$, for all $r, s \in \mathbf{Z}$). What does the number N count?

6) Show that the group of automorphisms of a smooth curve C of genus $g \geq 2$ is finite (*Hint:* consider the graph Γ of an automorphism of C in the surface $X = C \times C$, show that $(K_X \cdot \Gamma)$ is bounded, and use Example 5.1 and Theorem 4.10.b)).

Chapter 6

Parametrizing morphisms

We concentrate in this chapter on basically one object, whose construction dates back to Grothendieck in 1962: the space parametrizing curves on a given variety, or more precisely morphisms from a fixed projective curve C to a fixed smooth quasi-projective variety. Mori's techniques, which will be discussed in the next chapter, make systematic use of these spaces in a rather exotic way.

We will not reproduce Grothendieck's construction, since it is very nicely explained in [G2] and only the end product will be important for us. However, we will explain in some detail in what sense these spaces are *parameter spaces,* and work out their local structure. Roughly speaking, as in many deformation problems, the tangent space to such a parameter space at a point is $H^0(C, \mathscr{F})$, where $\mathscr{F}$ is some locally free sheaf on C, first-order deformations are obstructed by elements of $H^1(C, \mathscr{F})$, and the dimension of the parameter space is therefore bounded from below by the difference $h^0(C, \mathscr{F}) - h^1(C, \mathscr{F})$. The crucial point is that since C has dimension 1, this difference is the Euler characteristic of $\mathscr{F}$, which can be computed from numerical data by the Riemann-Roch theorem.

6.1 Parametrizing rational curves

Let $\mathbf{k}$ be a field. Any $\mathbf{k}$-morphism f from $\mathbf{P}^1_{\mathbf{k}}$ to $\mathbf{P}^N_{\mathbf{k}}$ can be written as

$$f(u, v) = (F_0(u, v), \dots, F_N(u, v)), \tag{6.1}$$

where $F_0, \dots, F_N$ are homogeneous polynomials in two variables, of the same degree d, with no nonconstant common factor in $\mathbf{k}[U, V]$ (or, equivalently, with no nonconstant common factor in $\bar{\mathbf{k}}[U, V]$, where $\bar{\mathbf{k}}$ is an algebraic closure of $\mathbf{k}$).

We are going to show that there exist universal integral polynomials in the coefficients of $F_0, \dots, F_N$ which vanish if and only if they have a nonconstant common factor in $\bar{\mathbf{k}}[U, V]$, i.e., a nontrivial common zero in $\mathbf{P}^1_{\bar{\mathbf{k}}}$. By the Nullstellensatz, the opposite holds if and only if the ideal generated by $F_0, \dots, F_N$ in $\bar{\mathbf{k}}[U, V]$ contains some power of the maximal ideal (U, V). This in turn means that for some m, the map

$$\begin{array}{ccc} (\bar{\mathbf{k}}[U, V]_{m-d})^{N+1} & \longrightarrow & \bar{\mathbf{k}}[U, V]_m \\ (G_0, \dots, G_N) & \longmapsto & \sum_{j=0}^N F_j G_j \end{array}$$

is surjective, hence of rank $m + 1$ (here $\mathbf{k}[U, V]_m$ is the vector space of homogeneous polynomials of degree m). This map being linear and defined over $\mathbf{k}$, we conclude that $F_0, \dots, F_N$ have a nonconstant common factor in $\mathbf{k}[U, V]$ if and only if, for all m, all $(m + 1)$-minors of some universal matrix whose entries are linear integral combinations of the coefficients of the F_i vanish. This defines a Zariski closed subset of the projective space $\mathbf{P}((\mathrm{Sym}^d\, \mathbf{k}^2)^{N+1})$, *defined over* $\mathbf{Z}$.

Therefore, morphisms of degree d from $\mathbf{P}^1_{\mathbf{k}}$ to $\mathbf{P}^N_{\mathbf{k}}$ are parametrized by a Zariski open set of the projective space $\mathbf{P}((\mathrm{Sym}^d\, \mathbf{k}^2)^{N+1})$; we denote this quasi-projective variety $\mathrm{Mor}_d(\mathbf{P}^1_{\mathbf{k}}, \mathbf{P}^N_{\mathbf{k}})$. Note that these morphisms fit together into a *universal morphism*

$$\begin{array}{cccc} f^{\mathrm{univ}}: & \mathbf{P}^1_{\mathbf{k}} \times \mathrm{Mor}_d(\mathbf{P}^1_{\mathbf{k}}, \mathbf{P}^N_{\mathbf{k}}) & \longrightarrow & \mathbf{P}^N_{\mathbf{k}} \\ & \big((u, v), f\big) & \longmapsto & \big(F_0(u, v), \dots, F_N(u, v)\big). \end{array}$$

Example 6.1 In the case $d = 1$, we can write $F_i(u,v) = a_i u + b_i v$, with $(a_0, \dots, a_N, b_0, \dots, b_N) \in \mathbf{P}^{2N+1}_{\mathbf{k}}$. The condition that $F_0, \dots, F_N$ have no common zeroes is equivalent to

$$\operatorname{rank}\begin{pmatrix} a_0 & \cdots & a_N \\ b_0 & \cdots & b_N \end{pmatrix} = 2.$$

Its complement Z in $\mathbf{P}^{2N+1}_{\mathbf{k}}$ is defined by the vanishing of all its 2×2-minors: $\begin{vmatrix} a_i & a_j \\ b_i & b_j \end{vmatrix} = 0$. The universal morphism is

$$\begin{array}{rccc} f^{\text{univ}}: & \mathbf{P}^1_{\mathbf{k}} \times (\mathbf{P}^{2N+1}_{\mathbf{k}} - Z) & \longrightarrow & \mathbf{P}^N_{\mathbf{k}} \\ & \big((u,v),(a_0,\dots,a_N,b_0,\dots,b_N)\big) & \longmapsto & \big(a_0u+b_0v,\dots,a_Nu+b_Nv\big). \end{array}$$

Finally, morphisms from $\mathbf{P}^1_{\mathbf{k}}$ to $\mathbf{P}^N_{\mathbf{k}}$ are parametrized by the disjoint union

$$\operatorname{Mor}(\mathbf{P}^1_{\mathbf{k}}, \mathbf{P}^N_{\mathbf{k}}) = \bigsqcup_{d \ge 0} \operatorname{Mor}_d(\mathbf{P}^1_{\mathbf{k}}, \mathbf{P}^N_{\mathbf{k}})$$

of quasi-projective schemes.

Let now X be a (closed) subscheme of $\mathbf{P}^N_{\mathbf{k}}$ defined by homogeneous equations $G_1, \dots, G_m$. Morphisms of degree d from $\mathbf{P}^1_{\mathbf{k}}$ to X are parametrized by the subscheme $\operatorname{Mor}_d(\mathbf{P}^1_{\mathbf{k}}, X)$ of $\operatorname{Mor}_d(\mathbf{P}^1_{\mathbf{k}}, \mathbf{P}^N_{\mathbf{k}})$ defined by the equations

$$G_j(F_0, \dots, F_N) = 0 \qquad \text{for all } \ j \in \{1, \dots, m\}.$$

Again, morphisms from $\mathbf{P}^1_{\mathbf{k}}$ to X are parametrized by the disjoint union

$$\operatorname{Mor}(\mathbf{P}^1_{\mathbf{k}}, X) = \bigsqcup_{d \ge 0} \operatorname{Mor}_d(\mathbf{P}^1_{\mathbf{k}}, X)$$

of quasi-projective schemes. The same conclusion holds for any quasi-projective variety X: embed X into some projective variety $\overline{X}$; there is a universal morphism

$$f^{\text{univ}}: \mathbf{P}^1_{\mathbf{k}} \times \operatorname{Mor}(\mathbf{P}^1_{\mathbf{k}}, \overline{X}) \longrightarrow \overline{X}$$

and $\operatorname{Mor}(\mathbf{P}^1_{\mathbf{k}}, X)$ is the complement in $\operatorname{Mor}(\mathbf{P}^1_{\mathbf{k}}, \overline{X})$ of the image by the (proper) second projection of the closed subscheme $(f^{\text{univ}})^{-1}(\overline{X} - X)$.

If now X can be defined by homogeneous equations $G_1, \dots, G_m$ with coefficients in a subring R of $\mathbf{k}$, the scheme $\operatorname{Mor}_d(\mathbf{P}^1_{\mathbf{k}}, X)$ has the same property. If $\mathfrak{m}$ is a maximal ideal of R, one may consider the reduction $X_{\mathfrak{m}}$ of X modulo $\mathfrak{m}$: this is the subscheme of $\mathbf{P}^N_{R/\mathfrak{m}}$ defined by the reductions of the G_j modulo $\mathfrak{m}$. Because the equations defining the complement of $\operatorname{Mor}_d(\mathbf{P}^1_{\mathbf{k}}, \mathbf{P}^N_{\mathbf{k}})$ in $\mathbf{P}((\operatorname{Sym}^d \mathbf{k}^2)^{N+1})$ are defined over $\mathbf{Z}$ and the same for all fields, $\operatorname{Mor}_d(\mathbf{P}^1_{\mathbf{k}}, X_{\mathfrak{m}})$ is the reduction of the R-scheme $\operatorname{Mor}_d(\mathbf{P}^1_{\mathbf{k}}, X)$ modulo $\mathfrak{m}$. In fancy terms, one may express this as follows: if $\mathscr{X}$ is a scheme over $\operatorname{Spec} R$, the R-morphisms $\mathbf{P}^1_R \to \mathscr{X}$ are parametrized by the R-points of a locally noetherian scheme

$$\operatorname{Mor}(\mathbf{P}^1_R, \mathscr{X}) \to \operatorname{Spec} R$$

and the fiber of a closed point $\mathfrak{m}$ is the space $\operatorname{Mor}(\mathbf{P}^1_{\mathbf{k}}, \mathscr{X}_{\mathfrak{m}})$.

6.2 Parametrizing morphisms

6.2. The space $\operatorname{Mor}(Y, X)$**.** Grothendieck vastly generalized the preceding construction: if X and Y are varieties over a field $\mathbf{k}$, with X quasi-projective and Y projective, he shows ([G2], 4.c) that $\mathbf{k}$-morphisms from Y to X are parametrized by a scheme $\operatorname{Mor}(Y, X)$ locally of finite type. As we saw in the case $Y = \mathbf{P}^1_{\mathbf{k}}$ and $X = \mathbf{P}^N_{\mathbf{k}}$, this scheme will in general have countably many components. One way to remedy that is to fix an ample divisor H on X and a polynomial P with rational coefficients: the subscheme $\operatorname{Mor}_P(Y, X)$ of $\operatorname{Mor}(Y, X)$ which parametrizes morphisms $f: Y \to X$ with fixed *Hilbert polynomial*

$$P(m) = \chi(Y, mf^*H)$$

is now quasi-projective over $\mathbf{k}$, and $\mathrm{Mor}(Y,X)$ is the disjoint (countable) union of the $\mathrm{Mor}_P(Y,X)$, for all polynomials P. Note that when Y is a curve, fixing the Hilbert polynomial amounts to fixing the degree of the 1-cycle f_*Y for the embedding of X defined by some multiple of H.

The fact that Y is projective is essential in this construction: the space $\mathrm{Mor}(\mathbf{A}^1_{\mathbf{k}},\mathbf{A}^N_{\mathbf{k}})$ is *not* a disjoint union of quasi-projective schemes.

Let us make more precise this notion of *parameter space*. We ask as above that there be a *universal morphism* (also called *evaluation map*)

$$f^{\mathrm{univ}}: Y\times \mathrm{Mor}(Y,X)\to X$$

such that for any $\mathbf{k}$-scheme T, the correspondance between

- morphisms $\varphi: T\to \mathrm{Mor}(Y,X)$ and
- morphisms $f: Y\times T\to X$

obtained by sending φ to

$$f(y,t)=f^{\mathrm{univ}}(y,\varphi(t))$$

is one-to-one.

In particular, if $L\supset \mathbf{k}$ is a field extension, L-points of $\mathrm{Mor}(Y,X)$ correspond to L-morphisms $Y_L\to X_L$ (where $X_L=X\times_{\mathrm{Spec}\,\mathbf{k}}\mathrm{Spec}\,L$ and similarly for Y_L).

Examples 6.3 1) The scheme $\mathrm{Mor}(\mathrm{Spec}\,\mathbf{k},X)$ is just X, the universal morphism being the second projection

$$f^{\mathrm{univ}}:\quad \mathrm{Spec}\,\mathbf{k}\times X\quad\longrightarrow\quad X.$$

2) When $Y=\mathrm{Spec}\,\mathbf{k}[\varepsilon]/(\varepsilon^2)$, a morphism $Y\to X$ corresponds to the data of a $\mathbf{k}$-point x of X and an element of the Zariski tangent space $T_{X,x}=(\mathfrak{m}_{X,x}/\mathfrak{m}^2_{X,x})^*$.

6.4. The tangent space to $\mathrm{Mor}(Y,X)$**.** We will use the universal property to determine the Zariski tangent space to $\mathrm{Mor}(Y,X)$ at a $\mathbf{k}$-point $[f]$. This vector space parametrizes by definition morphisms from $\mathrm{Spec}\,\mathbf{k}[\varepsilon]/(\varepsilon^2)$ to $\mathrm{Mor}(Y,X)$ with image $[f]$ ([H1], Ex. II.2.8), hence extensions of f to morphisms

$$f_\varepsilon: Y\times \mathrm{Spec}\,\mathbf{k}[\varepsilon]/(\varepsilon^2)\to X$$

which should be thought of as first-order infinitesimal deformations of f.

Proposition 6.5 *Let X and Y be varieties over a field $\mathbf{k}$, with X quasi-projective and Y projective, let $f: Y\to X$ be a $\mathbf{k}$-morphism, and let $[f]$ be the corresponding $\mathbf{k}$-point of $\mathrm{Mor}(Y,X)$. One has*

$$T_{\mathrm{Mor}(Y,X),[f]}\simeq H^0(Y,\mathscr{H}om(f^*\Omega_X,\mathscr{O}_Y)).$$

PROOF. Assume first that Y and X are affine and write $Y=\mathrm{Spec}(B)$ and $X=\mathrm{Spec}(A)$ (where A and B are finitely generated $\mathbf{k}$-algebras). Let $f^\sharp: A\to B$ be the morphism corresponding to f, making B into an A-algebra; we are looking for $\mathbf{k}$-algebra homomorphisms $f^\sharp_\varepsilon: A\to B[\varepsilon]$ of the type

$$\forall a\in A\quad f^\sharp_\varepsilon(a)=f(a)+\varepsilon g(a).$$

The equality $f^\sharp_\varepsilon(aa')=f^\sharp_\varepsilon(a)f^\sharp_\varepsilon(a')$ is equivalent to

$$\forall a,a'\in A\quad g(aa')=f^\sharp(a)g(a')+f^\sharp(a')g(a).$$

In other words, $g: A\to B$ must be a $\mathbf{k}$-derivation of the A-module B, hence must factor as $g: A\to\Omega_A\to B$ ([H1], §II.8). Such extensions are therefore parametrized by $\mathrm{Hom}_A(\Omega_A,B)=\mathrm{Hom}_B(\Omega_A\otimes_A B,B)$.

In general, cover X by affine open subsets $U_i = \operatorname{Spec}(A_i)$ and Y by affine open subsets $V_i = \operatorname{Spec}(B_i)$ such that $f(V_i)$ is contained in U_i. First-order extensions of $f|_{V_i} : V_i \to U_i$ are parametrized by

$$g_i \in \operatorname{Hom}_{B_i}(\Omega_{A_i} \otimes_{A_i} B_i, B_i) = H^0(V_i, \mathscr{H}om(f^*\Omega_X, \mathscr{O}_Y)).$$

To glue these, we need the compatibility condition

$$g_i|_{V_i \cap V_j} = g_j|_{V_i \cap V_j},$$

which is exactly saying that the g_i define a global section on Y. □

In particular, when X is smooth along the image of f,

$$T_{\operatorname{Mor}(Y,X),[f]} \simeq H^0(Y, f^*T_X).$$

Example 6.6 When Y is smooth, the proposition proves that $H^0(Y, T_Y)$ is the tangent space at the identity to the group of automorphisms of Y. The image of the canonical morphism $H^0(Y, T_Y) \to H^0(Y, f^*T_X)$ corresponds to the deformations of f by reparametrizations.

6.7. The local structure of $\operatorname{Mor}(Y, X)$**.** We prove the result mentioned in the introduction of this chapter. Its main use will be to provide a lower bound for the dimension of $\operatorname{Mor}(Y, X)$ at a point $[f]$, thereby allowing us in certain situations to produce many deformations of f. This lower bound is very accessible, via the Riemann-Roch theorem, when Y is a curve (see 6.12).

Theorem 6.8 *Let X and Y be projective varieties over a field $\mathbf{k}$ and let $f : Y \to X$ be a $\mathbf{k}$-morphism such that X is smooth along $f(Y)$. Locally around $[f]$, the scheme $\operatorname{Mor}(Y, X)$ can be defined by $h^1(Y, f^*T_X)$ equations in a smooth scheme of dimension $h^0(Y, f^*T_X)$. In particular, any (geometric) irreducible component of $\operatorname{Mor}(Y, X)$ through $[f]$ has dimension at least*

$$h^0(Y, f^*T_X) - h^1(Y, f^*T_X).$$

In particular, under the hypotheses of the theorem, a sufficient condition for $\operatorname{Mor}(Y, X)$ to be smooth at $[f]$ is $H^1(Y, f^*T_X) = 0$. We will give in 6.13 an example that shows that this condition is not necessary.

Proof. Locally around the $\mathbf{k}$-point $[f]$, the $\mathbf{k}$-scheme $\operatorname{Mor}(Y, X)$ can be defined by certain polynomial equations $P_1, \dots, P_m$ in an affine space $\mathbf{A}^n_{\mathbf{k}}$. The rank r of the corresponding Jacobian matrix $((\partial P_i/\partial x_j)([f]))$ is the codimension of the Zariski tangent space $T_{\operatorname{Mor}(Y,X),[f]}$ in $\mathbf{k}^n$. The subvariety V of $\mathbf{A}^n_{\mathbf{k}}$ defined by r equations among the P_i for which the corresponding rows have rank r is smooth at $[f]$ with the same Zariski tangent space as $\operatorname{Mor}(Y, X)$.

Letting $h^i = h^i(Y, f^*T_X)$, we are going to show that $\operatorname{Mor}(Y, X)$ can be locally around $[f]$ defined by h^1 equations inside the smooth h^0-dimensional variety V. For that, it is enough to show that in the regular local $\mathbf{k}$-algebra $R = \mathscr{O}_{V,[f]}$, the ideal I of functions vanishing on $\operatorname{Mor}(Y, X)$ can be generated by h^1 elements. Note that since the Zariski tangent spaces are the same, I is contained in the square of the maximal ideal $\mathfrak{m}$ of R. Finally, by Nakayama's lemma ([M], Theorem 2.3), it is enough to show that the $\mathbf{k}$-vector space $I/\mathfrak{m}I$ has dimension at most h^1.

The canonical morphism $\operatorname{Spec}(R/I) \to \operatorname{Mor}(Y, X)$ corresponds to an extension $f_{R/I} : Y \times \operatorname{Spec}(R/I) \to X$ of f. Since $I^2 \subset \mathfrak{m}I$, the obstruction to extending it to a morphism $f_{R/\mathfrak{m}I} : Y \times \operatorname{Spec}(R/\mathfrak{m}I) \to X$ lies by Lemma 6.9 below (applied to the ideal $I/\mathfrak{m}I$ in the $\mathbf{k}$-algebra $R/\mathfrak{m}I$) in

$$H^1(Y, f^*T_X) \otimes_{\mathbf{k}} (I/\mathfrak{m}I).$$

Write this obstruction as

$$\sum_{i=1}^{h^1} a_i \otimes \bar{b}_i,$$

where $(a_1, \dots, a_{h^1})$ is a basis for $H^1(Y, f^*T_X)$ and $b_1, \dots, b_{h^1}$ are in I. The obstruction vanishes modulo the ideal $(b_1, \dots, b_{h^1})$, which means that the morphism $\operatorname{Spec}(R/I) \to \operatorname{Mor}(Y, X)$ lifts to a morphism

$\mathrm{Spec}(R/I') \to \mathrm{Mor}(Y, X)$, where $I' = \mathfrak{m}I + (b_1, \dots, b_{h^1})$. The image of this lift lies in $\mathrm{Spec}(R) \cap \mathrm{Mor}(Y, X)$, which is $\mathrm{Spec}(R/I)$. This means that the identity $R/I \to R/I$ factors as

$$R/I \to R/I' \xrightarrow{\pi} R/I,$$

where π is the canonical projection. By Lemma 6.10 below (applied to the ideal I/I' in the $\mathbf{k}$-algebra R/I'), since $I \subset \mathfrak{m}^2$, we obtain

$$I = I' = \mathfrak{m}I + (b_1, \dots, b_{h^1}),$$

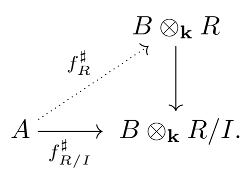
which means that $I/\mathfrak{m}I$ is generated by the classes of $b_1, \dots, b_{h^1}$. □

We now prove the two lemmas used in the proof above.

Lemma 6.9 *Let R be a noetherian local $\mathbf{k}$-algebra with maximal ideal $\mathfrak{m}$ and residue field $\mathbf{k}$ and let I be an ideal contained in $\mathfrak{m}$ such that $\mathfrak{m}I = 0$. Let $f : Y \to X$ be a $\mathbf{k}$-morphism and let $f_{R/I} : Y \times \mathrm{Spec}(R/I) \to X$ be an extension of f. Assume X is smooth along the image of f. The obstruction to extending $f_{R/I}$ to a morphism $f_R : Y \times \mathrm{Spec}(R) \to X$ lies in*

$$H^1(Y, f^*T_X) \otimes_{\mathbf{k}} I.$$

Proof. In the case where Y and X are affine, and with the notation of the proof of Proposition 6.5, we are looking for $\mathbf{k}$-algebra liftings $f_R^\sharp$ fitting into the diagram



Because $X = \mathrm{Spec}(A)$ is smooth along the image of f and $I^2 = 0$, such a lifting exists,[1] and two liftings differ by a $\mathbf{k}$-derivation of A into $B \otimes_{\mathbf{k}} I$,[2] that is by an element of

$$\begin{aligned}
\mathrm{Hom}_A(\Omega_A, B \otimes_{\mathbf{k}} I) &\simeq \mathrm{Hom}_A(\Omega_A, B \otimes_{\mathbf{k}} I)\\
&\simeq \mathrm{Hom}_B(B \otimes_{\mathbf{k}} \Omega_A, B \otimes_{\mathbf{k}} I)\\
&\simeq H^0(Y, \mathscr{H}om(f^*\Omega_X, \mathscr{O}_Y)) \otimes_{\mathbf{k}} I\\
&\simeq H^0(Y, f^*T_X) \otimes_{\mathbf{k}} I.
\end{aligned}$$

To pass to the global case, one needs to patch up various local extensions to get a global one. There is an obstruction to doing that: on each intersection $V_i \cap V_j$, two extensions differ by an element of $H^0(V_i \cap V_j, f^*T_X) \otimes_{\mathbf{k}} I$; these elements define a 1-cocycle, hence an element in $H^1(Y, f^*T_X) \otimes_{\mathbf{k}} I$ whose vanishing is necessary and sufficient for a global extension to exist.[3] □

Lemma 6.10 *Let A be a noetherian local ring with maximal ideal $\mathfrak{m}$ and let J be an ideal in A contained in $\mathfrak{m}^2$. If the canonical projection $\pi : A \to A/J$ has a section, $J = 0$.*

Proof. Let σ be a section of π: if a and b are in A, we can write $\sigma \circ \pi(a) = a + a'$ and $\sigma \circ \pi(b) = b + b'$, where a' and b' are in I. If a and b are in $\mathfrak{m}$, we have

$$(\sigma \circ \pi)(ab) = (\sigma \circ \pi)(a)\ (\sigma \circ \pi)(b) = (a + a')(b + b') \in ab + \mathfrak{m}J.$$

[1] In [Bo], this is the definition of *formally smooth* $\mathbf{k}$-algebras (§7, n°2, déf. 1). Then it is shown that for local noetherian $\mathbf{k}$-algebras with residue field $\mathbf{k}$, this is equivalent to absolute regularity (§7, n°5, cor. 1)

[2] This is very simple and has nothing to do with smoothness. For simplicity, change the notation and assume that we have R-algebras A and B, an ideal I of B with $I^2 = 0$, and a morphism $f : A \to B/I$ of R-algebras. Since $I^2 = 0$, the ideal I is a B/I-module, hence also an A-module via f. Let $g, g' : A \to B$ be two liftings of f. For any a and a' in A, we have

$$(g - g')(aa') = g(a')(g(a) - g'(a)) + g'(a)(g(a') - g'(a')) = a' \cdot (g - g')(a) + a \cdot (g - g')(a').$$

hence $g - g'$ is indeed an R-derivation of A into I.

In our case, since $\mathfrak{m}I = 0$, the structure of A-module on $B \otimes_{\mathbf{k}} I$ just come from the structure of A-module on B.

[3] On a separated noetherian scheme, the cohomology of a coherent sheaf is isomorphic to its Čech cohomology relative to any open affine covering ([H1], Theorem III.4.5).

Since J is contained in $\mathfrak{m}^2$, we get, for any x in J,

$$0 = \sigma \circ \pi(x) \in x + \mathfrak{m}J,$$

hence $J \subset \mathfrak{m}J$. Nakayama's lemma ([M], Theorem 2.2) implies $J = 0$. □

6.3 Parametrizing morphisms with fixed points

6.11. Morphisms with fixed points. We will need a slightly more general situation: fix a finite subset $B = \{y_1, \dots, y_r\}$ of Y and points $x_1, \dots, x_r$ of X; we want to study morphisms $f : Y \to X$ which map each y_i to x_i. These morphisms can be parametrized by the fiber over $(x_1, \dots, x_r)$ of the map

$$\begin{aligned} \rho : \operatorname{Mor}(Y, X) &\longrightarrow X^r \\ [f] &\longmapsto (f(y_1), \dots, f(y_r)). \end{aligned}$$

We denote this space by $\operatorname{Mor}(Y, X; y_i \mapsto x_i)$. At a point $[f]$ such that X is smooth along $f(Y)$, the tangent map to ρ is the evaluation

$$H^0(Y, f^*T_X) \to \bigoplus_{i=1}^r (f^*T_X)_{y_i} \simeq \bigoplus_{i=1}^r T_{X,x_i},$$

hence *the tangent space to* $\operatorname{Mor}(Y, X; y_i \mapsto x_i)$ *is* its kernel $H^0(Y, f^*T_X \otimes \mathscr{I}_{y_1,\dots,y_r})$, where $\mathscr{I}_{y_1,\dots,y_r}$ is the ideal sheaf of $y_1, \dots, y_r$ in Y.

Note also that by classical theorems on the dimension of fibers and Theorem 6.8, locally at a point $[f]$ such that X is smooth along $f(Y)$, *the scheme* $\operatorname{Mor}(Y, X; y_i \mapsto x_i)$ *can be defined by* $h^1(Y, f^*T_X) + r \dim(X)$ *equations in a smooth scheme of dimension* $h^0(Y, f^*T_X)$. *In particular, its irreducible components at* $[f]$ *are all of dimension at least*

$$h^0(Y, f^*T_X) - h^1(Y, f^*T_X) - r \dim(X).$$

In fact, one can show that more precisely, as in the case when there are no fixed points, the scheme $\operatorname{Mor}(Y, X; y_i \mapsto x_i)$ can be defined by $h^1(Y, f^*T_X \otimes \mathscr{I}_{y_1,\dots,y_r})$ equations in a smooth scheme of dimension $h^0(Y, f^*T_X \otimes \mathscr{I}_{y_1,\dots,y_r})$.

6.12. Morphisms from a curve. Everything takes a particularly simple form when Y is a curve C: for any $f : C \to X$, one has by Riemann-Roch

$$\begin{aligned} \dim_{[f]} \operatorname{Mor}(C, X) &\geq \chi(C, f^*T_X) \\ &= -K_X \cdot f_*C + (1 - g(C)) \dim(X), \end{aligned}$$

where $g(C) = 1 - \chi(C, \mathscr{O}_C)$, and, for $c_1, \dots, c_r \in C$,

$$\begin{aligned} \dim_{[f]} \operatorname{Mor}(C, X; c_i \mapsto f(c_i)) &\geq \chi(C, f^*T_X) - r \dim(X) \\ &= -K_X \cdot f_*C + (1 - g(C) - r) \dim(X). \end{aligned} \tag{6.2}$$

6.4 Lines on a subvariety of a projective space

We will describe lines on complete intersections in a projective space *over an algebraically closed field* $\mathbf{k}$ to illustrate the concepts developed above.

Let X be a subvariety of $\mathbf{P}^N_{\mathbf{k}}$ of dimension n. By associating its image to a rational curve, we define a morphism

$$\operatorname{Mor}_1(\mathbf{P}^1_{\mathbf{k}}, X) \to G(1, \mathbf{P}^N_{\mathbf{k}}),$$

where $G(1, \mathbf{P}^N_{\mathbf{k}})$ is the Grassmannian of lines in $\mathbf{P}^N_{\mathbf{k}}$. Its image parametrizes lines in X; it has a natural scheme structure and we will denote it by $F(X)$. It is simpler to study $F(X)$ instead of $\operatorname{Mor}_1(\mathbf{P}^1_{\mathbf{k}}, X)$.

The induced map $\rho : \mathrm{Mor}_1(\mathbf{P}^1_{\mathbf{k}}, X) \to F(X)$ is the quotient by the action of the automorphism group of $\mathbf{P}^1_{\mathbf{k}}$. Let $f : \mathbf{P}^1_{\mathbf{k}} \to X$ be a one-to-one parametrization of a line ℓ. *Assume X is smooth of dimension n along ℓ*; using Proposition 6.5, the tangent map to ρ at the point $[f]$ of $\mathrm{Mor}_1(\mathbf{P}^1_{\mathbf{k}}, X)$ fits into an exact sequence

$$0 \longrightarrow H^0(\mathbf{P}^1_{\mathbf{k}}, T_{\mathbf{P}^1_{\mathbf{k}}}) \longrightarrow H^0(\mathbf{P}^1_{\mathbf{k}}, f^*T_X) \xrightarrow{T_{\rho,[f]}} H^0(\mathbf{P}^1_{\mathbf{k}}, f^*N_{\ell/X}) \longrightarrow 0,$$

where $N_{\ell/X}$ is the *normal bundle* to ℓ in X. Since f induces an isomorphism onto its image, we may as well consider the same exact sequence on ℓ. The tangent space to $F(X)$ at $[\ell]$ is therefore $H^0(\ell, N_{\ell/X})$.

Similarly, given a point x on X and a parametrization $f : \mathbf{P}^1_{\mathbf{k}} \to X$ of a line contained in X with $f(0) = x$, the group of automorphisms of $\mathbf{P}^1_{\mathbf{k}}$ fixing 0 acts on the scheme

$$\mathrm{Mor}(\mathbf{P}^1_{\mathbf{k}}, X; 0 \mapsto x)$$

(notation of 6.11), with quotient the subscheme $F(X, x)$ of $F(X)$ consisting of lines passing through x and contained in X. Lines through x are parametrized by a hyperplane in $\mathbf{P}^N_{\mathbf{k}}$ of which $F(X, x)$ is a subscheme. From 6.11, it follows that the tangent space to $F(X, x)$ at $[\ell]$ is isomorphic to $H^0(\ell, N_{\ell/X}(-1))$.

There is an exact sequence of normal bundles

$$0 \to N_{\ell/X} \to \mathscr{O}_\ell(1)^{\oplus(N-1)} \to (N_{X/\mathbf{P}^N_{\mathbf{k}}})|_\ell \to 0. \tag{6.3}$$

Since any locally free sheaf on $\mathbf{P}^1_{\mathbf{k}}$ is isomorphic to a direct sum of invertible sheaf (compare with Example 5.7), we can write

$$N_{\ell/X} \simeq \bigoplus_{i=1}^{n-1} \mathscr{O}_\ell(a_i), \tag{6.4}$$

where $a_1 \geq \cdots \geq a_{n-1}$. By (6.3), we have $a_1 \leq 1$. If $a_{n-1} \geq -1$, the scheme $F(X)$ is smooth at $[\ell]$ (Theorem 6.8). If $a_{n-1} \geq 0$, the scheme $F(X, x)$ is smooth at $[\ell]$ for any point x on ℓ (see 6.11).

6.13. Fermat hypersurfaces. The Fermat hypersurface X^d_N is the hypersurface in $\mathbf{P}^N_{\mathbf{k}}$ defined by the equation

$$x_0^d + \cdots + x_N^d = 0.$$

It is smooth if and only if the characteristic p of $\mathbf{k}$ does not divide d. *Assume $p > 0$ and $d = p^r + 1$* for some $r > 0$. The line joining two points x and y is contained in X^d_N if and only if

$$\begin{aligned} 0 &= \sum_{j=0}^{N} (x_j + ty_j)^{p^r+1} \\ &= \sum_{j=0}^{N} (x_j^{p^r} + t^{p^r} y_j^{p^r})(x_j + ty_j) \\ &= \sum_{j=0}^{N} (x_j^{p^r+1} + t x_j^{p^r} y_j + t^{p^r} x_j y_j^{p^r} + t^{p^r+1} y_j^{p^r+1}) \end{aligned}$$

for all $t \in \bar{\mathbf{k}}$. It follows that the scheme

$$\{(x, y) \in X \times X \mid \langle x, y \rangle \subset X\}$$

is defined by the two equations

$$0 = \sum_{j=0}^{n+1} x_j^{p^r} y_j = \Big(\sum_{j=0}^{n+1} x_j^{p^{-r}} y_j \Big)^{p^r}$$

in $X \times X$, hence has everywhere dimension $\geq 2N - 4$. Since this scheme (minus the diagonal of $X \times X$) is fibered over $F(X^d_N)$ with fibers $\mathbf{P}^1_{\mathbf{k}} \times \mathbf{P}^1_{\mathbf{k}}$ (minus the diagonal), it follows that $F(X^d_N)$ has everywhere dimension $\geq 2N - 6$. With the notation of (6.4), this implies

$$2N - 6 \leq \dim(T_{F(X^d_N),[\ell]}) = h^0(\ell, N_{\ell/X^d_N}) = \dim \sum_{a_i \geq 0} (a_i + 1). \tag{6.5}$$

Since $a_i \le 1$ and $a_1 + \cdots + a_{N-2} = N - 1 - d$ by (6.3), the only possibility is, when $d \ge 4$,

$$N_{\ell/X_N^d} \simeq \mathscr{O}_\ell(1)^{\oplus(N-3)} \oplus \mathscr{O}_\ell(2-d)$$

and there is equality in (6.5). It follows that $F(X_N^d)$ *is everywhere smooth of dimension* $2N-6$*, although* $H^1(\ell, N_{\ell/X_N^d})$ *is nonzero.* Considering parametrizations of these lines, we get an example of a scheme $\mathrm{Mor}_1(\mathbf{P}^1_{\mathbf{k}}, X_N^d)$ *smooth at all points* $[f]$ *although* $H^1(\mathbf{P}^1_{\mathbf{k}}, f^*T_{X_N^d})$ *never vanishes.*

The scheme

$$\{(x,[\ell]) \in X \times F(X_N^d) \mid x \in \ell\}$$

is therefore smooth of dimension $2N-5$, hence the fiber $F(X_N^d, x)$ of the first projection has dimension $N-4$ for x general in X.[4] On the other hand, the calculation above shows that the scheme $F(X_N^d, x)$ is defined (in some fixed hyperplane not containing x) by the three equations

$$0 = \sum_{j=0}^{n+1} x_j^{p^r} y_j = \Big(\sum_{j=0}^{n+1} x_j^{p^{-r}} y_j\Big)^{p^r} = \sum_{j=0}^{n+1} y_j^{p^r+1}.$$

It is clear from these equations that the tangent space to $F(X_N^d, x)$ at every point has dimension $\ge N-3$. For $N \ge 4$, it follows that for x general in X, the scheme $F(X_N^d, x)$ *is nowhere reduced* and similarly, $\mathrm{Mor}_1(\mathbf{P}^1_{\mathbf{k}}, X_N^d; 0 \mapsto x)$ *is nowhere reduced.*

6.5 Exercises

1) Let X be a subscheme of $\mathbf{P}^N_{\mathbf{k}}$ defined by equations of degrees $d_1, \dots, d_s$ over an algebraically closed field. Assume $d_1 + \cdots + d_s < N$. Show that through any point of X, there is a line contained in X (we say that *X is covered by lines*).

[4] This is actually true for all $x \in X$.

Chapter 7

“Bend-and-break” lemmas

We now enter Mori’s world. The whole story began in 1979, with Mori’s astonishing proof of a conjecture of Hartshorne characterizing projective spaces as the only smooth projective varieties with ample tangent bundle ([Mo1]). The techniques that Mori introduced to solve this conjecture have turned out to have more far reaching applications than Hartshorne’s conjecture itself.

Mori’s first idea is that if a curve deforms on a projective variety X while passing through a fixed point, it must at some point break up with at least one rational component, hence the name “bend-and-break”. This is a relatively easy result, but now comes the really tricky part: when X is smooth, to ensure that a morphism $f : C \to X$ deforms fixing a point, the natural thing to do is to use the lower bound (6.2)

$$(-K_X \cdot f_*C) - g(C)\dim(X)$$

for the dimension of the space of deformations. How can one make this number positive? The divisor $-K_X$ had better have some positivity property, but even if it does, simple-minded constructions like ramified covers never lead to a positive bound. Only in positive characteristic can Frobenius operate its magic: increase the degree of f (hence the intersection number $(-K_X \cdot f_*C)$ if it is positive) *without changing the genus of* C.

The most favorable situation is when X is a Fano variety, which means that $-K_X$ is ample: in that case, any curve has positive $(-K_X)$-degree and the Frobenius trick combined with Mori’s bend-and-break lemma produces a rational curve through any point of X. Another bend-and-break-type result universally bounds the $(-K_X)$-degree of this rational curve and allows a proof in all characteristics of the fact that Fano varieties are covered by rational curves by reducing to the positive characteristic case (Theorem 7.5).

We then prove a finer version of the bend-and-break lemma (Proposition 7.6) and deduce a result which will be essential for the description of the cone of curves of any projective smooth variety (Theorem 8.1): if K_X has negative degree on a curve C, the variety X contains a rational curve that meets C (Theorem 7.7). We give a direct application in Theorem 7.9 by showing that varieties for which $-K_X$ is nef but not numerically trivial are also covered by rational curves.

We work here over an *algebraically closed field* $\mathbf{k}$.

Recall that a 1-cycle on X is a formal sum $\sum_{i=1}^{s} n_i C_i$, where the n_i are integers and the C_i are integral curves on X. It is called rational if the C_i are rational curves. If C is a curve with irreducible components $C_1, \dots, C_r$ and $f : C \to X$ a morphism, we will write f_*C for the effective 1-cycle $\sum_{i=1}^{r} d_i f(C_i)$, where d_i is the degree of $f|_{C_i}$ onto its image (as in 3.17). Note that for any Cartier divisor D on X, one has $(D \cdot f_*C) = \deg(f^*D)$.

7.1 Producing rational curves

The following is the original bend-and-break lemma, which can be found in [Mo1] (Theorems 5 and 6). It says that a curve deforming nontrivially while keeping a point fixed must break into an effective 1-cycle with a rational component passing through the fixed point.

Proposition 7.1 (Mori) *Let X be a projective variety, let $f : C \to X$ be a smooth curve and let c be a point on C. If $\dim_{[f]} \operatorname{Mor}(C, X; c \mapsto f(c)) \geq 1$, there exists a rational curve on X through $f(c)$.*

According to (6.2), *when X is smooth along $f(C)$*, the hypothesis is fulfilled whenever

$$(-K_X \cdot f_*C) - g(C)\dim(X) \geq 1.$$

The proof actually shows that there exists a morphism $f' : C \to X$ and a connected nonzero effective rational 1-cycle Z on X passing through $f(c)$ such that

$$f_*C \underset{\text{num}}{\equiv} f'_*C + Z.$$

(This numerical equivalence comes from the fact that these two cycles appear as fibers of a morphism from a surface to a curve and follows from the projection formula (3.6)).

PROOF. Let T be the normalization of a 1-dimensional subvariety of $\operatorname{Mor}(C, X; c \mapsto f(c))$ passing through $[f]$ and let $\overline{T}$ be a smooth compactification of T. By Theorem 5.18, the indeterminacies of the rational map

$$\operatorname{ev} : C \times \overline{T} \dashrightarrow X$$

coming from the morphism $T \to \operatorname{Mor}(C, X; c \mapsto f(c))$ can be resolved by blowing up points to get a morphism

$$e : S \xrightarrow{\varepsilon} C \times \overline{T} \overset{\operatorname{ev}}{\dashrightarrow} X.$$

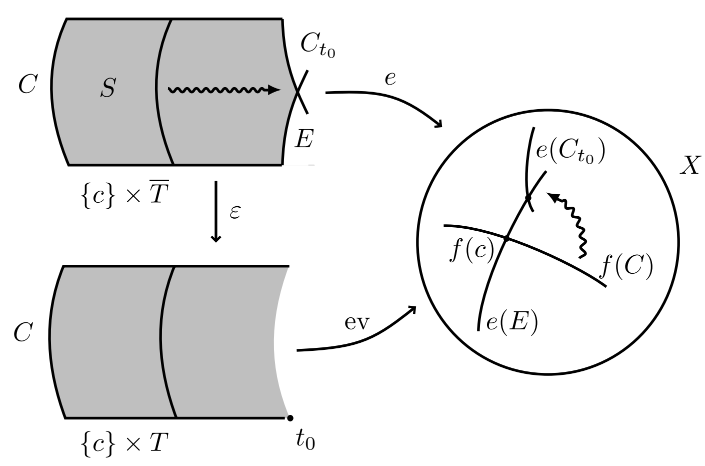
If ev is defined at every point of $\{c\} \times \overline{T}$, Lemma 4.22.a) implies that there exist a neighborhood V of c in C and a factorization

$$\operatorname{ev}|_{V \times \overline{T}} : V \times \overline{T} \xrightarrow{p_1} V \xrightarrow{g} X.$$

The morphism g must then be equal to $f|_V$. It follows that ev and $f \circ p_1$ coincide on $V \times T$, hence on $C \times T$. But this means that the image of T in $\operatorname{Mor}(C, X; c \mapsto f(c))$ is just the point $[f]$, and this is absurd.

Hence there exists a point t_0 in $\overline{T}$ such that ev is not defined at (c, t_0). The fiber of t_0 under the projection $S \to \overline{T}$ is the union of the strict transform of $C \times \{t_0\}$ and a (connected) exceptional rational 1-cycle E which is not entirely contracted by e and meets the strict transform of $\{c\} \times \overline{T}$, which is contracted by e to the point $f(c)$. Since the latter is contracted by e to the point $f(c)$, the rational nonzero 1-cycle e_*E passes through $f(c)$.

The following picture sums up our constructions:



The 1-cycle f_*C degenerates to a 1-cycle with a rational component $e(E)$.

□

Remark 7.2 It is interesting to remark that the conclusion of the proposition fails for curves on compact complex manifolds (although one expects that it should still hold for compact *Kähler* manifolds). An example can be constructed as follows: let E be an elliptic curve, let $\mathscr{L}$ be a very ample invertible sheaf on E, and let s and s' be sections of $\mathscr{L}$ that generate it at each point. The sections (s, s'), $(is, -is')$, $(s', -s)$ and (is', is) of $\mathscr{L} \oplus \mathscr{L}$ are independent over $\mathbf{R}$ in each fiber. They generate a discrete subgroup of the total space of $\mathscr{L} \oplus \mathscr{L}$ and the quotient X is a compact complex threefold with a morphism $\pi : X \to E$ whose fibers are 2-dimensional complex tori. There is a 1-dimensional family of sections $\sigma_t : E \to X$ of π defined by $\sigma_t(x) = (ts(x), 0)$, for $t \in \mathbf{C}$, and they all pass through the points of the zero section where s vanishes. However, X contains no rational curves, because they would have to be contained in a fiber of π, and complex tori contain no rational curves. The variety X is of course not algebraic, and not even bimeromorphic to a Kähler manifold.

Once we know there is a rational curve, it may under certain conditions be broken up into several components. More precisely, if it deforms nontrivially while keeping two points fixed, it must break up (into an effective 1-cycle with rational components).

Proposition 7.3 (Mori) *Let X be a projective variety and let $f : \mathbf{P}^1_{\mathbf{k}} \to X$ be a rational curve. If*

$$\dim_{[f]}(\mathrm{Mor}(\mathbf{P}^1_{\mathbf{k}}, X; 0 \mapsto f(0), \infty \mapsto f(\infty))) \geq 2,$$

the 1-cycle $f_*\mathbf{P}^1_{\mathbf{k}}$ is numerically equivalent to a connected nonintegral effective 1-cycle with rational components passing through $f(0)$ and $f(\infty)$.

According to (6.2), *when X is smooth along $f(\mathbf{P}^1_{\mathbf{k}})$*, the hypothesis is fulfilled whenever

$$(-K_X \cdot f_*\mathbf{P}^1_{\mathbf{k}}) - \dim(X) \geq 2.$$

Proof. The group of automorphisms of $\mathbf{P}^1_{\mathbf{k}}$ fixing two points is the multiplicative group $\mathbf{G}_m$. Let T be the normalization of a 1-dimensional subvariety of $\mathrm{Mor}(\mathbf{P}^1_{\mathbf{k}}, X; 0 \mapsto f(0), \infty \mapsto f(\infty))$ passing through $[f]$ but not contained in its $\mathbf{G}_m$-orbit. The corresponding map

$$F : \mathbf{P}^1_{\mathbf{k}} \times T \to X \times T$$

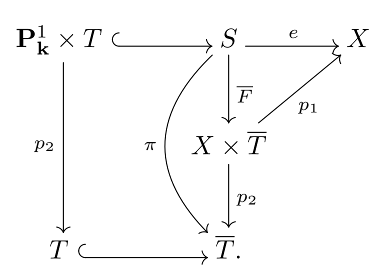
is finite. Let $\overline{T}$ be a smooth compactification of T, let

$$S' \to \mathbf{P}^1_{\mathbf{k}} \times \overline{T} \dashrightarrow X \times \overline{T}$$

be a resolution of indeterminacies (Theorem 5.18) of the rational map $\mathbf{P}^1_{\mathbf{k}} \times \overline{T} \dashrightarrow X \times \overline{T}$ and let

$$S' \longrightarrow S \xrightarrow{\overline{F}} X \times \overline{T}$$

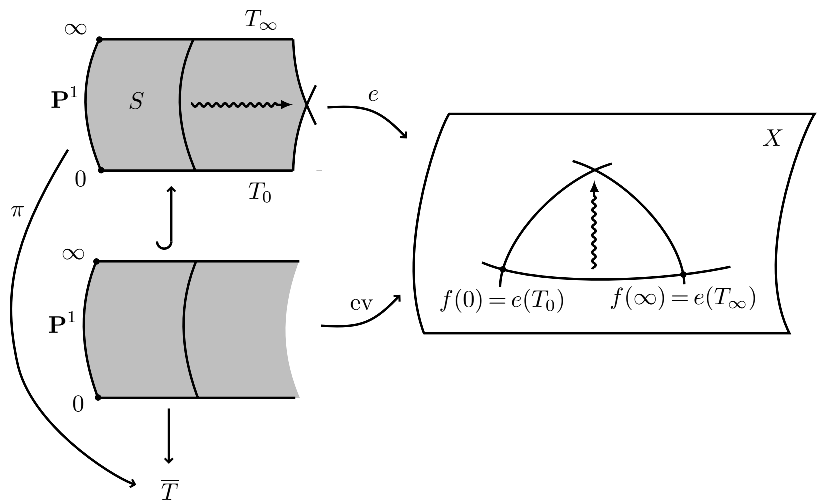
be its Stein factorization, where the surface S is normal and $\overline{F}$ is finite. By uniqueness of the Stein factorization, F factors through $\overline{F}$, so that there is a commutative diagram[1]



Since $\overline{T}$ is a smooth curve and S is integral, π is flat ([H1], Proposition III.9.7). Assume that its fibers are all integral. Their genus is then constant ([H1], Corollary III.9.10) hence equal to 0. Therefore, each fiber is a smooth rational curve and S is a ruled surface (Definition 5.4). Let T_0 be the closure of $\{0\} \times T$ in S and

[1]This construction is similar to the one we performed in the last proof; however, S might not be smooth but on the other hand, we know that no component of a fiber of π is contracted by e (because it would then be contracted by $\overline{F}$). In other words, the surface S is obtained from the surface S' by contracting all curves in the fibers of $S' \to \overline{T}$ that are contracted on X.

let T_∞ be the closure of $\{\infty\} \times T$. These sections of π are contracted by e (to $f(0)$ and $f(\infty)$ respectively). The following picture sums up our constructions:



The rational 1*-cycle* f_*C *bends and breaks.*

If H is an ample divisor on $e(S)$, which is a surface by construction, we have $((e^*H)^2) > 0$ and $(e^*H \cdot T_0) = (e^*H \cdot T_\infty) = 0$, hence (T_0^2) and (T_∞^2) are negative by the Hodge index theorem (Exercise 5.7.2)).

However, since T_0 and T_∞ are both sections of π, their difference is linearly equivalent to the pull-back by π of a divisor on $\overline{T}$ (Proposition 5.5). In particular,

$$0 = ((T_0 - T_\infty)^2) = (T_0^2) + (T_\infty^2) - 2(T_0 \cdot T_\infty) < 0,$$

which is absurd.

It follows that at least one fiber F of π is not integral: it is either reducible or has a multiple component. Let $S'' \to S$ be a resolution of singularities.[2] Each component of F is dominated by a component of the corresponding fiber of the morphism $\pi'' : S'' \to \overline{T}$. By the minimal model program for surfaces (see §5.6), S'' is obtained by successively blowing up points on a ruled surface $S''_0 \to \overline{T}$ (see §5.2), hence all the components of all the fibers of π'' are rational. It follows that the components of F_{red} are all rational curves, and they are not contracted by e. The direct image of F on X is the required 1-cycle. □

7.2 Rational curves on Fano varieties

A Fano variety is a smooth projective variety X (over the algebraically closed field $\mathbf{k}$) with ample anticanonical divisor; K_X is therefore as far as possible from being nef: it has negative degree on any curve.

Examples 7.4 1) The projective space is a Fano variety. Any smooth complete intersection in $\mathbf{P}^n$ defined by equations of degrees $d_1, \dots, d_s$ with $d_1 + \cdots + d_s \le n$ is a Fano variety. A finite product of Fano varieties is a Fano variety.

2) Let Y be a Fano variety, let $D_1, \dots, D_r$ be nef divisors on Y such that $-K_Y - D_1 - \cdots - D_r$ is ample, and let $\mathscr{E}$ be the locally free sheaf $\bigoplus_{i=1}^r \mathscr{O}_Y(D_i)$ on Y. Then $X = \mathbf{P}(\mathscr{E})$ is a Fano variety.[3] Indeed, if D is a divisor on X associated with the invertible sheaf $\mathscr{O}_{\mathbf{P}(\mathscr{E})}(1)$ and $\pi : X \to Y$ is the canonical map, one gets as in [H1], Lemma V.2.10,

$$-K_X = rD + \pi^*(-K_Y - D_1 - \cdots - D_r).$$

[2] The fact that a projective surface can always be desingularized is an important result proved by Walker over $\mathbf{C}$ (1935), by Zariski over any field of characteristic 0 (1939), and by Abhyankar over any field of positive characteristic (1956).

[3] As in §5.2, we follow Grothendieck's notation: for a locally free sheaf $\mathscr{E}$, the projectivization $\mathbf{P}(\mathscr{E})$ is the space of *hyperplanes* in the fibers of $\mathscr{E}$.

Since each D_i is nef, the divisor D is nef on X; since each $-K_Y - D_1 - \cdots - D_r + D_i$ is ample (4.3), the divisor $D + \pi^*(-K_Y - D_1 - \cdots - D_r)$ is ample. It follows that $-K_X$ is ample (4.3).

We will apply the bend-and-break lemmas to show that any Fano variety X is covered by rational curves. We start from any curve $f : C \to X$ and want to show, using the estimate (6.2), that it deforms nontrivially while keeping a point x fixed. As explained in the introduction, we only know how to do that in positive characteristic, where the Frobenius morphism allows to increase the degree of f without changing the genus of C. This gives in that case the required rational curve through x. Using the second bend-and-break lemma, we can bound the degree of this curve by a constant depending only on the dimension of X, and this will be essential for the remaining step: reduction of the characteristic zero case to positive characteristic.

Assume for a moment that X and x are defined over $\mathbf{Z}$; for almost all prime numbers p, the reduction of X modulo p is a Fano variety of the same dimension hence there is a rational curve (defined over the algebraic closure of $\mathbf{Z}/p\mathbf{Z}$) through x. This means that the scheme $\mathrm{Mor}(\mathbf{P}^1_{\mathbf{k}}, X; 0 \to x)$, which is defined over $\mathbf{Z}$, has a geometric point modulo almost all primes p. Since we can moreover bound the degree of the curve by a constant independent of p, we are in fact dealing with a quasi-projective scheme, and this implies that it has a point over $\bar{\mathbf{Q}}$, hence over $\mathbf{k}$. In general, X and x are defined over some finitely generated ring and a similar reasoning yields the existence of a $\mathbf{k}$-point of $\mathrm{Mor}(\mathbf{P}^1_{\mathbf{k}}, X; 0 \to x)$, i.e., of a rational curve on X through x.

Theorem 7.5 (Mori) *Let X be a Fano variety of positive dimension n. Through any point of X there is a rational curve of $(-K_X)$-degree at most $n+1$.*

There is no known proof of this theorem that uses only transcendental methods.

Proof. Let x be a point of X. To construct a rational curve through x, it is enough by Proposition 7.1 to produce a curve $f : C \to X$ and a point c on C such that $f(c) = x$ and $\dim_{[f]} \mathrm{Mor}(C, X; c \mapsto f(c)) \geq 1$. By the dimension estimate of (6.2), it is enough to have

$$(-K_X \cdot f_* C) - ng(C) \geq 1.$$

Unfortunately, there is no known way to achieve that, except in positive characteristic. Here is how it works.

Assume that the field $\mathbf{k}$ has characteristic $p > 0$; choose a smooth curve $f : C \to X$ through x and a point c of C such that $f(c) = x$. Consider the ($\mathbf{k}$-linear) Frobenius morphism $C_1 \to C$;[4] it has degree p, but C_1 and C being isomorphic as abstract schemes have the same genus. Iterating the construction, we get a morphism $F_m : C_m \to C$ of degree p^m between curves of the same genus. But

$$(-K_X \cdot (f \circ F_m)_* C_m) - ng(C_m) = -p^m (K_X \cdot f_* C) - ng(C)$$

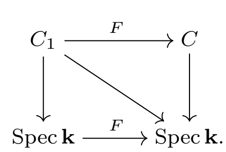
is positive for m large enough. By Proposition 7.1, there exists a rational curve $f' : \mathbf{P}^1_{\mathbf{k}} \to X$, with say $f'(0) = x$. If

$$(-K_X \cdot f'_* \mathbf{P}^1_{\mathbf{k}}) - n \geq 2,$$

the scheme $\mathrm{Mor}(\mathbf{P}^1_{\mathbf{k}}, X; f'|_{\{0,1\}}))$ has dimension at least 2 at $[f']$. By Proposition 7.3, one can break up the rational curve $f'(\mathbf{P}^1_{\mathbf{k}})$ into at least two (rational) pieces. Since $-K_X$ is ample, the component passing through x has smaller $(-K_X)$-degree, and we can repeat the process as long as $(-K_X \cdot \mathbf{P}^1_{\mathbf{k}}) - n \geq 2$, until we get to a rational curve of degree no more than $n+1$.

This proves the theorem in positive characteristic. Assume now that $\mathbf{k}$ has characteristic 0. Embed X in some projective space, where it is defined by a finite set of equations, and let R be the (finitely generated) subring of $\mathbf{k}$ generated by the coefficients of these equations and the coordinates of x. There is a projective

[4] If $F : \mathbf{k} \to \mathbf{k}$ is the Frobenius morphism, the $\mathbf{k}$-scheme C_1 fits into the Cartesian diagram



In other words, C_1 is the scheme C, but $\mathbf{k}$ acts on $\mathscr{O}_{C_1}$ via pth powers.

scheme $\mathscr{X} \to \mathrm{Spec}(R)$ with an R-point x_R, such that X is obtained from its generic fiber by base change from the quotient field $K(R)$ of R to $\mathbf{k}$. The geometric generic fiber is a Fano variety of dimension n, defined over $\overline{K(R)}$. There is a dense open subset U of $\mathrm{Spec}(R)$ over which $\mathscr{X}$ is smooth of dimension n ([G4], th. 12.2.4.(iii)). Since ampleness is an open property ([G4], cor. 9.6.4), we may even, upon shrinking U, assume that the dual $\omega^*_{\mathscr{X}_U/U}$ of the relative dualizing sheaf is ample on all fibers. It follows that for each maximal ideal $\mathfrak{m}$ of R in U, the geometric fiber $X_{\mathfrak{m}}$ is a Fano variety of dimension n, defined over $\overline{R/\mathfrak{m}}$.

Let us take a short break and use a little commutative algebra to show that the finitely generated domain R has the following properties:

- for each maximal ideal $\mathfrak{m}$ of R, the field $R/\mathfrak{m}$ is finite;
- maximal ideals are dense in $\mathrm{Spec}(R)$.

The first item is proved as follows. The field $R/\mathfrak{m}$ is a finitely generated $(\mathbf{Z}/\mathbf{Z}\cap\mathfrak{m})$-algebra, hence is finite over the quotient field of $\mathbf{Z}/\mathbf{Z}\cap\mathfrak{m}$ by the Nullstellensatz (which says that if k is a field and K a finitely generated k-algebra which is a field, K is a finite extension of k; see [M], Theorem 5.2). If $\mathbf{Z}\cap\mathfrak{m}=0$, the field $R/\mathfrak{m}$ is a finite dimensional $\mathbf{Q}$-vector space with basis say $(e_1,\dots,e_m)$. If $x_1,\dots,x_r$ generate the $\mathbf{Z}$-algebra $R/\mathfrak{m}$, there exists an integer q such that qx_j belongs to $\mathbf{Z}e_1\oplus\cdots\oplus\mathbf{Z}e_m$ for each j. This implies

$$\mathbf{Q}e_1\oplus\cdots\oplus\mathbf{Q}e_m = R/\mathfrak{m} \subset \mathbf{Z}[1/q]e_1\oplus\cdots\oplus\mathbf{Z}[1/q]e_m,$$

which is absurd; therefore, $\mathbf{Z}/\mathbf{Z}\cap\mathfrak{m}$ is finite and so is $R/\mathfrak{m}$.

For the second item, we need to show that the intersection of all maximal ideals of R is $\{0\}$. Let a be a nonzero element of R and let $\mathfrak{n}$ be a maximal ideal of the localization R_a. The field $R_a/\mathfrak{n}$ is finite by the first item hence its subring $R/R\cap\mathfrak{n}$ is a finite domain hence a field. Therefore $R\cap\mathfrak{n}$ is a maximal ideal of R which is in the open subset $\mathrm{Spec}(R_a)$ of $\mathrm{Spec}(R)$ (in other words, $a\notin\mathfrak{n}$).

Now back to the proof of the theorem. As proved in §6.1, there is a quasi-projective scheme

$$\rho : \mathrm{Mor}_{\le n+1}(\mathbf{P}^1_R, \mathscr{X}; 0\mapsto x_R) \to \mathrm{Spec}(R)$$

which parametrizes morphisms of degree at most $n+1$.

Let $\mathfrak{m}$ be a maximal ideal of R. Since the field $R/\mathfrak{m}$ is finite, hence of positive characteristic, what we just saw implies that the (geometric) fiber over a closed point of the dense open subset U of $\mathrm{Spec}(R)$ is nonempty; it follows that the image of ρ, which is a constructible[5] subset of $\mathrm{Spec}(R)$ by Chevalley's theorem ([H1], Exercise II.3.19), contains all closed points of U, therefore is dense by the second item, hence contains the generic point ([H1], Exercise II.3.18.(b)). This implies that the generic fiber is nonempty; it has therefore a geometric point, which corresponds to a rational curve on X through x, of degree at most $n+1$, defined over an algebraic closure of the quotient field of R, hence over $\mathbf{k}$.[6] □

7.3 A stronger bend-and-break lemma

We will need the following generalization of the bend-and-break lemma (Proposition 7.1) which gives some control over the degree of the rational curve that is produced. We start from a curve that deforms nontrivially with any (nonzero) number of fixed points. The more points are fixed, the better the bound on the degree. The ideas are the same as in the original bend-and-break, with additional computations of intersection numbers thrown in.

Proposition 7.6 *Let X be a projective variety and let H be an ample Cartier divisor on X. Let $f : C\to X$ be a smooth curve and let B be a finite nonempty subset of C such that*

$$\dim_{[f]}\mathrm{Mor}(C,X;B\mapsto f(B))\ge 1.$$

[5]Recall that a constructible subset is a finite union of locally closed subsets.

[6]It is important to remark that the "universal" bound on the degree of the rational curve is essential for the proof. By the way, for those who know something about logic, the statement that there exists a rational curve of $(-K_X)$-degree at most $\dim(X)+1$ on a projective Fano variety X is a first-order statement, so Lefschetz principle tells us that if it is valid on all algebraically closed fields of positive characteristics, it is valid over all algebraically closed fields.

There exists a rational curve Γ *on* X *which meets* $f(B)$ *and such that*

$$(H \cdot \Gamma) \leq \frac{2(H \cdot f_* C)}{\operatorname{Card}(B)}.$$

According to (6.2), *when* X *is smooth along* $f(C)$, the hypothesis is fulfilled whenever

$$(-K_X \cdot f_* C) + (1 - g(C) - \operatorname{Card}(B)) \dim(X) \geq 1.$$

The proof actually shows that there exist a morphism $f' : C \to X$ and a nonzero effective rational 1-cycle Z on X such that

$$f_* C \underset{\text{num}}{\equiv} f'_* C + Z,$$

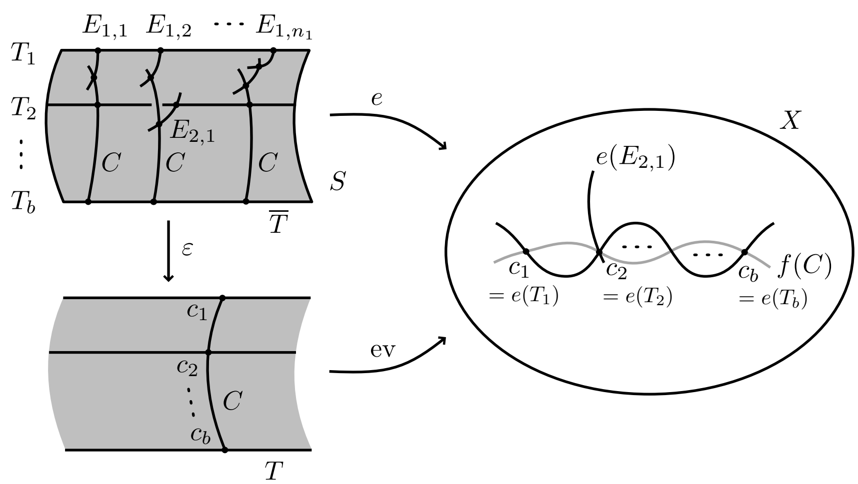
one component of which meets $f(B)$ and satisfies the degree condition above.

Proof. Set $B = \{c_1, \dots, c_b\}$. Let C' be the normalization of $f(C)$. If C' is rational and f has degree $\geq b/2$ onto its image, just take $\Gamma = C'$. From now on, we will assume that if C' is rational, f has degree $< b/2$ onto its image.

By 6.11, the dimension of the space of morphisms from C to $f(C)$ that send B to $f(B)$ is at most $h^0(C, f^* T_{C'} \otimes \mathscr{I}_B)$. When C' is irrational, $f^* T_{C'} \otimes \mathscr{I}_B$ has negative degree, and, under our assumption, this remains true when C' is rational. In both cases, the space is therefore 0-dimensional, hence any 1-dimensional subvariety of $\operatorname{Mor}(C, X; B \mapsto f(B))$ through $[f]$ corresponds to morphisms with varying images. Let $\overline{T}$ be a smooth compactification of the normalization of such a subvariety. Resolve the indeterminacies (Theorem 5.18) of the rational map $\operatorname{ev} : C \times \overline{T} \dashrightarrow X$ by blowing up points to get a morphism

$$e : S \xrightarrow{\varepsilon} C \times \overline{T} \overset{\operatorname{ev}}{\dashrightarrow} X$$

whose image is a *surface*.



The 1-cycle $f_* C$ *bends and breaks keeping* $c_1, \dots, c_b$ *fixed.*

For $i = 1, \dots, b$, we denote by $E_{i,1}, \dots, E_{i,n_i}$ the inverse images on S of the (-1)-exceptional curves that appear every time some point lying on the strict transform of $\{c_i\} \times \overline{T}$ is blown up. We have

$$(E_{i,j} \cdot E_{i',j'}) = -\delta_{i,i'} \delta_{j,j'}.$$

Write the strict transform T_i of $\{c_i\} \times \overline{T}$ on S as

$$T_i \underset{\text{num}}{\equiv} \varepsilon^* \overline{T} - \sum_{j=1}^{n_i} E_{i,j},$$

Write also

$$e^* H \underset{\text{num}}{\equiv} a \varepsilon^* C + d \varepsilon^* \overline{T} - \sum_{i=1}^{b} \sum_{j=1}^{n_i} a_{i,j} E_{i,j} + G,$$

where G is orthogonal to the $\mathbf{R}$-vector subspace of $N^1(S)_{\mathbf{R}}$ generated by ε^*C, $\varepsilon^*\overline{T}$ and the $E_{i,j}$. Note that e^*H is nef, hence

$$a = (e^*H \cdot \varepsilon^*\overline{T}) \geq 0 \quad , \quad a_{i,j} = (e^*H \cdot E_{i,j}) \geq 0.$$

Since T_i is contracted by e to $f(c_i)$, we have for each i

$$0 = (e^*H \cdot T_i) = a - \sum_{j=1}^{n_i} a_{i,j}.$$

Summing up over i, we get

$$ba = \sum_{i,j} a_{i,j}. \tag{7.1}$$

Moreover, since $(\varepsilon^*C \cdot G) = 0 = ((\varepsilon^*C)^2)$ and ε^*C is nonzero, the Hodge index theorem (Exercise 5.7.2)) implies $(G^2) \leq 0$, hence (using (7.1))

$$\begin{aligned} ((e^*H)^2) &= 2ad - \sum_{i,j} a_{i,j}^2 + (G^2) \\ &\leq 2ad - \sum_{i,j} a_{i,j}^2 \\ &= \frac{2d}{b} \sum_{i,j} a_{i,j} - \sum_{i,j} a_{i,j}^2 \\ &\leq \frac{2d}{b} \sum_{i,j} a_{i,j} - \sum_{i,j} a_{i,j}^2 \\ &= \sum_{i,j} a_{i,j} (\frac{2d}{b} - a_{i,j}). \end{aligned}$$

Since $e(S)$ is a surface, this number is positive, hence there exist indices i_0 and j_0 such that $0 < a_{i_0,j_0} < \frac{2d}{b}$.

But $d = (e^*H \cdot \varepsilon^*C) = (H \cdot C)$, and $(e^*H \cdot E_{i_0,j_0}) = a_{i_0,j_0}$ is the H-degree of the rational 1-cycle $e_*(E_{i_0,j_0})$. The latter is nonzero since $a_{i_0,j_0} > 0$, and it passes through $f(c_{i_0})$ since E_{i_0,j_0} meets T_{i_0} (their intersection number is 1) and the latter is contracted by e to $f(c_{i_0})$. This proves the proposition: take for Γ a component of $e_*E_{i_0,j_0}$ which passes through $f(c_{i_0})$. □

7.4 Rational curves on varieties whose canonical divisor is not nef

We proved in Theorem 7.5 that when X is a smooth projective variety such that $-K_X$ is ample (i.e., X is a Fano variety), there is a rational curve through any point of X. The following result considerably weakens the hypothesis: assuming only that K_X has negative degree on *one* curve C, we still prove that there is a rational curve through any point of C.

Note that the proof of Theorem 7.5 goes through in positive characteristic under this weaker hypothesis and does prove the existence of a rational curve through any point of C. However, to pass to the characteristic 0 case, one needs to bound the degree of this rational curve with respect to some ample divisor by some "universal" constant so that we deal only with a quasi-projective part of a morphism space. Apart from that, the ideas are essentially the same as in Theorem 7.5. This theorem is the main result of [MiM].

Theorem 7.7 (Miyaoka-Mori) *Let X be a projective variety, let H be an ample divisor on X, and let $f : C \to X$ be a smooth curve such that X is smooth along $f(C)$ and $(K_X \cdot f_*C) < 0$. Given any point x on $f(C)$, there exists a rational curve Γ on X through x with*

$$(H \cdot \Gamma) \leq 2\dim(X)\ \frac{(H \cdot f_*C)}{(-K_X \cdot f_*C)}.$$

When X is smooth, the rational curve can be broken up, using Proposition 7.3 and (6.2), into several pieces (of lower H-degree) keeping any two points fixed (one of which being on $f(C)$), until one gets a rational curve Γ which satisfies $(-K_X \cdot \Gamma) \leq \dim(X) + 1$ in addition to the bound on the H-degree.

It is nevertheless useful to have a more general statement allowing X to be singular. It implies for example that a normal projective variety X with ample ($\mathbf{Q}$-Cartier) anticanonical divisor is covered by rational curves of $(-K_X)$-degree at most $2\dim(X)$.

Finally, a simple corollary of this theorem is that *the canonical divisor of a smooth projective complex variety which contains no rational curves is nef.*

Proof. The idea is to take b as big as possible in Proposition 7.6, in order to get the lowest possible degree for the rational curve. As in the proof of Theorem 7.5, we first assume that the characteristic of the ground field $\mathbf{k}$ is positive, and use the Frobenius morphism to construct sufficiently many morphisms from C to X.

Assume then that the characteristic of the base field is $p > 0$. We compose f with m Frobenius morphisms to get $f_m : C_m \to X$ of degree $p^m \deg(f)$ onto its image. For any subset B_m of C_m with b_m elements, we have by 6.12

$$\dim_{[f_m]} \operatorname{Mor}(C_m, X; B_m \mapsto f_m(B_m)) \geq p^m(-K_X \cdot f_*C) + (1 - g(C) - b_m)\dim(X),$$

which is positive if we take

$$b_m = \left\lceil \frac{p^m(-K_X \cdot f_*C)}{\dim(X)} - g(C) \right\rceil,$$

which is positive for m sufficiently large. This is what we need to apply Proposition 7.6. It follows that there exists a rational curve Γ_m through some point of $f_m(B_m)$, such that

$$(H \cdot \Gamma_m) \leq \frac{2(H \cdot (f_m)_*C_m)}{b_m} = \frac{2p^m}{b_m}(H \cdot f_*C).$$

As m goes to infinity, p^m/b_m goes to $\dim(X)/(-K_X \cdot f_*C)$. Since the left-hand side is an integer, we get

$$(H \cdot \Gamma_m) \leq \frac{2\dim(X)}{(-K_X \cdot f_*C)}(H \cdot f_*C)$$

for $m \gg 0$. By the lemma below, the set of points of $f(C)$ through which passes a rational curve of degree at most $2\dim(X)\frac{(H \cdot f_*C)}{(-K_X \cdot f_*C)}$ is *closed* (it is the intersection of $f(C)$ and the image of the evaluation map); it cannot be finite since we could then take B_m such that $f_m(B_m)$ lies outside of that locus, hence it is equal to $f(C)$. This finishes the proof when the characteristic is positive.

As in the proof of Theorem 7.5, the characteristic 0 case is done by considering a finitely generated domain R over which X, C, f, H and a point x of $f(C)$ are defined. The family of rational curves mapping 0 to x and of H-degree at most $2\dim(X)\frac{(H \cdot f_*C)}{(-K_X \cdot f_*C)}$ is nonempty modulo any maximal ideal, hence is nonempty over an algebraic closure in $\mathbf{k}$ of the quotient field of R. □

Lemma 7.8 *Let X be a projective variety and let d be a positive integer. Let M_d be the quasi-projective scheme that parametrizes morphisms $\mathbf{P}^1_{\mathbf{k}} \to X$ of degree at most d with respect to some ample divisor. The image of the evaluation map*

$$\operatorname{ev}_d : \mathbf{P}^1_{\mathbf{k}} \times M_d \to X$$

is closed in X.

The image of ev_d is the set of points of X through which passes a rational curve of degree at most d.

Proof. The idea is that a rational curve can only degenerate into a union of rational curves of lower degrees.

Let x be a point in $\overline{\operatorname{ev}_d(\mathbf{P}^1_{\mathbf{k}} \times M_d)} - \operatorname{ev}_d(\mathbf{P}^1_{\mathbf{k}} \times M_d)$. Since M_d is a quasi-projective scheme, there exists an irreducible component M of M_d such that $x \in \overline{\operatorname{ev}_d(\mathbf{P}^1_{\mathbf{k}} \times M)}$ and a projective compactification $\overline{\mathbf{P}^1_{\mathbf{k}} \times M}$ such that ev_d extends to $\overline{\operatorname{ev}}_d : \overline{\mathbf{P}^1_{\mathbf{k}} \times M} \to X$ and $x \in \overline{\operatorname{ev}}_d(\overline{\mathbf{P}^1_{\mathbf{k}} \times M})$.

Let $\overline{T}$ be the normalization of a curve in $\overline{\mathbf{P}^1_{\mathbf{k}} \times M}$ meeting $\overline{\mathrm{ev}}_d^{-1}(x)$ and $\mathbf{P}^1_{\mathbf{k}} \times M$.

The indeterminacies of the rational map $\mathrm{ev}_{\overline{T}} : \mathbf{P}^1_{\mathbf{k}} \times \overline{T} \overset{(\mathrm{Id},p_2)}{\dashrightarrow} \mathbf{P}^1_{\mathbf{k}} \times M \xrightarrow{\mathrm{ev}_d} X$ can be resolved (Theorem 5.18) by blowing up a finite number of points to get a morphism

$$e : S \xrightarrow{\varepsilon} \mathbf{P}^1_{\mathbf{k}} \times \overline{T} \overset{\mathrm{ev}_{\overline{T}}}{\dashrightarrow} X.$$

The image $e(S)$ contains x; it is covered by the images of the fibers of the projection $S \to \overline{T}$, which are unions of rational curves of degree at most d. This proves the lemma. □

Our next result generalizes Theorem 7.5 and shows that varieties with nef but not numerically trivial anticanonical divisor are also covered by rational curves. One should be aware that this class of varieties is much larger than the class of Fano varieties.

Theorem 7.9 *If X is a smooth projective variety with $-K_X$ nef,*

- *either K_X is numerically trivial,*
- *or there is a rational curve through any point of X.*

More precisely, in the second case, there exists an ample divisor H on X such that, through any point x of X, there exists a rational curve of H-degree $\le 2n\frac{(H^n)}{(-K_X \cdot H^{n-1})}$, where $n = \dim(X)$. It follows that X is uniruled in the sense of Definition 9.3.

Proof. Let H be a very ample divisor on X, corresponding to a hyperplane section of an embedding of X in $\mathbf{P}^N_{\mathbf{k}}$. Assume $(K_X \cdot H^{n-1}) = 0$. For any curve $C \subset X$, there exist hypersurface $H_1, \dots, H_{n-1}$ in $\mathbf{P}^N_{\mathbf{k}}$, of respective degrees $d_1, \dots, d_{n-1}$, such that the scheme-theoretic intersection $Z := X \cap H_1 \cap \cdots \cap H_{n-1}$ has pure dimension 1 and contains C. Since $-K_X$ is nef, we have

$$0 \le (-K_X \cdot C) \le (-K_X \cdot Z) = d_1 \cdots d_{n-1}(-K_X \cdot H^{n-1}) = 0,$$

hence K_X is numerically trivial.

Assume now $(K_X \cdot H^{n-1}) < 0$. Let x be a point of X and let C be the normalization of the intersection of $n-1$ general hyperplane sections through x. By Bertini's theorem, C is an irreducible curve and $(K_X \cdot C) = (K_X \cdot H^{n-1}) < 0$. By Theorem 7.7, there is a rational curve on X which passes through x.□

Note that the canonical divisor of an abelian variety X is trivial, and that X contains no rational curves (see Example 5.10).

7.5 Exercise

1) Let X be a smooth projective variety with $-K_X$ big. Show that X is covered by rational curves.

Chapter 8

The cone of curves and the minimal model program

Let X be a smooth projective variety. We defined (Definition 4.8) the cone of curves $\mathrm{NE}(X)$ of X as the convex cone in $N_1(X)_{\mathbf{R}}$ generated by classes of effective curves. We prove here Mori's theorem on the structure of the closure $\overline{\mathrm{NE}}(X)$ of this cone, more exactly of the part where K_X is negative. We show that it is generated by countably many *extremal rays* and that these rays are generated by classes of rational curves and can only accumulate on the hyperplane $K_X = 0$.

Mori's method of proof works in any characteristic, and is a beautiful application of his bend-and-break results (more precisely of Theorem 7.7).

After proving the cone theorem, we study contractions of K_X-negative extremal rays (the existence of the contraction depends on a deep theorem which is only know to hold in characteristic zero, so we work from then on over the field $\mathbf{C}$). They are of three different kinds: fiber contractions (the general fiber is positive-dimensional), divisorial contractions (the exceptional locus is a divisor), small contractions (the exceptional locus has codimension at least 2). Small contractions are the most difficult to handle: their images are too singular, and the minimal model program can only continue if one can construct a *flip* of the contraction (see §8.6). The existence of flips is still unknown in general.

Everything takes place over an algebraically closed field $\mathbf{k}$.

8.1 The cone theorem

We recall the statement of the cone theorem for smooth projective varieties (Theorem 1.7).

If X is a projective scheme, D a divisor on X, and S a subset of $N_1(X)_{\mathbf{R}}$, we set

$$S_{D\geq 0} = \{z \in S \mid D \cdot z \geq 0\}$$

and similarly for $S_{D\leq 0}$, $S_{D>0}$ and $S_{D<0}$.

Theorem 8.1 (Mori's Cone Theorem) *Let X be a smooth projective variety. There exists a countable family $(\Gamma_i)_{i\in I}$ of rational curves on X such that*

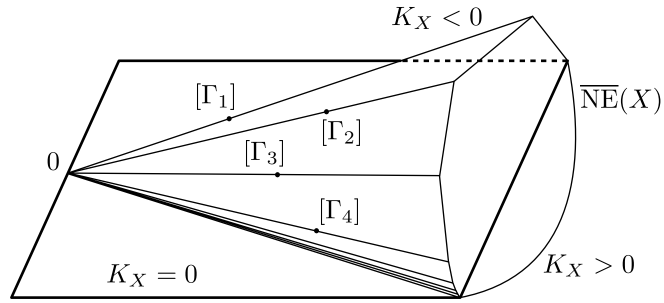
$$0 < (-K_X \cdot \Gamma_i) \leq \dim(X) + 1$$

and

$$\overline{\mathrm{NE}}(X) = \overline{\mathrm{NE}}(X)_{K_X\geq 0} + \sum_{i\in I} \mathbf{R}^+[\Gamma_i], \tag{8.1}$$

where the $\mathbf{R}^+[\Gamma_i]$ are all the extremal rays of $\overline{\mathrm{NE}}(X)$ that meet $N_1(X)_{K_X<0}$; these rays are locally discrete in that half-space.

An extremal ray that meets $N_1(X)_{K_X<0}$ is called K_X*-negative.*



The closed cone of curves

Proof. The idea of the proof is quite simple: if $\overline{\mathrm{NE}}(X)$ is not equal to the closure of the right-hand side of (8.1), there exists a divisor M on X which is nonnegative on $\overline{\mathrm{NE}}(X)$ (hence nef), positive on the closure of the right-hand side, and vanishes at some nonzero point z of $\overline{\mathrm{NE}}(X)$, which must therefore satisfy $K_X \cdot z < 0$. We approximate M by an ample divisor, z by an effective 1-cycle and use the bend-and-break Theorem 7.7 to get a contradiction. In the third and last step, we prove that the right-hand side is closed by a formal argument with no geometric content.

As we saw in §6.1, there are only countably many families of, hence classes of, rational curves on X. Pick a representative Γ_i for each such class z_i that satisfies $0 < -K_X \cdot z_i \leq \dim(X) + 1$.

First step: the rays $\mathbf{R}^+ z_i$ *are locally discrete in the half-space* $N_1(X)_{K_X<0}$.

Let H be an ample divisor on X. It is enough to show that for each $\varepsilon > 0$, there are only finitely many classes z_i in the half-space $N_1(X)_{K_X+\varepsilon H<0}$, since the union of these half-spaces is $N_1(X)_{K_X<0}$. If $((K_X + \varepsilon H) \cdot \Gamma_i) < 0$, we have

$$(H \cdot \Gamma_i) < \frac{1}{\varepsilon}(-K_X \cdot \Gamma_i) \leq \frac{1}{\varepsilon}(\dim(X) + 1)$$

and there are finitely many such classes of curves on X (Theorem 4.10.b)).

Second step: $\overline{\mathrm{NE}}(X)$ *is equal to the closure of*

$$V = \overline{\mathrm{NE}}(X)_{K_X \geq 0} + \sum_i \mathbf{R}^+ z_i.$$

If this is not the case, there exists by Lemma 4.24.d) (since $\overline{\mathrm{NE}}(X)$ contains no lines) an $\mathbf{R}$-divisor M on X which is nonnegative on $\overline{\mathrm{NE}}(X)$ (it is in particular nef), positive on $\overline{V} - \{0\}$ and which vanishes at some nonzero point z of $\overline{\mathrm{NE}}(X)$. This point cannot be in V, hence $K_X \cdot z < 0$.

Choose a norm on $N_1(X)_{\mathbf{R}}$ such that $\|[C]\| \geq 1$ for each irreducible curve C (this is possible since the set of classes of irreducible curves is discrete). We may assume, upon replacing M with a multiple, that $M \cdot v \geq 2\|v\|$ for all v in $\overline{V}$. We have

$$2\dim(X)(M \cdot z) = 0 < -K_X \cdot z.$$

Since the class $[M]$ is a limit of classes of ample $\mathbf{Q}$-divisors, and z is a limit of classes of effective rational 1-cycles, there exist an ample $\mathbf{Q}$-divisor H and an effective 1-cycle Z such that

$$2\dim(X)(H \cdot Z) < (-K_X \cdot Z) \qquad \text{and} \qquad H \cdot v \geq \|v\| \tag{8.2}$$

for all v in $\overline{V}$. We may further assume, by throwing away the other components, that each component C of Z satisfies $(-K_X \cdot C) > 0$.

Since the class of every rational curve Γ on X such that $(-K_X \cdot \Gamma) \leq \dim(X) + 1$ is in $\overline{V}$ (either it is in $\overline{\mathrm{NE}}(X)_{K_X \geq 0}$, or $(-K_X \cdot \Gamma) > 0$ and $[\Gamma]$ is one of the z_i), we have $(H \cdot \Gamma) \geq \|[\Gamma]\| \geq 1$ by (8.2) and the choice of the norm. Since X is smooth, the bend-and-break Theorem 7.7 implies

$$2\dim(X)\ \frac{(H \cdot C)}{(-K_X \cdot C)} \geq 1$$

for every component C of Z. This contradicts the first inequality in (8.2) and finishes the proof of the second step.

Third step: for any set J of indices, the cone

$$\overline{\mathrm{NE}}(X)_{K_X\geq 0}+\sum_{j\in J}\mathbf{R}^+z_j$$

is closed.

Let V_J be this cone. By Lemma 4.24.b), it is enough to show that any extremal ray $\mathbf{R}^+r$ in $\overline{V_J}$ satisfying $K_X\cdot r<0$ is in V_J. Let H be an ample divisor on X and let ε be a positive number such that $(K_X+\varepsilon H)\cdot r<0$. By the first step, there are only finitely many classes $z_{j_1},\dots,z_{j_q}$, with $j_\alpha\in J$, such that $(K_X+\varepsilon H)\cdot z_{j_\alpha}<0$.

Write r as the limit of a sequence $(r_m+s_m)_{m\geq 0}$, where $r_m\in\overline{\mathrm{NE}}(X)_{K_X+\varepsilon H\geq 0}$ and $s_m=\sum_{\alpha=1}^q\lambda_{\alpha,m}z_{j_\alpha}$. Since $H\cdot r_m$ and $H\cdot z_{j_\alpha}$ are positive, the sequences $(H\cdot r_m)_{m\geq 0}$ and $(\lambda_{\alpha,m})_{m\geq 0}$ are bounded, hence we may assume, after taking subsequences, that all sequences $(r_m)_{m\geq 0}$ and $(\lambda_{\alpha,m})_{m\geq 0}$ have limits (Theorem 4.10.b)). Because r spans an extremal ray in $\overline{V_J}$, the limits must be nonnegative multiples of r, and since $(K_X+\varepsilon H)\cdot r<0$, the limit of $(r_m)_{m\geq 0}$ must vanish. Moreover, r is a multiple of one the z_{j_α}, hence is in V_J.

If we choose a set I of indices such that $(\mathbf{R}^+z_j)_{j\in I}$ is the set of all (distinct) extremal rays among all $\mathbf{R}^+z_i$, the proof shows that any extremal ray of $\overline{\mathrm{NE}}(X)_{K_X<0}$ is spanned by a z_i, with $i\in I$. This finishes the proof of the cone theorem. □

Corollary 8.2 *Let X be a smooth projective variety and let R be a K_X-negative extremal ray. There exists a nef divisor M_R on X such that*

$$R=\{z\in\overline{\mathrm{NE}}(X)\mid M_R\cdot z=0\}.$$

For any such divisor, mM_R-K_X is ample for all $m\gg 0$.

Any such divisor M_R will be called a *supporting divisor* for R.

PROOF. With the notation of the proof of the cone theorem, there exists a (unique) element i_0 of I such that $R=\mathbf{R}^+z_{i_0}$. By the third step of the proof of the theorem, the cone

$$V=V_{I-\{i_0\}}=\overline{\mathrm{NE}}(X)_{K_X\geq 0}+\sum_{i\in I,\,i\neq i_0}\mathbf{R}^+z_i$$

is closed and is strictly contained in $\overline{\mathrm{NE}}(X)$ since it does not contain R. By Lemma 4.24.d), there exists a linear form which is nonnegative on $\overline{\mathrm{NE}}(X)$, positive on $V-\{0\}$ and which vanishes at some nonzero point of $\overline{\mathrm{NE}}(X)$, hence on R since $\overline{\mathrm{NE}}(X)=V+R$. The intersection of the interior of the dual cone V^* and the *rational* hyperplane $R^\perp$ is therefore nonempty, hence contains an integral point: there exists a divisor M_R on X which is positive on $V-\{0\}$ and vanishes on R. It is in particular nef and the first statement of the corollary is proved.

Choose a norm on $N_1(X)_\mathbf{R}$ and let a be the (positive) minimum of M_R on the set of elements of V with norm 1. If b is the maximum of K_X on the same compact, the divisor mM_R-K_X is positive on $V-\{0\}$ for m rational greater than b/a, and positive on $R-\{0\}$ for $m\geq 0$, hence ample for $m>\max(b/a,0)$ by Kleiman's criterion (Theorem 4.10.a)). This finishes the proof of the corollary. □

8.2 Contractions of K_X-negative extremal rays

The fact that extremal rays can be contracted is essential to the realization of Mori's minimal model program. This is only known in characteristic 0 (so say over $\mathbf{C}$) in all dimensions (and in any characteristic for surfaces; see §5.4) as a consequence of the following powerful theorem, whose proof is beyond the intended scope (and methods) of these notes.

Theorem 8.3 (Base-point-free theorem (Kawamata)) *Let X be a smooth complex projective variety and let D be a nef divisor on X such that $aD - K_X$ is nef and big for some $a \in \mathbf{Q}^{+*}$. The divisor mD is generated by its global sections for all $m \gg 0$.*

Corollary 8.4 *Let X be a smooth complex projective variety and let R be a K_X-negative extremal ray.*

a) The contraction $c_R : X \twoheadrightarrow Y$ of R exists, where Y is a normal projective variety. It is given by the Stein factorization of the morphism defined by any sufficiently high multiple of any supporting divisor of R.

b) Let C be any integral curve on X with class in R. There is an exact sequence

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathrm{Pic}(Y) & \xrightarrow{c_R^*} & \mathrm{Pic}(X) & \longrightarrow & \mathbf{Z} \\ & & & & [D] & \longmapsto & (D \cdot C) \end{array}$$

and $\rho_Y = \rho_X - 1$.

Remarks 8.5 1) The same result holds (with the same proof) for any K_X-negative extremal subcone V of $\overline{\mathrm{NE}}(X)$ instead of R (in which case the Picard number of $c_V(X)$ is $\rho_X - \dim(\langle V \rangle)$).

2) Item b) implies that there are dual exact sequences

$$0 \to N^1(Y)_{\mathbf{R}} \xrightarrow{c_R^*} N^1(X)_{\mathbf{R}} \xrightarrow{\mathrm{rest}} \langle R \rangle^* \to 0$$

and

$$0 \to \langle R \rangle \to N_1(X)_{\mathbf{R}} \xrightarrow{c_{R*}} N_1(Y)_{\mathbf{R}} \to 0.$$

3) By the relative Kleiman criterion (Exercise 4.18), $-K_X$ is c_R-ample.

4) For a contraction $c : X \to Y$ of an extremal ray which is *not* K_X-negative, the complex appearing in b) is in general not exact: take for example the second projection $c : E \times E \to E$, where E is a very general elliptic curve. The vector space $N_1(E \times E)_{\mathbf{Q}}$ has dimension 3, generated by the classes of $E \times \{0\}$, $\{0\} \times E$ and the diagonal ([Ko1], Exercise II.4.16). In this basis, $\overline{\mathrm{NE}}(E \times E)$ is the cone $xy + yz + zx \geq 0$ and $x + y + z \geq 0$, and c is the contraction of the extremal ray spanned by $(1, 0, 0)$. However, the complex

$$\begin{array}{ccccccc} 0 & \to & \mathbf{Q}(1,0,0) & \to & N_1(E \times E)_{\mathbf{Q}} & \xrightarrow{c_*} & N_1(E)_{\mathbf{Q}} \\ & & & & (x,y,z) & \longmapsto & y - z \end{array}$$

is not exact.

Proof of the Corollary. Let M_R be a supporting divisor for R, as in Corollary 8.2. By the same corollary and Theorem 8.3, mM_R is generated by its global sections for $m \gg 0$. The contraction c_R is given by the Stein factorization of the induced morphism $X \to \mathbf{P}^N_{\mathbf{k}}$. This proves a). Note for later use that there exists a Cartier divisor D_m on Y such that $mM_R \underset{\mathrm{lin}}{\equiv} c_R^* D_m$.

For b), note first that since $c_{R*}\mathscr{O}_X \simeq \mathscr{O}_Y$, we have for any invertible sheaf L on Y, by the projection formula ([H1], Exercise II.5.1.(d)),

$$c_{R*}(c_R^* L) \simeq L \otimes c_{R*}\mathscr{O}_X \simeq L.$$

This proves that c_R^* is injective. Let now D be a divisor on X such that $(D \cdot C) = 0$. Proceeding as in the proof of Corollary 8.2, we see that the divisor $mM_R + D$ is nef for all $m \gg 0$ and vanishes only on R. It is therefore a supporting divisor for R hence some multiple $m'(mM_R + D)$ also defines its contraction. Since the contraction is unique, it is c_R and there exists a Cartier divisor $E_{m,m'}$ on Y such that $m'(mM_R + D) \underset{\mathrm{lin}}{\equiv} c_R^* E_{m,m'}$. We obtain $D \underset{\mathrm{lin}}{\equiv} c_R^*(E_{m,m'+1} - E_{m,m'} - D_m)$ and this finishes the proof of the corollary. □

8.3 Different types of contractions

Let X be a smooth complex projective variety and let R be a K_X-negative extremal ray, with contraction $c_R : X \twoheadrightarrow Y$. The morphism c_R contracts all curves whose class lies in R: the *relative cone of curves* of the contraction (Definition 4.15) is therefore R. Since $c_{R*}\mathscr{O}_X \simeq \mathscr{O}_Y$, either $\dim(Y) < \dim(X)$, or c_R is birational.

8.6. Exceptional locus of a morphism. Let $\pi : X \to Y$ be a proper birational morphism. The *exceptional locus* $\mathrm{Exc}(\pi)$ of π is the locus of points of X where π is not a local isomorphism. It is closed and we endow it with its reduced structure. We will denote it here by E.

If Y is *normal,* Zariski's Main Theorem says that $E = \pi^{-1}(\pi(E))$ and the fibers of $E \to \pi(E)$ are connected and everywhere positive-dimensional. In particular, $\pi(E)$ has codimension at least 2 in Y. The largest open set over which $\pi^{-1} : Y \dashrightarrow X$ is defined is $Y \smallsetminus \pi(E)$.

The exceptional locus of c_R is called the *locus* of R and will be denoted by $\mathrm{locus}(R)$. It is the union of all curves in X whose classes belong to R.

There are 3 cases:

- the locus of R is X, $\dim(c_R(X)) < \dim(X)$, and c_R is a *fiber contraction;*
- the locus of R is a divisor, and c_R is a *divisorial contraction;*
- the locus of R has codimension at least 2, and c_R is a *small contraction.*

Proposition 8.7 *Let X be a smooth complex projective variety and let R be a K_X-negative extremal ray of $\overline{\mathrm{NE}}(X)$. If Z is an irreducible component of* $\mathrm{locus}(R)$*,*

a) Z is covered by rational *curves contracted by c_R;*

b) if Z has codimension 1*, it is equal to* $\mathrm{locus}(R)$*;*

c) the following inequality holds

$$\dim(Z) \geq \frac{1}{2}(\dim(X) + \dim(c_R(Z))).$$

The locus of R may be disconnected (see 8.22; the contraction c_R is then necessarily small). The inequality in c) is sharp (Example 8.21) but can be made more precise (see 8.8).

PROOF. Any point x in $\mathrm{locus}(R)$ is on some irreducible curve C whose class is in R. Let M_R be a (nef) supporting divisor for R (as in Corollary 8.2), let H be an ample divisor on X, and let m be an integer such that

$$m > 2\dim(X)\frac{(H\cdot C)}{(-K_X\cdot C)}.$$

By Proposition 7.7, applied with the ample divisor $mM_R + H$, there exists a rational curve Γ through x such that

$$\begin{aligned} 0 &< ((mM_R + H)\cdot\Gamma) \\ &\leq 2\dim(X)\frac{((mM_R+H)\cdot C)}{(-K_X\cdot C)} \\ &= 2\dim(X)\frac{(H\cdot C)}{(-K_X\cdot C)} \\ &< m, \end{aligned}$$

from which it follows that the integer $(M_R\cdot\Gamma)$ must vanish, and $(H\cdot\Gamma) < m$: the class $[\Gamma]$ is in R hence Γ is contained in $\mathrm{locus}(R)$, hence in Z. This proves a).

Assume $\mathrm{locus}(R) \neq X$. Then c_R is birational and M_R is nef and big. As in the proof of Corollary 4.14, for $m \gg 0$, $mM_R - H$ is linearly equivalent to an effective divisor D. A nonzero element in R has negative intersection with D, hence with some irreducible component D' of D. Any irreducible curve with class in R must then be contained in D', which therefore contains the locus of R. This implies b).

Assume now that x is general in Z and pick a rational curve Γ in Z through x with class in R and minimal (positive) $(-K_X)$-degree. Let $f : \mathbf{P}^1_{\mathbf{k}} \to \Gamma \subset X$ be the normalization, with $f(0) = x$.

Let T be a component of $\mathrm{Mor}(\mathbf{P}^1_{\mathbf{k}}, X)$ passing through $[f]$ and let $e_0 : T \to X$ be the map $t \mapsto f_t(0)$. By (6.2), T has dimension at least $\dim(X) + 1$. Each curve $f_t(\mathbf{P}^1_{\mathbf{k}})$ has same class as Γ hence is contained in Z. In particular, $e_0(T) \subset Z$ and for any component T_x of $e_0^{-1}(x)$, we have

$$\begin{aligned} \dim(Z) &\geq \dim(T) - \dim(T_x) \\ &\geq \dim(X) + 1 - \dim(T_x). \end{aligned} \tag{8.3}$$

Consider the evaluation $e_\infty : T_x \to X$ and let $y \in X$. *If $e_\infty^{-1}(y)$ has dimension at least* 2*,* Proposition 7.3 implies that Γ is numerically equivalent to a connected effective rational nonintegral 1-cycle $\sum_i a_i \Gamma_i$ passing through x and y. Since R is extremal, each $[\Gamma_i]$ must be in R, hence $0 < (-K_X \cdot \Gamma_i) < (-K_X \cdot \Gamma)$ for each i. This contradicts the choice of Γ.

It follows that the fibers of e_∞ have dimension at most 1. Since the curve $f_t(\mathbf{P}^1_{\mathbf{k}})$, for $t \in T_x$, passes through x hence has same image as x by c_R,

$$e_\infty(T_x) = \bigcup_{t \in T_x} \{f_t(\infty)\} = \bigcup_{t \in T_x} f_t(\mathbf{P}^1_{\mathbf{k}})$$

is irreducible and contained in the fiber $c_R^{-1}(c_R(x))$. We get

$$\dim_x(c_R^{-1}(c_R(x))) \geq \dim(\overline{e_\infty(T_x)}) \geq \dim(T_x) - 1. \tag{8.4}$$

Since the left-hand side is $\dim(Z) - \dim(c_R(Z))$, item c) follows from (8.3). □

8.8. Length of an extremal ray. Inequality (6.2) actually yields

$$\dim(Z) \geq \dim(X) + (-K_X \cdot \Gamma) - \dim(T_x)$$

instead of (8.3), for any rational curve Γ contained in the fiber of c_R through x. The integer

$$\ell(R) = \min\{(-K_X \cdot \Gamma) \mid \Gamma \text{ rational curve on } X \text{ with class in } R\}$$

is called the *length* of the extremal ray R. Together with (8.4), we get the following improvement of Proposition 8.7.c), due to Wiśniewski: any positive-dimensional irreducible component F of a fiber of c_R satisfies

$$\begin{aligned} \dim(F) &\geq \dim(T_x) - 1 \\ &\geq \dim(X) + \ell(R) - \dim(\mathrm{locus}(R)) - 1 \\ &= \mathrm{codim}(\mathrm{locus}(R)) + \ell(R) - 1, \end{aligned} \tag{8.5}$$

and F is covered by rational curves of $(-K_X)$-degree at most $\dim(F) + 1 - \mathrm{codim}(\mathrm{locus}(R))$.

8.4 Fiber contractions

Let X be a smooth complex projective variety and let R be a K_X-negative extremal ray with contraction $c_R : X \twoheadrightarrow Y$ *of fiber type,* i.e., $\dim(Y) < \dim(X)$. It follows from Proposition 8.7.a) that X *is covered by rational curves* (contained in fibers of c_R). Moreover, a general fiber F of c_R is smooth and $-K_F = (-K_X)|_F$ is ample (Remark 8.5.3)): F is a Fano variety as defined in §7.2.

The normal variety Y may be singular, but not too much. Recall that a variety is *locally factorial* if its local rings are unique factorization domains. This is equivalent to saying that all Weil divisors are Cartier divisors.

Proposition 8.9 *Let X be a smooth complex projective variety and let R be a K_X-negative extremal ray. If the contraction $c_R : X \twoheadrightarrow Y$ is of fiber type, Y is locally factorial.*

PROOF. Let C be an irreducible curve whose class generates R (Theorem 8.1). Let D be a prime Weil divisor on Y. Let c_R^0 be the restriction of c_R to $c_R^{-1}(Y_{\mathrm{reg}})$ and let D_X be the closure in X of $(c_R^0)^*(D \cap Y_{\mathrm{reg}})$.

The Cartier divisor D_X is disjoint from a general fiber of c_R hence has intersection 0 with C. By Corollary 8.4.b), there exists a Cartier divisor D_Y on Y such that $D_X \underset{\mathrm{lin}}{\equiv} c_R^* D_Y$. Since $c_{R*}\mathscr{O}_X \simeq \mathscr{O}_Y$, by the projection formula, the Weil divisors D and D_Y are linearly equivalent on Y_{reg} hence on Y ([H1], Proposition II.6.5.(b)). This proves that Y is locally factorial. □

Example 8.10 (A projective bundle is a fiber contraction) Let $\mathscr{E}$ be a locally free sheaf of rank r over a smooth projective variety Y and let $X = \mathbf{P}(\mathscr{E})$,[1] with projection $\pi : X \to Y$. If ξ is the class of the invertible sheaf $\mathscr{O}_X(1)$, we have

$$K_X = -r\xi + \pi^*(K_Y + \det(\mathscr{E})).$$

If L is a line contained in a fiber of π, we have $(K_X \cdot L) = -r$. The class $[L]$ spans a K_X-negative ray whose contraction is π: indeed, a curve is contracted by π if and only if it is numerically equivalent to a multiple of L (by Proposition 4.21.a), this implies that the ray spanned by $[L]$ is extremal).

Example 8.11 (A fiber contraction which is not a projective bundle) Let C be a smooth curve of genus g, let d be a positive integer, and let $J^d(C)$ be the Jacobian of C which parametrizes isomorphism classes of invertible sheaves of degree d on C.

Let C_d be the symmetric product of d copies of C; the Abel-Jacobi map $\pi_d : C_d \to J^d(C)$ is a $\mathbf{P}^{d-g}$-bundle for $d \geq 2g-1$ hence is the contraction of a K_{C_d}-negative extremal ray by 8.10. All fibers of π_d are projective spaces. If L_d is a line in a fiber, we have

$$(K_{C_d} \cdot L_d) = g - d - 1.$$

Indeed, the formula holds for $d \geq 2g-1$ by 8.10. Assume it holds for d; use a point of C to get an embedding $\iota : C_{d-1} \to C_d$. Then $(\iota^* C_{d-1} \cdot L_d) = 1$ and the adjunction formula yields

$$\begin{aligned}(K_{C_{d-1}} \cdot L_{d-1}) &= (\iota^*(K_{C_d} + C_{d-1}) \cdot L_{d-1}) \\ &= ((K_{C_d} + C_{d-1}) \cdot \iota_* L_{d-1}) \\ &= ((K_{C_d} + C_{d-1}) \cdot L_d), \\ &= (g-d-1)+1,\end{aligned}$$

which proves the formula by descending induction on d.

It follows that for $d \geq g$, the (surjective) map π_d is the contraction of the K_{C_d}-negative extremal ray $\mathbf{R}^+[L_d]$. It is a fiber contraction for $d > g$. For $d = g+1$, the generic fiber is $\mathbf{P}^1_{\mathbf{k}}$, but there are larger-dimensional fibers when $g \geq 3$, so the contraction is not a projective bundle.

8.5 Divisorial contractions

Let X be a smooth complex projective variety and let R be a K_X-negative extremal ray whose contraction $c_R : X \twoheadrightarrow Y$ is *divisorial.* It follows from Proposition 8.7.b) and its proof that the locus of R is an irreducible divisor E such that $E \cdot z < 0$ for all $z \in R - \{0\}$.

Again, Y may be singular (see Example 8.16), but not too much. We say that a scheme is *locally* $\mathbf{Q}$*-factorial* if any Weil divisor has a nonzero multiple which is a Cartier divisor. One can still intersect any Weil divisor D with a curve C on such a variety: choose a positive integer m such that mD is a Cartier divisor and set

$$(D \cdot C) = \frac{1}{m} \deg \mathscr{O}_C(mD).$$

This number is however only rational (see 3.20).

[1] As usual, we follow Grothendieck's notation: for a locally free sheaf $\mathscr{E}$, the projectivization $\mathbf{P}(\mathscr{E})$ is the space of *hyperplanes* in the fibers of $\mathscr{E}$.

Proposition 8.12 *Let X be a smooth complex projective variety and let R be a K_X-negative extremal ray. If the contraction $c_R : X \twoheadrightarrow Y$ is divisorial, Y is locally $\mathbf{Q}$-factorial.*

PROOF. Let C be an irreducible curve whose class generates R (Theorem 8.1). Let D be a prime Weil divisor on Y. Let $c_R^0 : c_R^{-1}(Y_{\text{reg}}) \to Y_{\text{reg}}$ be the morphism induces by c_R and let D_X be the closure in X of $c_R^{0*}(D \cap Y_{\text{reg}})$.

Let E be the exceptional locus of c_R. Since $(E \cdot C) \neq 0$, there exist integers $a \neq 0$ and b such that $aD_X + bE$ has intersection 0 with C. By Corollary 8.4.b), there exists a Cartier divisor D_Y on Y such that $aD_X + bE \underset{\text{lin}}{\equiv} c_R^* D_Y$.

Lemma 8.13 *Let X and Y be varieties, with Y normal, and let $\pi : X \to Y$ be a proper birational morphism. Let F an effective Cartier divisor on X whose support is contained in the exceptional locus of π. We have*

$$\pi_* \mathscr{O}_X(F) \simeq \mathscr{O}_Y.$$

PROOF. Since this is a statement which is local on Y, it is enough to prove $H^0(Y, \mathscr{O}_Y) \simeq H^0(Y, \pi_* \mathscr{O}_X(F))$ when Y is affine. By Zariski's Main Theorem, we have $H^0(Y, \mathscr{O}_Y) \simeq H^0(Y, \pi_* \mathscr{O}_X) \simeq H^0(X, \mathscr{O}_X)$, hence

$$H^0(Y, \mathscr{O}_Y) \simeq H^0(X, \mathscr{O}_X) \subset H^0(X, \mathscr{O}_X(F)) \subset H^0(X - E, \mathscr{O}_X(F))$$

and

$$H^0(X - E, \mathscr{O}_X(F)) \simeq H^0(X - E, \mathscr{O}_X) \simeq H^0(Y - \pi(E), \mathscr{O}_Y) \simeq H^0(Y, \mathscr{O}_Y),$$

the last isomorphism holding because Y is normal and $\pi(E)$ has codimension at least 2 in Y (8.6 and [H1], Exercise III.3.5). All these spaces are therefore isomorphic, hence the lemma. □

Using the lemma, we get:

$$\begin{aligned} \mathscr{O}_{Y_{\text{reg}}}(D_Y) &\simeq c^0_{R*} \mathscr{O}_{c_R^{-1}(Y_{\text{reg}})}(aD_X + bE) \\ &\simeq \mathscr{O}_{Y_{\text{reg}}}(aD) \otimes c^0_{R*} \mathscr{O}_{X^0}(bE) \\ &\simeq \mathscr{O}_{Y_{\text{reg}}}(aD), \end{aligned}$$

hence the Weil divisors aD and D_Y are linearly equivalent on Y. It follows that Y is locally $\mathbf{Q}$-factorial. □

Example 8.14 (A smooth blow-up is a divisorial contraction) Let Y be a smooth projective variety, let Z be a smooth subvariety of Y of codimension c, and let $\pi : X \to Y$ be the blow-up of Z, with exceptional divisor E. We have ([H1], Exercise II.8.5.(b))

$$K_X = \pi^* K_Y + (c - 1)E.$$

Any fiber F of $E \to Z$ is isomorphic to $\mathbf{P}^{c-1}$, and $\mathscr{O}_F(E)$ is isomorphic to $\mathscr{O}_F(-1)$. If L is a line contained in F, we have $(K_X \cdot L) = -(c-1)$; the class $[L]$ therefore spans a K_X-negative ray whose contraction is π: a curve is contracted by π if and only if it lies in a fiber of $E \to Z$, hence is numerically equivalent to a multiple of L.

Example 8.15 (A divisorial contraction which is not a smooth blow-up) We keep the notation of Example 8.11. The (surjective) map $\pi_g : C_g \twoheadrightarrow J^g(C)$ is the contraction of the K_{C_g}-negative extremal ray $\mathbf{R}^+[L_g]$. Its locus is, by Riemann-Roch, the divisor

$$\{D \in C_g \mid h^0(C, K_C - D) > 0\}$$

and its image in $J^g(C)$ has dimension $g - 2$. The general fiber over this image is $\mathbf{P}^1_{\mathbf{k}}$, but there are bigger fibers when $g \geq 6$, because the curve C has a g^1_{g-2}, and the contraction is not a smooth blow-up.

Example 8.16 (A divisorial contraction with singular image) Let Z be a smooth projective threefold and let C be an irreducible curve in Z whose only singularity is a node. The blow-up Y of Z along C is normal and its only singularity is an ordinary double point q. This is checked by a local calculation: locally analytically, the ideal of C is generated by xy and z, where x, y, z form a system of parameters. The blow-up is

$$\{((x,y,z),[u,v]) \in \mathbf{A}^3_{\mathbf{k}} \times \mathbf{P}^1_{\mathbf{k}} \mid xyv = zu\}.$$

It is smooth except at the point $q = ((0,0,0),[0,1])$. The exceptional divisor is the $\mathbf{P}^1_{\mathbf{k}}$-bundle over C with local equations $xy = z = 0$.

The blow-up X of Y at q is smooth. It contains the proper transform E of the exceptional divisor of Y and an exceptional divisor Q, which is a smooth quadric. The intersection $E \cap Q$ is the union of two lines L_1 and L_2 belonging to the two different rulings of Q. Let $\tilde{E} \to E$ and $\tilde{C} \to C$ be the normalizations; each fiber of $\tilde{E} \to \tilde{C}$ is a smooth rational curve, except over the preimages of the node of C, where it is the union of two rational curves meeting transversally. One of these curves maps to L_i, the other one to the same rational curve L. *It follows that L_1 and L_2 are algebraically, hence numerically, equivalent on X*; they have the same class ℓ.

Any curve contracted by the blow-up $\pi : X \to Y$ is contained in Q hence its class is a multiple of ℓ. A local calculation shows that $\mathscr{O}_Q(K_X)$ is of type $(-1,-1)$, hence $K_X \cdot \ell = -1$. The ray $\mathbf{R}^+\ell$ is K_X-negative and its (divisorial) contraction is π (hence $\mathbf{R}^+\ell$ is extremal).[2]

8.6 Small contractions and flips

Let X be a smooth complex projective variety and let R be a K_X-negative extremal ray whose contraction $c_R : X \twoheadrightarrow Y$ is *small.*

The following proposition shows that Y is very singular: it is not even locally $\mathbf{Q}$-factorial, which means that one cannot do intersection theory on Y.

Proposition 8.17 *Let Y be a normal and locally $\mathbf{Q}$-factorial variety and let $\pi : X \to Y$ be a birational proper morphism. Every irreducible component of the exceptional locus of π has codimension 1 in X.*

PROOF. This can be seen as follows. Let E be the exceptional locus of π and let $x \in E$ and $y = \pi(x)$; identify the quotient fields $K(Y)$ and $K(X)$ by the isomorphism π^*, so that $\mathscr{O}_{Y,y}$ is a proper subring of $\mathscr{O}_{X,x}$. Let t be an element of $\mathfrak{m}_{X,x}$ not in $\mathscr{O}_{Y,y}$, and write its divisor as the difference of two effective (Weil) divisors D' and D'' on Y without common components. There exists a positive integer m such that mD' and mD'' are Cartier divisors, hence define elements u and v of $\mathscr{O}_{Y,y}$ such that $t^m = \frac{u}{v}$. Both are actually in $\mathfrak{m}_{Y,y}$: v because t^m is not in $\mathscr{O}_{Y,y}$ (otherwise, t would be since $\mathscr{O}_{Y,y}$ is integrally closed), and $u = t^m v$ because it is in $\mathfrak{m}_{X,x} \cap \mathscr{O}_{Y,y} = \mathfrak{m}_{Y,y}$. But $u = v = 0$ defines a subscheme Z of Y containing y of codimension 2 in some neighborhood of y (it is the intersection of the codimension 1 subschemes mD' and mD''), whereas $\pi^{-1}(Z)$ is defined by $t^m v = v = 0$ hence by the sole equation $v = 0$: it has codimension 1 in X, hence is contained in E. It follows that there is a codimension 1 component of E through every point of E, which proves the proposition. □

Fibers of c_R contained in locus(R) have dimension at least 2 (see (8.5)) and

$$\dim(X) \geq \dim(c_R(\operatorname{locus}(R))) + 4$$

(Proposition 8.7.c)). In particular, there are no small extremal contractions on smooth varieties in dimension 3 (see Example 8.20 for an example with a locally $\mathbf{Q}$-factorial threefold).

Since it is impossible to do anything useful with Y, Mori's idea is that there should exist instead another (mildly singular) projective variety X^+ with a small contraction $c^+ : X^+ \to Y$ such that K_{X^+} has positive degree on curves contracted by c^+. The map c^+ (or sometimes the resulting rational map

[2]This situation is very subtle: although the completion of the local ring $\mathscr{O}_{Y,q}$ is not factorial (it is isomorphic to $\mathbf{k}[[x,y,z,u]]/(xy - zu)$, and the equality $xy = zu$ is a decomposition in a product of irreducibles in two different ways) the fact that L_1 is numerically equivalent to L_2 implies that the ring $\mathscr{O}_{Y,q}$ is factorial (see [Mo2], (3.31)).

$(c^+)^{-1} \circ c : X \dashrightarrow X^+)$ is called a *flip* (see Definition 8.18 for more details and Example 8.20 for an example).

Definition 8.18 *Let $c : X \twoheadrightarrow Y$ be a small contraction between normal projective varieties. Assume that K_X is $\mathbf{Q}$-Cartier and $-K_X$ is c-ample. A* flip *of c is a small contraction $c^+ : X^+ \to Y$ such that*

- *X^+ is a projective normal variety;*
- *K_{X^+} is $\mathbf{Q}$-Cartier and c^+-ample.*

The main problem here is the *existence* of a flip of the small contraction of a negative extremal ray, which has only been shown very recently ([BCHM]; see also [Dr], cor. 2.5).

Proposition 8.19 *Let X be a locally $\mathbf{Q}$-factorial complex projective variety and let $c : X \twoheadrightarrow Y$ be a* small *contraction of a K_X-negative extremal ray R. If the flip $X^+ \twoheadrightarrow Y$ exists, the variety X^+ is locally $\mathbf{Q}$-factorial with Picard number ρ_X.*

Proof. The composition $\varphi = c^{-1} \circ c^+ : X^+ \dashrightarrow X$ is an isomorphism in codimension 1, hence induces an isomorphism between the Weil divisor class groups of X and X^+ ([H1], Proposition II.6.5.(b)). Let D^+ be a Weil divisor on X^+ and let D be the corresponding Weil divisor on X. Let C be an irreducible curve whose class generates R and let r be a rational number such that $((D + rK_X) \cdot C) = 0$ and let m be an integer such that mD, mrK_X, and mrK_{X^+} are Cartier divisors (the fact that K_{X^+} is $\mathbf{Q}$-Cartier is part of the definition of a flip!). By Corollary 8.4.b), there exists a Cartier divisor D_Y on Y such that $m(D + rK_X) \underset{\text{lin}}{\equiv} c^* D_Y$, and

$$mD^+ = \varphi^*(mD) \underset{\text{lin}}{\equiv} (c^+)^* D_Y - \varphi^*(mrK_X) \underset{\text{lin}}{\equiv} (c^+)^* D_Y - mrK_{X^+}$$

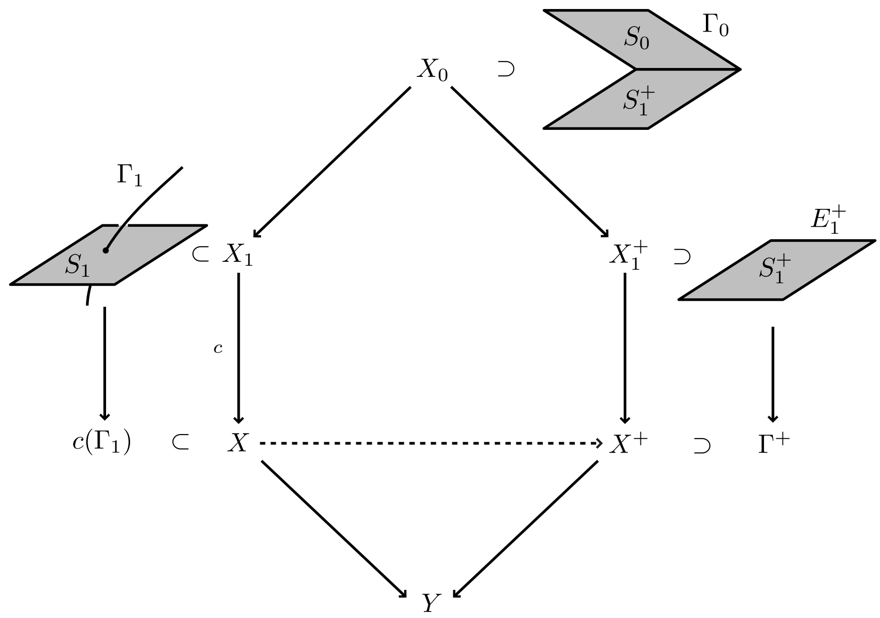
is a Cartier divisor. This proves that X^+ is locally $\mathbf{Q}$-factorial. Moreover, φ^* induces an isomorphism between $N^1(X)_{\mathbf{R}}$ and $N^1(X^+)_{\mathbf{R}}$, hence the Picard numbers are the same. □

Contrary to the case of a divisorial contraction, the Picard number stays the same after a flip. So the second main problem is the *termination* of flips: can there exist an infinite chain of flips? It is conjectured that the answer is negative, but this is still unknown in general.

Example 8.20 (A flip in dimension 3) We start from the end product of the flip, which is a smooth complex variety X^+ containing a smooth rational curve Γ^+ with normal bundle $\mathscr{O}(-1) \oplus \mathscr{O}(-2)$, such that the K_{X^+}*-positive* ray $\mathbf{R}^+[\Gamma^+]$ can be contracted by a morphism $X^+ \to Y$.[3]

[3] Take for example $X^+ = \mathbf{P}(\mathscr{O}_{\mathbf{P}^1_{\mathbf{k}}} \oplus \mathscr{O}_{\mathbf{P}^1_{\mathbf{k}}}(1) \oplus \mathscr{O}_{\mathbf{P}^1_{\mathbf{k}}}(2))$ and take for Γ^+ the image of the section of the projection $X^+ \to \mathbf{P}^1_{\mathbf{k}}$ corresponding to the trivial quotient of $\mathscr{O}_{\mathbf{P}^1_{\mathbf{k}}} \oplus \mathscr{O}_{\mathbf{P}^1_{\mathbf{k}}}(1) \oplus \mathscr{O}_{\mathbf{P}^1_{\mathbf{k}}}(2)$. It is contracted by the base-point-free linear system $|\mathscr{O}_{X^+}(1)|$.

Let us first summarize all the notation in the following diagram.



A flip

Let $X_1^+ \to X^+$ be the blow-up of Γ^+. The exceptional divisor is the ruled surface

$$S_1^+ = \mathbf{P}(N^*_{\Gamma^+/X^+}) = \mathbf{P}(\mathscr{O}_{\mathbf{P}^1_{\mathbf{k}}} \oplus \mathscr{O}_{\mathbf{P}^1_{\mathbf{k}}}(1))$$

which has a section E_1^+ with self-intersection -1, whose normal bundle in X_1^+ can be shown to be isomorphic to $\mathscr{O}(-1) \oplus \mathscr{O}(-1)$. Blow-up the curve E_1^+ in X_1^+ to get a smooth threefold X_0; the exceptional divisor is now the ruled surface $S_0 = \mathbf{P}^1_{\mathbf{k}} \times \mathbf{P}^1_{\mathbf{k}}$, and its normal bundle is of type $(-1,-1)$. Let Γ_0 be a fiber of $S_0 \to E_1^+$; a section is given by the intersection of the strict transform of S_1^+ (which we will still denote by S_1^+) with S_0, which we will also denote by E_1^+.

The K_{X_0}-negative ray $\mathbf{R}^+[E_1^+]$ is extremal. Indeed, the relative cone of the morphism $X_0 \to X_1^+ \to X^+ \to Y$, generated by $[E_1^+]$, $[\Gamma_0]$, and the class of the strict transform F_0 of a fiber of $S_1^+ \to \Gamma^+$, is extremal by Proposition 4.21.a). If $\mathbf{R}^+[E_1^+]$ is *not* extremal, one can therefore write $[E_1^+] = a[F_0] + b[\Gamma_0]$ with a and b positive. Intersecting with S_0, we get $-1 = a - b$; intersecting with (the strict transform of) S_1^+, we get the relation $-1 = -a + b$, which is absurd.

One checks that its contraction is the blow-up of a smooth threefold X_1 along a smooth rational curve Γ_1 with normal bundle $\mathscr{O}(-1) \oplus \mathscr{O}(-1)$, so that $(K_{X_1} \cdot \Gamma_1) = 0$; the exceptional curve E_1^+ of S_1^+ gets blown-down so S_1^+ maps onto a projective plane S_1.

To compute the normal bundle to S_1 in X_1, we restrict to a line F_1 in S_1 which does not meet Γ_1. This restriction is the same as the restriction of $N_{S_1^+/X_0}$ to a line in S_1^+ disjoint from E_1^+, and this can be shown to have degree -2. Hence $N_{S_1/X_1} \simeq \mathscr{O}(-2)$ and $(K_{X_1})|_{S_1} \simeq \mathscr{O}_{S_1}(-1)$.

In particular, $(K_{X_1} \cdot F_1) = -1$, and the extremal ray $\mathbf{R}^+[F_1]$ can be contracted by $c : X_1 \to X$. A local study shows that locally analytically at $c(S_1)$, the variety X is isomorphic to the quotient of $\mathbf{A}^3_{\mathbf{k}}$ by the involution $x \mapsto -x$. The corresponding complete local ring is *not* factorial, but its Weil divisor class group has order 2. It follows that $2K_X$ is a Cartier divisor. Write $K_{X_1} = c^*K_X + a[S_1]$, for some rational a. By restricting to S_1, we get $a = 1/2$, hence $(K_X \cdot c(\Gamma_1)) = -1/2$.

The morphism $X \to Y$ is the contraction of the ray $\mathbf{R}^+[c(\Gamma_1)]$, which is therefore extremal. The corresponding flip is the composition $X \dashrightarrow X^+$: the "K_X-negative" rational curve $c(\Gamma_1)$ is replaced with the "K_{X^+}-positive" rational curve Γ^+.

Example 8.21 (A flip in dimension 4) We discuss in more details the example of 1.9. Recall that we started from the Segre embedding $\mathbf{P}^1_{\mathbf{k}} \times \mathbf{P}^2_{\mathbf{k}} \subset \mathbf{P}^5_{\mathbf{k}}$, then defined $Y \subset \mathbf{P}^6$ as the cone over $\mathbf{P}^1_{\mathbf{k}} \times \mathbf{P}^2_{\mathbf{k}}$, and

$\varepsilon : X \to Y$ as the blow-up of the vertex of Y, with exceptional divisor $E \subset X$. There is a projection $\pi : X \to \mathbf{P}^1_{\mathbf{k}} \times \mathbf{P}^2_{\mathbf{k}}$ which identifies X with $\mathbf{P}(\mathscr{O}_{\mathbf{P}^1_{\mathbf{k}} \times \mathbf{P}^2_{\mathbf{k}}} \oplus \mathscr{O}_{\mathbf{P}^1_{\mathbf{k}} \times \mathbf{P}^2_{\mathbf{k}}}(1,1))$ and E is a section (we write $\mathscr{O}_{\mathbf{P}^1_{\mathbf{k}} \times \mathbf{P}^2_{\mathbf{k}}}(a,b)$ for $p_1^*\mathscr{O}_{\mathbf{P}^1_{\mathbf{k}}}(a) \otimes p_2^*\mathscr{O}_{\mathbf{P}^2_{\mathbf{k}}}(b)$).

Let ℓ_1 be the class in X of the curve $\{\star\}\times\{\text{line}\} \subset E \subset X$, let ℓ_2 be the class in X of $\mathbf{P}^1_{\mathbf{k}}\times\{\star\} \subset E \subset X$, and let ℓ_0 be the class of a fiber of π. The Picard number of X is 3 and

$$N_1(X)_{\mathbf{R}} = \mathbf{R}\ell_0 \oplus \mathbf{R}\ell_1 \oplus \mathbf{R}\ell_2.$$

For $i \in \{1,2\}$, let h_i be the nef class of $\pi^*p_i^*\mathscr{O}_{\mathbf{P}^i}(1)$. Since $\mathscr{O}_E(E) \simeq \mathscr{O}_E(-1,-1)$, we have the following multiplication table

$$\begin{array}{ccc} h_1 \cdot \ell_1 = 0, & h_1 \cdot \ell_2 = 1, & h_1 \cdot \ell_0 = 0, \\ h_2 \cdot \ell_1 = 1, & h_2 \cdot \ell_2 = 0, & h_2 \cdot \ell_0 = 0, \\ [E] \cdot \ell_1 = -1, & [E] \cdot \ell_2 = -1, & [E] \cdot \ell_0 = 1. \end{array}$$

Let $a_0\ell_0 + a_1\ell_1 + a_2\ell_2$ be the class of an irreducible curve C contained in X but not in E. We have

$$a_1 = h_2 \cdot C \geq 0 \quad , \quad a_2 = h_1 \cdot C \geq 0 \quad , \quad a_0 - a_1 - a_2 = (E \cdot C) \geq 0$$

hence, since any curve in E is algebraically equivalent to some nonnegative linear combination of ℓ_1 and ℓ_2, we obtain

$$\mathrm{NE}(X_{r\cdot s}) = \overline{\mathrm{NE}}(X_{r\cdot s}) = \mathbf{R}^+\ell_0 + \mathbf{R}^+\ell_1 + \mathbf{R}^+\ell_2 \tag{8.6}$$

and the rays $R_i = \mathbf{R}^+\ell_i$ are extremal. Furthermore, it follows from Example 7.4.2) that X is a Fano variety, hence all extremal subcones of X can be contracted (at least in characteristic zero).

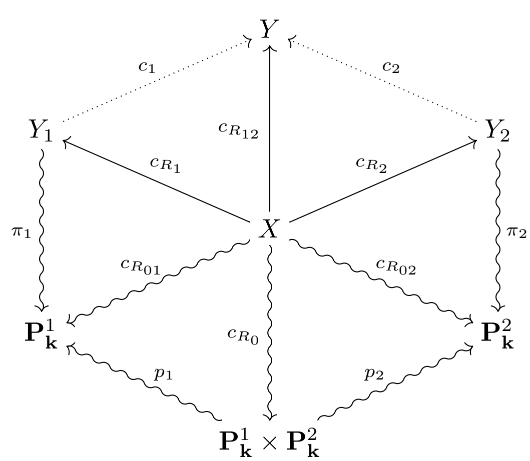
Set $R_{ij} = R_i + R_j$. The contraction of R_0 is π and the contraction of R_{12} is ε. It follows easily that for $i \in \{1,2\}$, the contraction of R_{0i} is $p_i \circ \pi : X \to \mathbf{P}^i$ and this map must factor through the contraction of R_i. Note that the divisor E is contained in the locus of R_i. Let us define the fourfolds

$$\pi_1 : Y_1 := \mathbf{P}(\mathscr{O}_{\mathbf{P}^1_{\mathbf{k}}} \oplus \mathscr{O}_{\mathbf{P}^1_{\mathbf{k}}}(1)^{\oplus 3}) \to \mathbf{P}^1_{\mathbf{k}}$$

and

$$\pi_2 : Y_2 := \mathbf{P}(\mathscr{O}_{\mathbf{P}^2_{\mathbf{k}}} \oplus \mathscr{O}_{\mathbf{P}^2_{\mathbf{k}}}(1)^{\oplus 2}) \to \mathbf{P}^2_{\mathbf{k}}.$$

Then there is a map $X \to Y_i$ which is the contraction c_{R_i}. The divisor E is therefore the locus of R_i and is mapped onto the image P_i of the section of π_i corresponding to the trivial quotient of the defining locally free sheaf on $\mathbf{P}^i$. All contractions are displayed in the following commutative diagram:

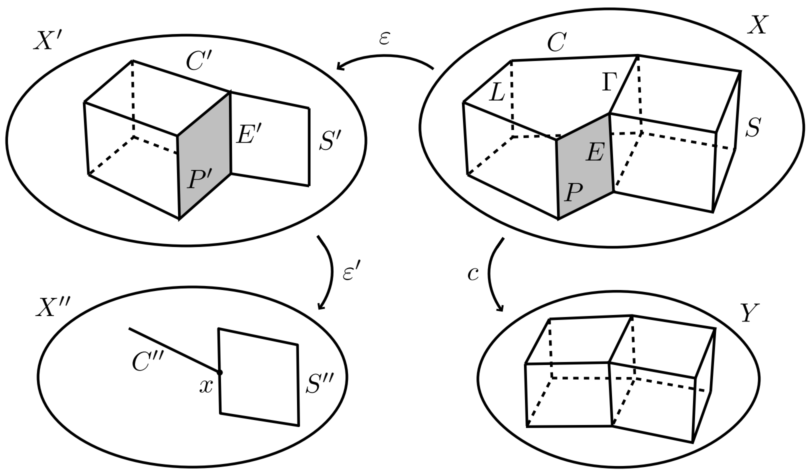


Straight arrows are divisorial contractions, wiggly arrows are contractions of fiber type, and dotted arrows are small contractions (the map c_i contracts P_i to the vertex of Y).

By Example 7.4.2) again, Y_2 is a Fano variety, hence c_2 is the contraction of a K_{Y_2}-negative extremal ray (which gives an example where there is equality in Proposition 8.7.c)). However, one checks that the ray contracted by c_1 is K_{Y_1}-positive. It follows that c_1 *is the flip of* c_2.

Example 8.22 (A small contraction with disconnected exceptional locus (Kawamata)) Start from a smooth complex fourfold X'' that contains a smooth curve C'' and a smooth surface S'' meeting transversely at points $x_1, \dots, x_r$. Let $\varepsilon' : X' \to X''$ be the blow-up of C''. The exceptional divisor C' is a smooth threefold which is a $\mathbf{P}^2_{\mathbf{k}}$-bundle over C''. The strict transform S' of S'' is the blow-up of S'' at the points $x_1, \dots, x_r$; let $E'_1, \dots, E'_r$ be the corresponding exceptional curves and let $P'_1, \dots, P'_r$ be the corresponding $\mathbf{P}^2_{\mathbf{k}}$ that contain them, i.e., $P'_i = {\varepsilon'}^{-1}(x_i)$. Let $\varepsilon : X \to X'$ be the blow-up of S'. The exceptional divisor S is a smooth threefold which is a $\mathbf{P}^1_{\mathbf{k}}$-bundle over S'; let Γ_i be the fiber over a point of E'_i and let P_i be the strict transform of P'_i. Finally, let L be a line in one of the $\mathbf{P}^2_{\mathbf{k}}$ in the inverse image C of C'.

For $r = 1$, the picture is something like the following diagram.



A small contraction

The curves Γ_i are all algebraically equivalent in X (they are fibers of the $\mathbf{P}^1_{\mathbf{k}}$-bundle $S \to S'$) hence have the same class $[\Gamma]$. Let $\alpha = \varepsilon' \circ \varepsilon$; the relative effective cone $\mathrm{NE}(\alpha)$ is generated by the classes $[\Gamma]$, $[L]$, and $[E_i]$. Since the vector space $N_1(X)_{\mathbf{R}}/\alpha^* N_1(X'')_{\mathbf{R}}$ has dimension 2, there must be a relation

$$E_i \underset{\text{num}}{\equiv} a_i L + b_i \Gamma.$$

One checks

$$(C \cdot E_i) = (C' \cdot E'_i) = -1 = (C' \cdot \varepsilon_*(L)) = (C \cdot L).$$

Moreover, $(C \cdot \Gamma) = 0$ (because Γ is contracted by ε'), $(S \cdot L) = 0$ (because S and L are disjoint), and $(S \cdot E_i) = 1$ (because S and P_i meets transversally in E_i). This implies $a_i = -b_i = 1$ and the E_i are all numerically equivalent to $L - \Gamma$. The relative cone $\mathrm{NE}(\alpha)$ is therefore generated by $[\Gamma]$ and $[L - \Gamma]$. Since it is an extremal subcone of $\mathrm{NE}(X)$, the class $[L - \Gamma]$ spans an extremal ray, which is moreover K_X-negative (one checks $(K_X \cdot (L - \Gamma)) = -1$), hence can be contracted (at least in characteristic zero). The corresponding contraction $X \to Y$ maps each P_i to a point. Its exceptional locus is the disjoint union $P_1 \sqcup \dots \sqcup P_r$.

8.7 The minimal model program

Let X be a smooth complex projective variety. We saw in §5.6 that when X is a surface, it has a smooth minimal model $X_{\min}$ obtained by contracting all exceptional curves on X. If X is covered by rational curves, this minimal model is not unique, and is either a ruled surface or $\mathbf{P}^2_{\mathbf{k}}$. Otherwise, the minimal model is unique and has nef canonical divisor.

In higher dimensions, Mori's idea is to try to simplify X by contracting K_X-negative extremal rays, hoping to end up with a variety X_0 which either has a contraction of fiber type (in which case X_0, hence also X, is covered by rational curves (see §8.4)) or has nef canonical divisor (hence no K_{X_0}-negative extremal rays). Three main problems arise:

- the end-product of a contraction is usually singular. This means that to continue Mori's program, *we must allow singularities.* This is very bad from our point of view, since most of our methods do not work on singular varieties. Completely different methods are required.
- One must determine what kind of singularities must be allowed. But in any event, the singularities of the target of a small contraction are too severe and one needs to perform a flip. So we have the problem of *existence of flips.*
- One needs to know that the process terminates. In case of surfaces, we used that the Picard number decreases when an exceptional curve is contracted. This is still the case for a fiber-type or divisorial contraction, but not for a flip! So we have the additional problem of *termination of flips:* do there exist infinite sequences of flips?

The first two problems have been overcome: the first one by the introduction of cohomological methods to prove the cone theorem on (mildly) singular varieties, the second one more recently in [BCHM] (see [Dr], cor. 2.5). The third point is still open in full generality (see however [Dr], cor. 2.8).

8.8 Minimal models

Let $\mathscr{C}$ be a birational equivalence class of smooth projective varieties, modulo isomorphisms. One aims at finding a "simplest" member in $\mathscr{C}$. If X_0 and X_1 are members of $\mathscr{C}$, we write $X_1 \preceq X_0$ if there is a birational *morphism* $X_0 \to X_1$. This defines an ordering on $\mathscr{C}$ (use Exercise 4.8.5)).

We explain here one reason why we are interested in varieties with nef canonical bundles (and why we called them *minimal models*), by proving:

- any member of $\mathscr{C}$ with nef canonical bundle is minimal (Proposition 8.25);
- any member of $\mathscr{C}$ which contains no rational curves is the smallest element of $\mathscr{C}$ (Corollary 8.24).

However, here are a few warnings about minimal models:

- a minimal model can only exist if the variety is not covered by rational curves (Example 9.14);
- there exist smooth projective varieties which are not covered by rational curves but which are not birational to any smooth projective variety with nef canonical bundle;[4]
- in dimension at least 3, minimal models may not be unique, but any two are isomorphic in codimension 1 ([D1], 7.18).

Proposition 8.23 *Let X and Y be varieties, with X smooth, and let $\pi : Y \to X$ be a birational morphism. Any component of* $\operatorname{Exc}(\pi)$ *is birational to a product $\mathbf{P}^1_{\mathbf{k}} \times Z$, where π contracts the $\mathbf{P}^1_{\mathbf{k}}$-factor.*

In particular, if π is moreover projective, there is, through any point of $\operatorname{Exc}(\pi)$, a rational curve contracted by π (use Lemma 7.8).

PROOF. Let E be a component of $\operatorname{Exc}(\pi)$. Upon replacing Y with its normalization, we may assume that Y is smooth in codimension 1. Upon shrinking Y, we may also assume that Y is smooth and that $\operatorname{Exc}(\pi)$ is smooth, equal to E.

Let $U_0 = X - \operatorname{Sing}(\overline{\pi(E)})$ and let $V_1 = \pi^{-1}(U_0)$. The complement of V_1 in Y has codimension ≥ 2, V_1 and $E \cap V_1$ are smooth, and so is the closure in U_0 of the image of $E \cap V_1$. Let $\varepsilon_1 : X_1 \to U_0$ be its blow-up; by the universal property of blow-ups ([H1], Proposition II.7.14), since the ideal of $E \cap V_1$ in $\mathscr{O}_{V_1}$ is invertible, there exists a factorization

$$\pi|_{V_1} : V_1 \xrightarrow{\pi_1} X_1 \xrightarrow{\varepsilon_1} U_0 \subset X$$

[4]This is the case for any desingularization of the quotient X of an abelian variety of dimension 3 by the involution $x \mapsto -x$ ([U], 16.17); of course, a minimal model here is X itself, but it is singular.

where $\overline{\pi_1(E \cap V_1)}$ is contained in the support of the exceptional divisor of ε_1. If the codimension of $\overline{\pi_1(E \cap V_1)}$ in X_1 is at least 2, the divisor $E \cap V_1$ is contained in the exceptional locus of π_1 and, upon replacing V_1 by the complement V_2 of a closed subset of codimension at least 2 and X_1 by an open subset U_1, we may repeat the construction. After i steps, we get a factorization

$$\pi : V_i \xrightarrow{\pi_i} X_i \xrightarrow{\varepsilon_i} U_{i-1} \subset X_{i-1} \xrightarrow{\varepsilon_{i-1}} \cdots \xrightarrow{\varepsilon_2} U_1 \subset X_1 \xrightarrow{\varepsilon_1} U_0 \subset X$$

as long as the codimension of $\overline{\pi_{i-1}(E \cap V_{i-1})}$ in X_{i-1} is at least 2, where V_i is the complement in Y of a closed subset of codimension at least 2. Let $E_j \subset X_j$ be the exceptional divisor of ε_j. We have

$$\begin{aligned} K_{X_i} &= \varepsilon_i^* K_{U_{i-1}} + c_i E_i \\ &= (\varepsilon_1 \circ \cdots \circ \varepsilon_i)^* K_X + c_i E_i + c_{i-1} E_{i,i-1} + \cdots + c_1 E_{i,1}, \end{aligned}$$

where $E_{i,j}$ is the inverse image of E_j in X_i and

$$c_i = \operatorname{codim}_{X_{i-1}}(\overline{\pi_{i-1}(E \cap V_{i-1})}) - 1 > 0$$

([H1], Exercise II.8.5). Since π_i is birational, $\pi_i^* \mathscr{O}_{X_i}(K_{X_i})$ is a subsheaf of $\mathscr{O}_{V_i}(K_{V_i})$. Moreover, since $\pi_j(E \cap V_j)$ is contained in the support of E_j, the divisor $\pi_j^* E_j - E|_{V_j}$ is effective, hence so is $E_{i,j} - E|_{V_i}$.

It follows that $\mathscr{O}_Y(\pi^* K_X + (c_i + \cdots + c_1)E)|_{V_i}$ is a subsheaf of $\mathscr{O}_{V_i}(K_{V_i}) = \mathscr{O}_Y(K_Y)|_{V_i}$. Since Y is normal and the complement of V_i in Y has codimension at least 2, $\mathscr{O}_Y(\pi^* K_X + (c_i + \cdots + c_1)E)$ is also a subsheaf of $\mathscr{O}_Y(K_Y)$. Since there are no infinite ascending sequences of subsheaves of a coherent sheaf on a noetherian scheme, the process must terminate at some point: $\overline{\pi_i(E \cap V_i)}$ is a divisor in X_i for some i, hence $E \cap V_i$ is not contained in the exceptional locus of π_i (by 8.6 again). The morphism π_i then induces a dominant map between $E \cap V_i$ and E_i which, since, by Zariski's Main Theorem, the fibers of π are connected, must be birational. Since the latter is birationally isomorphic to $\mathbf{P}^{c_i - 1} \times (\pi_{i-1}(E \cap V_{i-1}))$, where ε_i contracts the $\mathbf{P}^{c_i-1}$-factor, this proves the proposition. □

Corollary 8.24 *Let Y and X be projective varieties. Assume that X is smooth and that Y contains no rational curves. Any rational map $X \dashrightarrow Y$ is defined everywhere.*

Proof. Let $X' \subset X \times Y$ be the graph of a rational map $\pi : X \dashrightarrow Y$ as defined in 5.17. The first projection induces a birational morphism $p : X' \to X$. Assume its exceptional locus $\operatorname{Exc}(p)$ is nonempty. By Proposition 8.23, there exists a rational curve on $\operatorname{Exc}(p)$ which is contracted by p. Since Y contains no rational curves, it must also be contracted by the second projection, which is absurd since it is contained in $X \times Y$. Hence $\operatorname{Exc}(p)$ is empty and π is defined everywhere. □

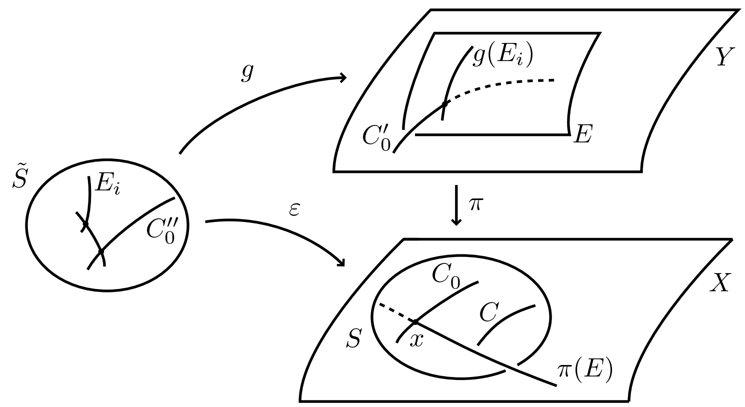
Under the hypotheses of the proposition, one can say more if Y also is smooth.

Proposition 8.25 *Let X and Y be smooth projective varieties and let $\pi : Y \to X$ be a birational morphism which is not an isomorphism. There exists a rational curve C on Y contracted by π such that $(K_Y \cdot C) < 0$.*

Proof. Let E be the exceptional locus of π; by 8.6, $\pi(E)$ has codimension at least 2 in X and $E = \pi^{-1}(\pi(E))$. Let x be a point of $\pi(E)$. By Bertini's theorem ([H1], Theorem II.8.18), a general hyperplane section of X passing through x is smooth and connected.

It follows that by taking $\dim(X) - 2$ hyperplane sections, we get a smooth surface S in X that meets $\pi(E)$ in a finite set containing x. Moreover, taking one more hyperplane section, we get on S a smooth curve

C_0 that meets $\pi(E)$ only at x and a smooth curve C that does not meet $\pi(E)$.



Construction of a rational curve $g(E_i)$ in the exceptional locus E of π

By construction,

$$(K_X \cdot C) = (K_X \cdot C_0).$$

One can write $K_Y \underset{\text{lin}}{\equiv} \pi^* K_X + R$, where the support of the divisor R is exactly E. Since the curve $C' = \pi^{-1}(C)$ does not meet E, we have

$$(K_Y \cdot C') = (K_X \cdot C).$$

On the other hand, since the strict transform

$$C_0' = \overline{\pi^{-1}(C_0 - \pi(E))}$$

of C_0 does meet $E = \pi^{-1}(\pi(E))$, we have

$$(K_Y \cdot C_0') = ((\pi^* K_X + R) \cdot C_0') > ((\pi^* K_X) \cdot C_0') = (K_X \cdot C_0)$$

hence

$$(K_Y \cdot C_0') > (K_Y \cdot C'). \tag{8.7}$$

The indeterminacies of the rational map $\pi^{-1} : S \dashrightarrow Y$ can be resolved (Theorem 5.18) by blowing-up a finite number of points of $S \cap \pi(E)$ to get a morphism

$$g : \tilde{S} \xrightarrow{\varepsilon} S \overset{\pi^{-1}}{\dashrightarrow} Y$$

whose image is the strict transform of S. The curve $C'' = \varepsilon^* C$ is irreducible and $g_* C'' = C'$; for C_0, we write

$$\varepsilon^* C_0 = C_0'' + \sum_i m_i E_i,$$

where the m_i are nonnegative integers, the E_i are exceptional divisors for ε (hence in particular rational curves), and $g_* C_0'' = C_0'$. Since C and C_0 are linearly equivalent on S, we have

$$C'' \underset{\text{lin}}{\equiv} C_0'' + \sum_i m_i E_i$$

on $\tilde{S}$ hence, by applying g_*,

$$C' \underset{\text{lin}}{\equiv} C_0' + \sum_i m_i (g_* E_i).$$

Taking intersections with K_Y, we get

$$(K_Y \cdot C') = (K_Y \cdot C_0') + \sum_i m_i (K_Y \cdot g_* E_i).$$

It follows from (8.7) that $(K_Y \cdot g_* E_i)$ is negative for some i. In particular, $g(E_i)$ is not a point hence is a rational curve on Y. Moreover, $\pi(g(E_i)) = \varepsilon(E_i) = \{x\}$ hence $g(E_i)$ is contracted by π. □

8.9 Exercises

1) Let X be a smooth projective variety and let $M_1, \dots, M_r$ be ample divisors on X. Show that $K_X + M_1 + \cdots + M_r$ is nef for all $r \geq \dim(X) + 1$ (*Hint:* use the cone theorem).

2) a) Let $X \to \mathbf{P}^2_{\mathbf{k}}$ be the blow-up of two distinct points. Determine the cone of curves of X, its extremal faces, and for each extremal face, describe its contraction.

b) Same questions for the blow-up of three noncolinear points.

3) Let V be a $\mathbf{k}$-vector space of dimension n and let $r \in \{1, \dots, n-1\}$. Let $G_r(V)$ be the Grassmanian that parametrizes vector subspaces of V of codimension r and set

$$X = \{(W, [u]) \in G_r(V) \times \mathbf{P}(\mathrm{End}(V)) \mid u(W) = 0\}.$$

a) Show that X is smooth irreducible of dimension $r(2n-r) - 1$, that $\mathrm{Pic}(X) \simeq \mathbf{Z}^2$, and that the projection $X \to G_r(V)$ is a K_X-negative extremal contraction.

b) Show that

$$Y = \{[u] \in \mathbf{P}(\mathrm{End}(V)) \mid \mathrm{rank}(u) \leq r\}$$

is irreducible of dimension $r(2n-r) - 1$. It can be proved that Y is normal. If $r \geq 2$, show that Y is not locally $\mathbf{Q}$-factorial and that $\mathrm{Pic}(Y) \simeq \mathbf{Z}[\mathscr{O}_Y(1)]$. What happens when $r = 1$?

4) Let X be a smooth complex projective Fano variety with Picard number ≥ 2. Assume that X has an extremal ray whose contraction $X \to Y$ maps a hypersurface $E \subset X$ to a point. Show that X also has an extremal contraction whose fibers are all of dimension ≤ 1 (*Hint:* consider a ray R such that $(E \cdot R) > 0$.)

5) Let X be a smooth complex projective variety of dimension n and let $\mathbf{R}^+ r_1, \dots, \mathbf{R}^+ r_s$ be distinct K_X-negative extremal rays, all of fiber type. Prove $s \leq n$ (*Hint:* show that each linear form $\ell_i(z) = z \cdot r_i$ on $N^1(X)_{\mathbf{R}}$ divides the polynomial $P(z) = (z^n)$.)

6) Let X be a smooth projective Fano variety of positive dimension n, let $f : \mathbf{P}^1_{\mathbf{k}} \to X$ be a (nonconstant) rational curve of $(-K_X)$-degree $\leq n+1$, let M_f be a component of $\mathrm{Mor}(\mathbf{P}^1_{\mathbf{k}}, X; 0 \mapsto f(0))$ containing $[f]$, and let

$$\mathrm{ev}_\infty : M_f \longrightarrow X$$

be the evaluation map at ∞. Assume that the $(-K_X)$-degree of any rational curve on X is $\geq (n+3)/2$.

a) Show that $Y_f := \mathrm{ev}(\mathbf{P}^1_{\mathbf{k}} \times M_f)$ is closed in X and that its dimension is at least $(n+1)/2$ (*Hint:* follow the proof of Proposition 8.7.c)).

b) Show that any curve contained in Y_f is numerically equivalent to a multiple of $f(\mathbf{P}^1_{\mathbf{k}})$ (*Hint:* use Proposition 5.5).

c) If $g : \mathbf{P}^1_{\mathbf{k}} \to X$ is another rational curve of $(-K_X)$-degree $\leq n+1$ such that $Y_f \cap Y_g \neq \varnothing$, show that the classes $[f(\mathbf{P}^1_{\mathbf{k}})]$ and $[g(\mathbf{P}^1_{\mathbf{k}})]$ are proportional in $N_1(X)_{\mathbf{Q}}$.

d) Conclude that $N_1(X)_{\mathbf{R}}$ has dimension 1 (*Hint:* use Theorem 7.5 to produce a g such that $Y_g = X$).

7) **Non-isomorphic minimal models in dimension 3.** Let S be a Del Pezzo surface, i.e., a smooth Fano surface. Set

$$P = \mathbf{P}(\mathscr{O}_S \oplus \mathscr{O}_S(-K_S)) \xrightarrow{\pi} S$$

and let S_0 be the image of the section of π that corresponds to the trivial quotient of $\mathscr{O}_S \oplus \mathscr{O}_S(-K_S)$, so that the restriction of $\mathscr{O}_P(1)$ to S_0 is trivial.

a) What is the normal bundle to S_0 in P?

b) By considering a cyclic cover of P branched along a suitable section of $\mathscr{O}_P(m)$, for m large, construct a smooth projective threefold of general type X with K_X nef that contains S as a hypersurface with normal bundle K_S.

c) Assume from now on that S contains an exceptional curve C (i.e., a smooth rational curve with self-intersection -1). What is the normal bundle of C in X?

d) Let $\tilde{X} \to X$ be the blow-up of C. Describe the exceptional divisor E.

e) Let C_0 be the image of a section $E \to C$. Show that the ray $\mathbf{R}^+[C_0]$ is extremal and $K_{X'}$-negative.

f) Assume moreover that the characteristic is zero. The ray $\mathbf{R}^+[C_0]$ can be contracted (according to Corollary 8.4) by a morphism $\tilde{X} \to X^+$. Show that X^+ is smooth, that K_{X^+} is nef and that X^+ is not isomorphic to X. The induced rational map $X \dashrightarrow X^+$ is called a *flop.*

8) **A rationality theorem.** Let X be a smooth projective variety whose canonical divisor is not nef and let M be a nef divisor on X. Set

$$r = \sup\{t \in \mathbf{R} \mid M + tK_X \text{ nef}\}.$$

a) Let $(\Gamma_i)_{i\in I}$ be the (nonempty and countable) set of rational curves on X that appears in the cone Theorem 8.1. Show

$$r = \inf_{i\in I} \frac{(M \cdot \Gamma_i)}{(-K_X \cdot \Gamma_i)}.$$

b) Deduce that one can write

$$r = \frac{u}{v},$$

with u and v relatively prime integers and $0 < v \leq \dim(X) + 1$, and that there exists a K_X-negative extremal ray R of $\overline{\mathrm{NE}}(X)$ such that

$$((M + rK_X) \cdot R) = 0.$$

Chapter 9

Varieties with many rational curves

9.1 Rational varieties

Let $\mathbf{k}$ be a field. A $\mathbf{k}$-variety X of dimension n is $\mathbf{k}$-*rational* if it is birationally isomorphic to $\mathbf{P}^n_{\mathbf{k}}$. It is *rational* if, for some algebraically closed extension $\mathbf{K}$ of $\mathbf{k}$, the variety $X_{\mathbf{K}}$ is $\mathbf{K}$-rational (this definition does not depend on the choice of the algebraically closed extension $\mathbf{K}$).

One can also say that a variety is $\mathbf{k}$-rational if its function field is a purely transcendental extension of $\mathbf{k}$.

A geometrically integral projective curve is rational if and only if it has genus 0. It is $\mathbf{k}$-rational if and only if it has genus 0 and has a $\mathbf{k}$-point.

9.2 Unirational and separably unirational varieties

Definition 9.1 A $\mathbf{k}$-variety X of dimension n is

- $\mathbf{k}$-*unirational* if there exists a dominant rational map $\mathbf{P}^n_{\mathbf{k}} \dashrightarrow X$;
- $\mathbf{k}$-*separably unirational* if there exists a dominant and separable[1] rational map $\mathbf{P}^n_{\mathbf{k}} \dashrightarrow X$.

In characteristic zero, both definitions are equivalent. We say that X is *(separably) unirational* if for some algebraically closed extension $\mathbf{K}$ of $\mathbf{k}$, the variety $X_{\mathbf{K}}$ is $\mathbf{K}$-(separably) unirational (this definition does not depend on the choice of the algebraically closed extension $\mathbf{K}$).

A variety is $\mathbf{k}$-(separably) unirational if its function field has a purely transcendental (separable) extension.

Rational points are Zariski-dense in a $\mathbf{k}$-unirational variety, hence a conic with no rational points is rational but not $\mathbf{k}$-unirational.

Example 9.2 (Fermat hypersurfaces) Recall from 6.13 that the Fermat hypersurface $X^d_N \subset \mathbf{P}^N_{\mathbf{k}}$ is defined by the equation

$$x_0^d + \cdots + x_N^d = 0.$$

Assume that the field $\mathbf{k}$ has characteristic $p > 0$, take $d = p^r + 1$ for some $r > 0$, and assume that $\mathbf{k}$ contains an element ω such that $\omega^d = -1$. Assume also $N \geq 3$. The hypersurface X^d_N is then $\mathbf{k}$-unirational (Exercise 9.11.1). However, when $d > N$, its canonical class is nef, hence it is not separably unirational (not even separably uniruled; see Example 9.14).

[1]Recall that a dominant rational map $f : Y \dashrightarrow X$ between integral schemes is *separable* if the extension $K(Y)/K(X)$ is separable. It implies that f is smooth on a dense open subset of Y.

Any unirational curve is rational (Lüroth theorem), and any separably unirational surface is rational. However, any smooth cubic hypersurface $X \subset \mathbf{P}^4_{\mathbf{k}}$ is unirational but not rational.

I will explain the classical construction of a double cover of X which is rational. Let ℓ be a line contained in X and consider the map $\varphi : \mathbf{P}(T_X|_\ell) \dashrightarrow X$ defined as follows:[2] let L be a tangent line to X at a point $x_1 \in \ell$; the divisor $X|_L$ can be written as $2x_1 + x$, and we set $\varphi(L) = x$. Given a general point $x \in X$, the intersection of the 2-plane $\langle \ell, x\rangle$ with X is the union of the line ℓ and a conic C_x. The points of $\varphi^{-1}(x)$ are the two points of intersection of ℓ and C_x, hence φ is dominant of degree 2.

Now $T_X|_\ell$ is a sum of invertible sheaves which are all trivial on the complement $\ell^0 \simeq \mathbf{A}^1_{\mathbf{k}}$ of any point of ℓ. It follows that $\mathbf{P}(T_X|_{\ell_0})$ is isomorphic to $\ell^0 \times \mathbf{P}^2_{\mathbf{k}}$ hence is rational. This shows that X is unirational. The fact that it is not rational is a difficult theorem of Clemens-Griffiths and Artin-Mumford.

9.3 Uniruled and separably uniruled varieties

We want to make a formal definition for varieties that are "covered by rational curves". The most reasonable approach is to make it a "geometric" property by defining it over an algebraic closure of the base field. Special attention has to be paid to the positive characteristic case, hence the two variants of the definition.

Definition 9.3 Let $\mathbf{k}$ be a field and let $\mathbf{K}$ be an algebraically closed extension of $\mathbf{k}$. A variety X of dimension n defined over a field $\mathbf{k}$ is

- *uniruled* if there exist a $\mathbf{K}$-variety M of dimension $n-1$ and a dominant rational map $\mathbf{P}^1_{\mathbf{K}} \times M \dashrightarrow X_{\mathbf{K}}$;
- *separably uniruled* if there exist a $\mathbf{K}$-variety M of dimension $n-1$ and a dominant and separable rational map $\mathbf{P}^1_{\mathbf{K}} \times M \dashrightarrow X_{\mathbf{K}}$.

These definitions do not depend on the choice of the algebraically closed extension $\mathbf{K}$, and in characteristic zero, both definitions are equivalent.

In the same way that a "unirational" variety is dominated by a rational variety, a "uniruled" variety is dominated by a ruled variety; hence the terminology.

Of course, (separably) unirational varieties of positive dimension are (separably) uniruled. For the converse, uniruled curves are rational; separably uniruled surfaces are birationally isomorphic to a ruled surface. As explained in Example 9.2, in positive characteristic, some Fermat hypersurfaces are unirational (hence uniruled), but not separably uniruled.

Also, smooth projective varieties X with $-K_X$ nef and not numerically trivial are uniruled (Theorem 7.9), but there are Fano varieties that are not separably uniruled ([Ko2]).

Here are various other characterizations and properties of (separably) uniruled varieties.

Remark 9.4 A point is not uniruled. Any variety birationally isomorphic to a (separably) uniruled variety is (separably) uniruled. The product of a (separably) uniruled variety with any variety is (separably) uniruled.

Remark 9.5 A variety X of dimension n is (separably) uniruled if and only if there exist a a $\mathbf{K}$-variety M, an open subset U of $\mathbf{P}^1_{\mathbf{K}} \times M$ and a dominant (and separable) morphism $e : U \to X_{\mathbf{K}}$ such that for some point m in M, the set $U \cap (\mathbf{P}^1_{\mathbf{K}} \times m)$ is nonempty and not contracted by e.

Remark 9.6 Let X be a *proper* (separably) uniruled variety, with a rational map $e : \mathbf{P}^1_{\mathbf{K}} \times M \dashrightarrow X_{\mathbf{K}}$ as in the definition. We may compactify M then normalize it. The map e is then defined outside of a subvariety of $\mathbf{P}^1_{\mathbf{K}} \times M$ of codimension at least 2, which therefore projects onto a proper closed subset of M. By shrinking M, we may therefore assume that e is a *morphism.*

[2] Here we *do not* follow Grothendieck's convention: $\mathbf{P}(T_X|_\ell)$ is the set of tangent directions to X at points of ℓ.

Remark 9.7 Assume $\mathbf{k}$ is algebraically closed. It follows from Remark 9.6 that there is a rational curve through a general point of a proper uniruled variety (actually, by Lemma 7.8, there is even a rational curve through *every* point). The converse holds *if* $\mathbf{k}$ *is uncountable.* Therefore, in the definition, it is often useful to choose an uncountable algebraically closed extension $\mathbf{K}$.

Indeed, we may, after shrinking and compactifying X, assume that it is projective. There is still a rational curve through a general point, and this is exactly saying that the evaluation map $\mathrm{ev} : \mathbf{P}^1_{\mathbf{k}} \times \mathrm{Mor}_{>0}(\mathbf{P}^1_{\mathbf{k}}, X) \to X$ is dominant. Since $\mathrm{Mor}_{>0}(\mathbf{P}^1_{\mathbf{k}}, X)$ has at most countably many irreducible components and X is not the union of countably many proper subvarieties, the restriction of ev to at least one of these components must be surjective, hence X is uniruled by Remark 9.5.

Remark 9.8 Let $X \to T$ be a proper and equidimensional morphism with irreducible fibers. The set $\{t \in T \mid X_t \text{ is uniruled}\}$ is closed ([Ko1], Theorem 1.8.2; see also Exercise 9.32).

Remark 9.9 A connected finite étale cover of a proper (separably) uniruled variety is (separably) uniruled.

Let X be a proper uniruled variety, let $e : \mathbf{P}^1_{\mathbf{K}} \times M \to X_{\mathbf{K}}$ be a dominant (and separable) morphism (Remark 9.6), and let $\pi : \tilde{X} \to X$ be a connected finite étale cover. Since $\mathbf{P}^1_{\mathbf{K}}$ is simply connected, the pull-back by e of $\pi_{\mathbf{K}}$ is an étale morphism of the form $\mathbf{P}^1_{\mathbf{K}} \times \tilde{M} \to \mathbf{P}^1_{\mathbf{K}} \times M$ and the morphism $\mathbf{P}^1_{\mathbf{K}} \times \tilde{M} \to \tilde{X}_{\mathbf{K}}$ is dominant (and separable).[3]

9.4 Free rational curves and separably uniruled varieties

Let X be a variety of dimension n and let $f : \mathbf{P}^1_{\mathbf{k}} \to X$ be a nonconstant morphism whose image is contained in the smooth locus of X. Since any locally free sheaf on $\mathbf{P}^1_{\mathbf{k}}$ is isomorphic to a direct sum of invertible sheaf, we can write

$$f^*T_X \simeq \mathscr{O}_{\mathbf{P}^1_{\mathbf{k}}}(a_1) \oplus \cdots \oplus \mathscr{O}_{\mathbf{P}^1_{\mathbf{k}}}(a_n), \tag{9.1}$$

with $a_1 \geq \cdots \geq a_n$. If f is separable, f^*T_X contains $T_{\mathbf{P}^1_{\mathbf{k}}} \simeq \mathscr{O}_{\mathbf{P}^1_{\mathbf{k}}}(2)$ and $a_1 \geq 2$. In general, decompose f as $\mathbf{P}^1_{\mathbf{k}} \xrightarrow{h} \mathbf{P}^1_{\mathbf{k}} \xrightarrow{g} X$ where g is separable and h is a composition of r Frobenius morphisms. Then $a_1(f) = p^r a_1(g) \geq 2p^r$.

If $H^1(\mathbf{P}^1_{\mathbf{k}}, f^*T_X)$ vanishes, the space $\mathrm{Mor}(\mathbf{P}^1_{\mathbf{k}}, X)$ is smooth at $[f]$ (Theorem 6.8). This happens exactly when $a_n \geq -1$.

Definition 9.10 Let X be a $\mathbf{k}$-variety. A $\mathbf{k}$-rational curve $f : \mathbf{P}^1_{\mathbf{k}} \to X$ is *free* if its image is a curve contained in the smooth locus of X and f^*T_X is generated by its global sections.

With our notation, this means $a_n \geq 0$.

Examples 9.11 1) For any $\mathbf{k}$-morphism $f : \mathbf{P}^1_{\mathbf{k}} \to X$ whose image is contained in the smooth locus of X, we have

$$\deg(\det(f^*T_X)) = \deg(f^* \det(T_X)) = -\deg(f^*K_X) = -(K_X \cdot f_*\mathbf{P}^1_{\mathbf{k}}).$$

Therefore, there are no free rational curves on a smooth variety whose canonical divisor is nef.

2) A rational curve with image C on a smooth surface is free if and only if $(C^2) \geq 0$.

Let $f : \mathbf{P}^1_{\mathbf{k}} \to C \subset X$ be the normalization and assume that f is free. Since

$$(K_X \cdot C) + (C^2) = 2h^1(C, \mathscr{O}_C) - 2,$$

we have, with the notation (9.1),

$$(C^2) = a_1 + a_2 + 2h^1(C, \mathscr{O}_C) - 2 \geq (a_1 - 2) + a_2 \geq a_2 \geq 0.$$

[3] For uniruledness, one can also work on an uncountable algebraically closed extension $\mathbf{K}$ and show that there is a rational curve through a general point of $\tilde{X}_{\mathbf{K}}$.

Conversely, assume $a := (C^2) \geq 0$. Since the ideal sheaf of C in X is invertible, there is an exact sequence

$$0 \to \mathscr{O}_C(-C) \to \Omega_X|_C \to \Omega_C \to 0$$

of locally free sheaves on C which pulls back to $\mathbf{P}^1_{\mathbf{k}}$ and dualizes to

$$0 \to \mathscr{H}om(f^*\Omega_C, \mathscr{O}_{\mathbf{P}^1_{\mathbf{k}}}) \to f^*T_X \to f^*\mathscr{O}_X(C) \to 0. \tag{9.2}$$

There is also a morphism $f^*\Omega_C \to \Omega_{\mathbf{P}^1_{\mathbf{k}}}$ which is an isomorphism on a dense open subset of $\mathbf{P}^1_{\mathbf{k}}$, hence dualizes to an injection $T_{\mathbf{P}^1_{\mathbf{k}}} \hookrightarrow \mathscr{H}om(f^*\Omega_C, \mathscr{O}_{\mathbf{P}^1_{\mathbf{k}}})$. In particular, the invertible sheaf $\mathscr{H}om(f^*\Omega_C, \mathscr{O}_{\mathbf{P}^1_{\mathbf{k}}})$ has degree $b \geq 2$, and we have an exact sequence

$$0 \to \mathscr{O}_{\mathbf{P}^1_{\mathbf{k}}}(b) \to f^*T_X \to \mathscr{O}_{\mathbf{P}^1_{\mathbf{k}}}(a) \to 0.$$

If $a_2 < 0$, the injection $\mathscr{O}_{\mathbf{P}^1_{\mathbf{k}}}(b) \to f^*T_X$ lands in $\mathscr{O}_{\mathbf{P}^1_{\mathbf{k}}}(a_1)$, and we have an isomorphism

$$\Big(\mathscr{O}_{\mathbf{P}^1_{\mathbf{k}}}(a_1)/\mathscr{O}_{\mathbf{P}^1_{\mathbf{k}}}(b)\Big) \oplus \mathscr{O}_{\mathbf{P}^1_{\mathbf{k}}}(a_2) \simeq \mathscr{O}_{\mathbf{P}^1_{\mathbf{k}}}(a),$$

which implies $a_1 = b$ and $a = a_2 < 0$, a contradiction. So we have $a_2 \geq 0$ and f is free.

3) One can show ([D1], 2.15) that the Fermat hypersurface (see 6.13) X^d_N of dimension at least 3 and degree $d = p^r + 1$ over a field of characteristic p is uniruled by lines, none of which are free (in fact, when $d > N$, there are no free rational curves on X by Example 9.11.1)). Moreover, $\mathrm{Mor}_1(\mathbf{P}^1_{\mathbf{k}}, X)$ is smooth, but the evaluation map

$$\mathrm{ev} : \mathbf{P}^1_{\mathbf{k}} \times \mathrm{Mor}_1(\mathbf{P}^1_{\mathbf{k}}, X) \longrightarrow X$$

is *not separable.*

Proposition 9.12 *Let X be a smooth quasi-projective variety defined over a field $\mathbf{k}$ and let $f : \mathbf{P}^1_{\mathbf{k}} \to X$ be a rational curve.*

a) *If f is free, the evaluation map*
$$\mathrm{ev} : \mathbf{P}^1_{\mathbf{k}} \times \mathrm{Mor}(\mathbf{P}^1_{\mathbf{k}}, X) \to X$$
is smooth at all points of $\mathbf{P}^1_{\mathbf{k}} \times \{[f]\}$.

b) *If there is a scheme M with a $\mathbf{k}$-point m and a morphism $e : \mathbf{P}^1_{\mathbf{k}} \times M \to X$ such that $e|_{\mathbf{P}^1_{\mathbf{k}} \times m} = f$ and the tangent map to e is surjective at some point of $\mathbf{P}^1_{\mathbf{k}} \times m$, the curve f is free.*

Geometrically speaking, item a) implies that the deformations of a free rational curve cover X. In b), the hypothesis that the tangent map to e is surjective is weaker than the smoothness of e, and does not assume anything on the smoothness, or even reducedness, of the scheme M.

The proposition implies that the set of free rational curves on a quasi-projective $\mathbf{k}$-variety X is a smooth open subset $\mathrm{Mor}^{\mathrm{free}}(\mathbf{P}^1_{\mathbf{k}}, X)$ of $\mathrm{Mor}(\mathbf{P}^1_{\mathbf{k}}, X)$, possibly empty.

Finally, *when* $\mathrm{char}(\mathbf{k}) = 0$, and there is an irreducible $\mathbf{k}$-scheme M and a *dominant* morphism $e : \mathbf{P}^1_{\mathbf{k}} \times M \to X$ which does not contract one $\mathbf{P}^1_{\mathbf{k}} \times m$, the rational curves corresponding to points in some nonempty open subset of M are free (by generic smoothness, the tangent map to e is surjective on some nonempty open subset of $\mathbf{P}^1_{\mathbf{k}} \times M$).

PROOF. The tangent map to ev at $(t, [f])$ is the map

$$\begin{array}{rcl} T_{\mathbf{P}^1_{\mathbf{k}},t} \oplus H^0(\mathbf{P}^1_{\mathbf{k}}, f^*T_X) & \longrightarrow & T_{X,f(t)} \simeq (f^*T_X)_t \\ (u, \sigma) & \longmapsto & T_t f(u) + \sigma(t). \end{array}$$

If f is free, it is surjective because the evaluation map

$$H^0(\mathbf{P}^1_{\mathbf{k}}, f^*T_X) \longrightarrow (f^*T_X)_t$$

is. Moreover, since $H^1(\mathbf{P}^1_{\mathbf{k}}, f^*T_X)$ vanishes, $\mathrm{Mor}(\mathbf{P}^1_{\mathbf{k}}, X)$ is smooth at $[f]$ (6.11). This implies that ev is smooth at $(t, [f])$ and proves a).

Conversely, the morphism e factors through ev, whose tangent map at $(t,[f])$ is therefore surjective. This implies that the map

$$H^0(\mathbf{P}^1_{\mathbf{k}}, f^*T_X) \to (f^*T_X)_t/\operatorname{Im}(T_tf) \tag{9.3}$$

is surjective. There is a commutative diagram

$$\begin{array}{ccc} H^0(\mathbf{P}^1_{\mathbf{k}}, f^*T_X) & \xrightarrow{a} & (f^*T_X)_t \\ \uparrow & & \uparrow{\scriptstyle T_tf} \\ H^0(\mathbf{P}^1_{\mathbf{k}}, T_{\mathbf{P}^1_{\mathbf{k}}}) & \xrightarrow{a'} & T_{\mathbf{P}^1_{\mathbf{k}},t}. \end{array}$$

Since a' is surjective, the image of a contains $\operatorname{Im}(T_tf)$. Since the map (9.3) is surjective, a is surjective. Hence f^*T_X is generated by global sections at one point. It is therefore generated by global sections and f is free. □

Corollary 9.13 *Let X be a quasi-projective variety defined over an algebraically closed field $\mathbf{k}$.*

a) If X contains a free rational curve, X is separably uniruled.

b) Conversely, if X is separably uniruled, smooth, and projective, there exists a free rational curve through a general point of X.

Proof. If $f : \mathbf{P}^1_{\mathbf{k}} \to X$ is free, the evaluation map ev is smooth at $(0,[f])$ by Proposition 9.12.a). It follows that the restriction of ev to the unique component of $\operatorname{Mor}_{>0}(\mathbf{P}^1_{\mathbf{k}}, X)$ that contains $[f]$ is separable and dominant and X is separably uniruled.

Assume conversely that X is separably uniruled, smooth, and projective. By Remark 9.6, there exists a $\mathbf{k}$-variety M and a dominant and separable, hence generically smooth, morphism $\mathbf{P}^1_{\mathbf{k}} \times M \to X$. The rational curve corresponding to a general point of M passes through a general point of X and is free by Proposition 9.12.b). □

Example 9.14 By Example 9.11 and Corollary 9.13.b), a smooth proper variety X with K_X nef is not separably uniruled.

On the other hand, we proved in Theorem 7.9 that smooth projective varieties X with $-K_X$ nef and not numerically trivial are uniruled. However, Kollár constructed Fano varieties that are not separably uniruled ([Ko2]).

Corollary 9.15 *If X is a smooth projective separably uniruled variety, the* plurigenera $p_m(X) := h^0(X, \mathscr{O}_X(mK_X))$ *vanish for all positive integers m.*

The converse is conjectured to hold: for curves, it is obvious since $p_1(X)$ is the genus of X; for surfaces, we have the more precise Castelnuovo criterion; $p_{12}(X) = 0$ if and only if X is birationally isomorphic to a ruled surface; in dimension three, it is known in characteristic zero.

Proof. We may assume that the base field $\mathbf{k}$ is algebraically closed. By Corollary 9.13.b), there is a free rational curve $f : \mathbf{P}^1_{\mathbf{k}} \to X$ through a general point of X. Since f^*K_X has negative degree, any section of $\mathscr{O}_X(mK_X)$ must vanish on $f(\mathbf{P}^1_{\mathbf{k}})$, hence on a dense subset of X, hence on X. □

The next results says that a rational curve through a very general point (i.e., outside the union of a countable number of proper subvarieties) of a smooth variety is free (in characteristic zero).

Proposition 9.16 *Let X be a smooth quasi-projective variety defined over a field of characteristic zero. There exists a subset X^{free} of X which is the intersection of countably many dense open subsets of X, such that any rational curve on X whose image meets X^{free} is free.*

Proof. The space $\mathrm{Mor}(\mathbf{P}^1_{\mathbf{k}}, X)$ has at most countably many irreducible components, which we denote by $(M_i)_{i\in\mathbf{N}}$. Let $e_i : \mathbf{P}^1_{\mathbf{k}} \times (M_i)_{\mathrm{red}} \to X$ be the morphisms induced by the evaluation maps.

By generic smoothness, there exists a dense open subset U_i of X such that the tangent map to e_i is surjective at each point of $e_i^{-1}(U_i)$ (if e_i is not dominant, one may simply take for U_i the complement of the closure of the image of e_i). We let X^{free} be the intersection $\bigcap_{i\in\mathbf{N}} U_i$.

Let $f : \mathbf{P}^1_{\mathbf{k}} \to X$ be a curve whose image meets X^{free}, and let M_i be an irreducible component of $\mathrm{Mor}(\mathbf{P}^1_{\mathbf{k}}, X)$ that contains $[f]$. By construction, the tangent map to e_i is surjective at some point of $\mathbf{P}^1_{\mathbf{k}} \times \{[f]\}$, hence f is free by Proposition 9.12.b). □

The proposition is interesting only when X is uniruled (otherwise, the set X^{free} is more or less the complement of the union of all rational curves on X); it is also useless when the ground field is countable, because X^{free} may be empty.

Examples 9.17 1) If $\varepsilon : \widetilde{\mathbf{P}}^2_{\mathbf{k}} \to \mathbf{P}^2_{\mathbf{k}}$ is the blow-up of one point, $(\widetilde{\mathbf{P}}^2_{\mathbf{k}})^{\mathrm{free}}$ is the complement of the exceptional divisor E: for any rational curve C other than E, write $C \underset{\mathrm{lin}}{\equiv} dH - mE$, where H is the inverse image of a line; we have $m = (C \cdot E) \geq 0$. The intersection of C with the strict transform of a line through the blown-up point, which has class $H - E$, is nonnegative, hence $d \geq m$. It implies $(C^2) = d^2 - m^2 \geq 0$, hence C is free by Example 9.11.2).

2) On the blow-up X of $\mathbf{P}^2_{\mathbf{C}}$ at nine general points, there are countably many rational curves with self-intersection -1 ([H1], Exercise V.4.15.(e)) hence X^{free} is not open.

9.5 Rationally connected and separably rationally connected varieties

We now want to make a formal definition for varieties for which there exists a rational curve through two general points. Again, this will be a geometric property.

Definition 9.18 Let $\mathbf{k}$ be a field and let $\mathbf{K}$ be an algebraically closed extension of $\mathbf{k}$. A $\mathbf{k}$-variety X is *rationally connected* (resp. *separably rationally connected*) if it is *proper* and if there exist a $\mathbf{K}$-variety M and a rational map $e : \mathbf{P}^1_{\mathbf{K}} \times M \dashrightarrow X_{\mathbf{K}}$ such that the rational map

$$\begin{array}{rccc} \mathrm{ev}_2 : & \mathbf{P}^1_{\mathbf{K}} \times \mathbf{P}^1_{\mathbf{K}} \times M & \dashrightarrow & X_{\mathbf{K}} \times X_{\mathbf{K}} \\ & (t, t', z) & \longmapsto & (e(t,z), e(t',z)) \end{array}$$

is dominant (resp. dominant and separable).

Again, this definition does not depend on the choice of the algebraically closed extension $\mathbf{K}$, and in characteristic zero, both definitions are equivalent. Moreover, the rational map e may be assumed to be a morphism (proceed as in Remark 9.6).

Of course, (separably) rationally connected varieties are (separably) uniruled, and (separably) unirational varieties are (separably) rationally connected. For the converse, rationally connected curves are rational, and separably rationally connected surfaces are rational. One does not expect, in dimension ≥ 3, rational connectedness to imply unirationality, but no examples are known!

It can be shown that Fano varieties are rationally connected,[4] although they are in general not even separably uniruled in positive characteristic (Example 9.2).

Remark 9.19 A point is separably rationally connected. (Separable) rational connectedness is a birational property (for proper varieties!); better, if X is a (separably) rationally connected variety and $X \dashrightarrow Y$ a (separable) dominant rational map, with Y proper, Y is (separably) rationally connected. A (finite) product of (separably) rationally connected varieties is (separably) rationally connected. A (separably) rationally connected variety is (separably) uniruled.

[4] This is a result due independently to Campana and Kollár-Miyaoka-Mori; see for example [D1], Proposition 5.16.

Remark 9.20 In the definition, one may replace the condition that ev_2 be dominant (resp. dominant and separable) by the condition that the map

$$\begin{array}{ccc} M & \dashrightarrow & X_{\mathbf{K}} \times X_{\mathbf{K}} \\ z & \longmapsto & (e(0,z), e(\infty,z)) \end{array}$$

be dominant (resp. dominant and separable).

Indeed, upon shrinking and compactifying X, we may assume that X is projective. The morphism e then factors through an evaluation map $\mathrm{ev} : \mathbf{P}^1_{\mathbf{K}} \times \mathrm{Mor}_d(\mathbf{P}^1_{\mathbf{K}}, X) \to X_{\mathbf{K}}$ for some $d > 0$ and the image of

$$\mathrm{ev}_2 : \mathbf{P}^1_{\mathbf{K}} \times \mathbf{P}^1_{\mathbf{K}} \times \mathrm{Mor}_d(\mathbf{P}^1_{\mathbf{K}}, X) \to X_{\mathbf{K}} \times X_{\mathbf{K}}$$

is then the same as the image of

$$\begin{array}{ccc} \mathrm{Mor}_d(\mathbf{P}^1_{\mathbf{K}}, X) & \to & X_{\mathbf{K}} \times X_{\mathbf{K}} \\ z & \longmapsto & (e(0,z), e(\infty,z)) \end{array}$$

(This is because $\mathrm{Mor}_d(\mathbf{P}^1_{\mathbf{K}}, X)$ is stable by reparametrizations, i.e., by the action of $\mathrm{Aut}(\mathbf{P}^1_{\mathbf{K}})$; for separable rational connectedness, there are some details to check.)

Remark 9.21 Assume $\mathbf{k}$ is algebraically closed. On a rationally connected variety, a general pair of points can be joined by a rational curve.[5] The converse holds *if* $\mathbf{k}$ *is uncountable* (with the same proof as in Remark 9.7).

Remark 9.22 Any proper variety which is an étale cover of a (separably) rationally connected variety is (separably) rationally connected (proceed as in Remark 9.9). In fact, Kollár proved that any such a cover of a smooth proper separably rationally connected variety is in fact trivial ([D3], cor. 3.6).

9.6 Very free rational curves and separably rationally connected varieties

Definition 9.23 Let X be a $\mathbf{k}$-variety. A $\mathbf{k}$-rational curve $f : \mathbf{P}^1_{\mathbf{k}} \to X$ is *r-free* if its image is contained in the smooth locus of X and $f^*T_X \otimes \mathscr{O}_{\mathbf{P}^1_{\mathbf{k}}}(-r)$ is generated by its global sections.

In particular, 0-free curves are free curves. We will say "very free" instead of "1-free". For easier statements, we will also agree that a constant morphism $\mathbf{P}^1_{\mathbf{k}} \to X$ is very free if and only if X is a point. Note that given a very free rational curve, its composition with a (ramified) finite map $\mathbf{P}^1_{\mathbf{k}} \to \mathbf{P}^1_{\mathbf{k}}$ of degree r is r-free.

Examples 9.24 1) Any $\mathbf{k}$-rational curve $f : \mathbf{P}^1_{\mathbf{k}} \to \mathbf{P}^n_{\mathbf{k}}$ is very free. This is because $T_{\mathbf{P}^n_{\mathbf{k}}}$ is a quotient of $\mathscr{O}_{\mathbf{P}^n_{\mathbf{k}}}(1)^{\oplus(n+1)}$, hence its inverse image by f is a quotient of $\mathscr{O}_{\mathbf{P}^1_{\mathbf{k}}}(d)^{\oplus(n+1)}$, where $d > 0$ is the degree of $f^*\mathscr{O}_{\mathbf{P}^n_{\mathbf{k}}}(1)$. With the notation of (9.1), each $\mathscr{O}_{\mathbf{P}^1_{\mathbf{k}}}(a_i)$ is a quotient of $\mathscr{O}_{\mathbf{P}^1_{\mathbf{k}}}(d)^{\oplus(n+1)}$ hence $a_i \geq d$.

2) A rational curve with image C on a smooth surface is very free if and only if $(C^2) > 0$ (proceed as in Example 9.11.2)).

Informally speaking, the freer a rational curve is, the more it can move while keeping points fixed. The precise result is the following. It generalizes Proposition 9.12 and its proof is similar.

Proposition 9.25 *Let X be a smooth quasi-projective* $\mathbf{k}$*-variety, let r be a nonnegative integer, let $f : \mathbf{P}^1_{\mathbf{k}} \to X$ be a rational curve and let B be a finite subset of $\mathbf{P}^1_{\mathbf{k}}$ of cardinality b.*

[5] We will prove in Theorem 9.40 that *any* two points of a *smooth* projective separably rationally connected variety can be joined by a rational curve.

a) *If f is r-free, for any integer s such that $0 < s \leq r+1-b$, the evaluation map*

$$\begin{array}{rccc} \mathrm{ev}_s : & (\mathbf{P}^1_{\mathbf{k}})^s \times \mathrm{Mor}(\mathbf{P}^1_{\mathbf{k}}, X; f|_B) & \longrightarrow & X^s \\ & (t_1, \dots, t_s, [g]) & \longmapsto & (g(t_1), \dots, g(t_s)) \end{array}$$

is smooth at all points $(t_1, \dots, t_s, [f])$ such that $\{t_1, \dots, t_s\} \cap B = \varnothing$.

b) *If there is a $\mathbf{k}$-scheme M with a $\mathbf{k}$-point m and a morphism $\varphi : M \to \mathrm{Mor}(\mathbf{P}^1_{\mathbf{k}}, X; f|_B)$ such that $\varphi(m) = [f]$ and the tangent map to the corresponding evaluation map*

$$\mathrm{ev}_s : (\mathbf{P}^1_{\mathbf{k}})^s \times M \longrightarrow X^s$$

is surjective at some point of $\mathbf{P}^1_{\mathbf{k}} \times m$ for some $s > 0$, the rational curve f is $\min(2, b+s-1)$-free.

Geometrically speaking, item a) implies that the deformations of an r-free rational curve keeping b points fixed ($b \leq r$) pass through $r+1-b$ general points of X.

The proposition implies that the set of very free rational curves on X is a smooth open subset $\mathrm{Mor}^{\mathrm{vfree}}(\mathbf{P}^1_{\mathbf{k}}, X)$ of $\mathrm{Mor}(\mathbf{P}^1_{\mathbf{k}}, X)$, possibly empty.

In §9.4, we studied the relationships between separable uniruledness and the existence of free rational curves on a smooth projective variety. We show here that there is an analogous relationship between separable rational connectedness and the existence of very free rational curves.

Corollary 9.26 *Let X be a proper variety defined over an algebraically closed field $\mathbf{k}$.*

a) If X contains a very free rational curve, there is a very free rational curve through a general finite subset of X. In particular, X is separably rationally connected.

b) Conversely, if X is separably rationally connected and smooth, there exists a very free rational curve through a general point of X.

The result will be strengthened in Theorem 9.40 where it is proved that on a smooth projective separably rationally connected variety, there is a very free rational curve through *any* given finite subset.

Proof. Assume there is a very free rational curve $f : \mathbf{P}^1_{\mathbf{k}} \to X$. By composing f with a finite map $\mathbf{P}^1_{\mathbf{k}} \to \mathbf{P}^1_{\mathbf{k}}$ of degree r, we get an r-free curve. By Proposition 9.12.a) (applied with $B = \varnothing$), there is a deformation of this curve that passes through $r+1$ general points of X. The rest of the proof is the same as in Corollary 9.13. □

Corollary 9.27 *If X is a smooth proper separably rationally connected variety, $H^0(X, (\Omega^p_X)^{\otimes m})$ vanishes for all positive integers m and p. In particular, in characteristic zero, $\chi(X, \mathscr{O}_X) = 1$.*

A converse is conjectured to hold (at least in characteristic zero): if $H^0(X, (\Omega^1_X)^{\otimes m})$ vanishes for all positive integers m, the variety X should be rationally connected. This is proved in dimensions at most 3 in [KMM], Theorem (3.2).

Note that the conclusion of the corollary does not hold in general for unirational varieties: some Fermat hypersurfaces X are unirational with $H^0(X, K_X) \neq 0$ (see Example 9.2).

Proof of the Corollary. For the first part, proceed as in the proof of Corollary 9.15. For the second part, $H^p(X, \mathscr{O}_X)$ then vanishes for $p > 0$ by Hodge theory,[6] hence $\chi(X, \mathscr{O}_X) = 1$. □

Corollary 9.28 *Let X be a proper normal rationally connected variety defined over an algebraically closed field $\mathbf{k}$.*

a) The algebraic fundamental group of X is finite.

[6] For a smooth separably rationally connected variety X, the vanishing of $H^m(X, \mathscr{O}_X)$ for $m > 0$ is not known in general.

b) If $\mathbf{k} = \mathbf{C}$ and X is smooth, X is topologically simply connected.

When X is smooth and separably rationally connected, Kollár proved that X is in fact algebraically simply connected ([D3], cor. 3.6).

Proof of the Corollary. By Remark 9.20, there exist a variety M and a point x of X such that the evaluation map

$$\mathrm{ev} : \mathbf{P}^1_{\mathbf{k}} \times M \longrightarrow X$$

is dominant and satisfies $\mathrm{ev}(0 \times M) = x$. The composition of ev with the injection $\iota : 0 \times M \hookrightarrow \mathbf{P}^1_{\mathbf{k}} \times M$ is then constant, hence

$$\pi_1(\mathrm{ev}) \circ \pi_1(\iota) = 0.$$

Since $\mathbf{P}^1_{\mathbf{k}}$ is simply connected, $\pi_1(\iota)$ is bijective, hence $\pi_1(\mathrm{ev}) = 0$. Since ev is dominant, the following lemma implies that the image of $\pi_1(\mathrm{ev})$ has finite index. This proves a).

Lemma 9.29 *Let X and Y be $\mathbf{k}$-varieties, with Y normal, and let $f : X \to Y$ be a dominant morphism. For any geometric point x of X, the image of the morphism $\pi_1(f) : \pi_1^{\mathrm{alg}}(X, x) \to \pi_1^{\mathrm{alg}}(Y, f(x))$ has finite index.*

When $\mathbf{k} = \mathbf{C}$, the same statement holds with topological fundamental groups.

Sketch of proof. The lemma is proved in [De] (lemme 4.4.17) when X and Y are smooth. The same proof applies in our case ([CL]).

We will sketch the proof when $\mathbf{k} = \mathbf{C}$. The first remark is that if A is an irreducible analytic space and B a proper closed analytic subspace, $A - B$ is connected. The second remark is that the universal cover $\pi : \tilde{Y} \to Y$ is irreducible; indeed, Y being normal is locally irreducible in the classical topology, hence so is $\tilde{Y}$. Since it is connected, it is irreducible.

Now if Z is a proper subvariety of Y, its inverse image $\pi^{-1}(Z)$ is a proper subvariety of $\tilde{Y}$, hence $\pi^{-1}(Y - Z)$ is connected by the two remarks above. This means exactly that the map $\pi_1(Y - Z) \to \pi_1(Y)$ is surjective. So we may replace Y with any dense open subset, and assume that Y is smooth.

We may also shrink X and assume that it is smooth and quasi-projective. Let $\overline{X}$ be a compactification of X. We may replace X with a desingularization $\overline{\overline{X}}$ of the closure in $\overline{X} \times Y$ of the graph of f and assume that f is *proper*. Since the map $\pi_1(X) \to \pi_1(\overline{\overline{X}})$ is surjective by the remark above, this does not change the cokernel of $\pi_1(f)$.

Finally, we may, by generic smoothness, upon shrinking Y again, assume that f is smooth. The finite morphism in the Stein factorization of f is then étale; we may therefore assume that the fibers of f are connected. It is then classical that f is locally $\mathscr{C}^\infty$-trivial with fiber F, and the long exact homotopy sequence

$$\cdots \to \pi_1(F) \to \pi_1(X) \to \pi_1(Y) \to \pi_0(F) \to 0$$

of a fibration gives the result. □

If $\mathbf{k} = \mathbf{C}$ and X is smooth, we have $\chi(X, \mathscr{O}_X) = 1$ by Corollary 9.27. Let $\pi : \tilde{X} \to X$ be a connected finite étale cover; $\tilde{X}$ is rationally connected by Remark 9.22, hence $\chi(\tilde{X}, \mathscr{O}_{\tilde{X}}) = 1$. But $\chi(\tilde{X}, \mathscr{O}_{\tilde{X}}) = \deg(\pi)\, \chi(X, \mathscr{O}_X)$ ([L], Proposition 1.1.28) hence π is an isomorphism. This proves b). □

We finish this section with an analog of Proposition 9.16: on a *smooth* projective variety defined over an algebraically closed field *of characteristic zero,* a rational curve through a fixed point and a very general point is very free.

Proposition 9.30 *Let X be a smooth quasi-projective variety defined over an algebraically closed field of characteristic zero and let x be a point in X. There exists a subset X_x^{free} of $X - \{x\}$ which is the intersection of countably many dense open subsets of X, such that any rational curve on X passing through x and whose image meets X_x^{vfree} is very free.*

PROOF. The space $\mathrm{Mor}(\mathbf{P}^1_{\mathbf{k}}, X; 0 \mapsto x)$ has at most countably many irreducible components, which we will denote by $(M_i)_{i\in\mathbf{N}}$. Let $e_i : \mathbf{P}^1_{\mathbf{k}} \times (M_i)_{\mathrm{red}} \to X$ be the morphisms induced by the evaluation maps.

Denote by U_i a dense open subset of $X - \{x\}$ over which e_i is smooth and let X_x^{vfree} be the intersection of the U_i. Let $f : \mathbf{P}^1_{\mathbf{k}} \to X$ be a curve with $f(0) = x$ whose image meets X_x^{vfree}, and let M_i be an irreducible component of $\mathrm{Mor}(\mathbf{P}^1_{\mathbf{k}}, X; 0 \mapsto x)$ that contains $[f]$. By construction, the tangent map to e_i is surjective at some point of $\mathbf{P}^1_{\mathbf{k}} \times \{[f]\}$, hence so is the tangent map to ev; it follows from Proposition 9.25 that f is very free. □

Again, this proposition is interesting only when X is rationally connected and the ground field is uncountable.

9.7 Smoothing trees of rational curves

9.31. Scheme of morphisms over a base. We explained in 6.2 that given a projective $\mathbf{k}$-variety Y and a quasi-projective $\mathbf{k}$-variety X, morphisms from Y to X are parametrized by a $\mathbf{k}$-scheme $\mathrm{Mor}(Y, X)$ locally of finite type. One can also impose fixed points (see 6.11).

All this can be done over an irreducible noetherian base scheme T ([Mo1], [Ko1], Theorem II.1.7): if $Y \to T$ is a projective flat T-scheme, with a subscheme $B \subset Y$ finite and flat over T, and $X \to T$ is a quasi-projective T-scheme with a T-morphism $g : B \to X$, the T-morphisms from Y to X that restrict to g on B can be parametrized by a locally noetherian T-scheme $\mathrm{Mor}_T(Y, X; g)$. The universal property implies in particular that for any point t of T, one has

$$\mathrm{Mor}_T(Y, X; g)_t \simeq \mathrm{Mor}(Y_t, X_t; g_t).$$

In other words, the schemes $\mathrm{Mor}(Y_t, X_t; g_t)$ fit together to form a scheme over T ([Mo1], Proposition 1, and [Ko1], Proposition II.1.5).

When moreover Y is a relative reduced curve C over T, with geometrically reduced fibers, and X is smooth over T, given a point t of T and a morphism $f : C_t \to X_t$ which coincides with g_t on B_t, we have

$$\begin{aligned}\dim_{[f]} \mathrm{Mor}_T(C, X; g) &\geq \chi(C_t, f^*T_{X_t} \otimes \mathscr{I}_{B_t}) + \dim(T)\\ &= (-K_{X_t} \cdot f_*C_t) + (1 - g(C_t) - \lg(B_t)) \dim(X_t) + \dim(T). \quad (9.4)\end{aligned}$$

Furthermore, *if* $H^1(C_t, f^*T_{X_t} \otimes \mathscr{I}_{B_t})$ *vanishes,* $\mathrm{Mor}_T(C, X; g)$ *is smooth over* T *at* $[f]$ ([Ko1], Theorem II.1.7).

Exercise 9.32 Let $X \to T$ be a smooth and proper morphism. Show that the sets

$$\{t \in T \mid X_t \text{ is separably uniruled}\}$$

and

$$\{t \in T \mid X_t \text{ is separably rationally connected}\}$$

are open.

9.33. Smoothing of trees. *We assume now that* $\mathbf{k}$ *is algebraically closed.*

Definition 9.34 A *rational* $\mathbf{k}$*-tree* is a connected projective nodal $\mathbf{k}$-curve C such that $\chi(C, \mathscr{O}_C) = 1$.

Exercise 9.35 Show that the irreducible components of a tree are smooth rational curves and that they can be numbered as $C_0, \dots, C_m$ in such a way that C_0 is any given component and, for each $0 \leq i \leq m - 1$, the curve C_{i+1} meets $C_0 \cup \cdots \cup C_i$ transversely in a single smooth point. We will always assume that the components of a rational tree are numbered in this fashion.

It is easy to construct a *smoothing* of a rational $\mathbf{k}$-tree C: let $T = \mathbf{P}^1_{\mathbf{k}}$ and blow up the smooth surface $C_0 \times T$ at the point $(C_0 \cap C_1) \times 0$, then at $((C_0 \cup C_1) \cap C_2) \times 0$ and so on. The resulting flat projective T-curve $\mathscr{C} \to T$ has fiber C above 0 and $\mathbf{P}^1_{\mathbf{k}}$ elsewhere.

Moreover, given a smooth point p of C, one can construct a section σ of the smoothing $\mathscr{C} \to T$ such that $\sigma(0) = p$: let C'_1 be the component of C that contains p. Each connected component of $\overline{C - C'_1}$ is a rational tree hence can be blown-down, yielding a birational T-morphism $\varepsilon : \mathscr{C} \to \mathscr{C}'$, where $\mathscr{C}'$ is a ruled smooth surface over T, with fiber of 0 the curve $\varepsilon(C'_1)$. Take a section of $\mathscr{C}' \to T$ that passes through $\varepsilon(p)$; its strict transform on $\mathscr{C}$ is a section of $\mathscr{C} \to T$ that passes through p.

Given a smooth $\mathbf{k}$-variety X and a rational $\mathbf{k}$-tree C, any morphism $f : C \to X$ defines a $\mathbf{k}$-point $[f]$ of the T-scheme $\mathrm{Mor}_T(\mathscr{C}, X \times T)$ above $0 \in T(\mathbf{k})$. By 9.31, if $H^1(C, f^*T_X) = 0$, this T-scheme is smooth at $[f]$. This means that f can be smoothed to a rational curve $\mathbf{P}^1_{\mathbf{k}} \to X_{\mathbf{k}}$.

It will often be useful to be able to fix points in this deformation. Let $B = \{p_1, \dots, p_r\}$ be a set of smooth points of C and let $\sigma_1, \dots, \sigma_r$ be sections of $\mathscr{C} \to T$ such that $\sigma_i(0) = p_i$; upon shrinking T, we may assume that they are disjoint. Let

$$g : \bigsqcup_{i=1}^{r} \sigma_i(T) \to X \times T$$

be the morphism $\sigma_i(t) \mapsto (f(p_i), t)$. Now, T-morphisms from $\mathscr{C}$ to $X \times T$ extending g are parametrized by the T-scheme $\mathrm{Mor}_T(\mathscr{C}, X \times T; g)$ whose fiber at 0 is $\mathrm{Mor}(C, X; p_i \mapsto f(p_i))$, and this scheme is smooth over T at $[f]$ when $H^1(C, (f^*T_X)(-p_1 - \cdots - p_r))$ vanishes.

It is therefore useful to have a criterion which ensures that this group vanish.

Lemma 9.36 *Let $C = C_0 \cup \cdots \cup C_m$ be a rational $\mathbf{k}$-tree. Let $\mathscr{E}$ be a locally free sheaf on C such that $(\mathscr{E}|_{C_i})(1)$ is nef for $i = 0$ and ample for each $i \in \{1, \dots, m\}$. We have $H^1(C, \mathscr{E}) = 0$.*

PROOF. We show this by induction on m, the result being obvious for $m = 0$. Set $C' = C_0 \cup \cdots \cup C_{m-1}$ and $C' \cap C_m = \{q\}$. There are exact sequences

$$0 \to (\mathscr{E}|_{C_m})(-q) \to \mathscr{E} \to \mathscr{E}|_{C'} \to 0$$

and

$$H^1(C_m, (\mathscr{E}|_{C_m})(-q)) \to H^1(C, \mathscr{E}) \to H^1(C', \mathscr{E}|_{C'}).$$

By hypothesis and induction, the spaces on both ends vanish, hence the lemma. □

Proposition 9.37 *Let X be a smooth projective variety, let C be a rational tree, both defined over an algebraically closed field, and let $f : C \to X$ be a morphism whose restriction to each component of C is free.*

a) *The morphism f is smoothable, keeping any smooth point of C fixed, into a free rational curve.*

b) *If moreover f is r-free on one component C_0 ($r \geq 0$), f is smoothable, keeping fixed any r points of C_0 smooth on C and any smooth point of $C - C_0$, into an r-free rational curve.*

PROOF. Item a) is a particular case of item b) (case $r = 0$). Let $p_1, \dots, p_r$ be smooth points of C on C_0 and let q be a smooth point of C, on the component C_i, with $i \neq 0$. The locally free sheaf $\big((f^*T_X)(-p_1 - \cdots - p_r - q)\big)|_{C_j}(1)$ is nef for $j = i$ and ample for $j \neq i$. The lemma implies $H^1(C, (f^*T_X)(-p_1 - \cdots - p_r - q)) = 0$, hence, by the discussion above,

- f is smoothable, keeping $f(p_0), \dots, f(p_r), f(q)$ fixed, to a rational curve $h : \mathbf{P}^1_{\mathbf{k}} \to X$;
- by semi-continuity, we may assume $H^1(\mathbf{P}^1_{\mathbf{k}}, (h^*T_X)(-r - 1)) = 0$, hence h is r-free.

This proves the proposition. □

We now take a special look at a certain kind of rational tree.

Definition 9.38 A *rational* $\mathbf{k}$*-comb* is a rational $\mathbf{k}$-tree with a distinguished irreducible component C_0 (the *handle*) isomorphic to $\mathbf{P}^1_{\mathbf{k}}$ and such that all the other irreducible components (the *teeth*) meet C_0 (transversely in a single point).

Proposition 9.37 tells us that a morphism f from a rational tree C to a smooth variety can be smoothed when the restriction of f to each component of C is free. When C is a rational comb, we can relax this assumption: we only assume that the restriction of f to each tooth is free, and we get a smoothing of a subcomb if there are enough teeth.

Theorem 9.39 *Let C be a rational comb with m teeth and let $p_1, \dots, p_r$ be points on its handle C_0 which are smooth on C. Let X be a smooth projective variety and let $f : C \to X$ be a morphism.*

a) *Assume that the restriction of f to each tooth of C is free, and that*

$$m > (K_X \cdot f_* C_0) + (r-1)\dim(X) + \dim_{[f|_{C_0}]} \mathrm{Mor}(\mathbf{P}^1_{\mathbf{k}}, X; f|_{\{p_1,\dots,p_r\}}).$$

There exists a subcomb C' of C with at least one tooth such that $f|_{C'}$ is smoothable, keeping $f(p_1), \dots, f(p_r)$ fixed.

b) *Let s be a nonnegative integer such that $((f^* T_X)|_{C_0})(s)$ is nef. Assume that the restriction of f to each tooth of C is very free and that*

$$m > s + (K_X \cdot f_* C_0) + (r-1)\dim(X) + \dim_{[f|_{C_0}]} \mathrm{Mor}(\mathbf{P}^1_{\mathbf{k}}, X; f|_{\{p_1,\dots,p_r\}}).$$

There exists a subcomb C' of C with at least one tooth such that $f|_{C'}$ is smoothable, keeping $f(p_1), \dots, f(p_r)$ fixed, to a very free curve.

Proof. We construct a "universal" smoothing of the comb C as follows. Let $\mathscr{C}_m \to C_0 \times \mathbf{A}^m_{\mathbf{k}}$ be the blow-up of the (disjoint) union of the subvarieties $\{q_i\} \times \{y_i = 0\}$, where $y_1, \dots, y_m$ are coordinates on $\mathbf{A}^m_{\mathbf{k}}$. Fibers of $\pi : \mathscr{C}_m \to \mathbf{A}^m_{\mathbf{k}}$ are subcombs of C, the number of teeth being the number of coordinates y_i that vanish at the point. Note that π is projective and flat, because its fibers are curves of the same genus 0. Let m' be a positive integer smaller than m, and consider $\mathbf{A}^{m'}_{\mathbf{k}}$ as embedded in $\mathbf{A}^m_{\mathbf{k}}$ as the subspace defined by the equations $y_i = 0$ for $m' < i \le m$. The inverse image $\pi^{-1}(\mathbf{A}^{m'}_{\mathbf{k}})$ splits as the union of $\mathscr{C}_{m'}$ and $m - m'$ disjoint copies of $\mathbf{P}^1_{\mathbf{k}} \times \mathbf{A}^{m'}_{\mathbf{k}}$. We set $\mathscr{C} = \mathscr{C}_m$.

Let σ_i be the constant section of π equal to p_i, and let

$$g : \bigsqcup_{i=1}^{r} \sigma_i(\mathbf{A}^m_{\mathbf{k}}) \to X \times \mathbf{A}^m_{\mathbf{k}}$$

be the morphism $\sigma_i(y) \mapsto (f(p_i), y)$. Since π is projective and flat, there is an $\mathbf{A}^m_{\mathbf{k}}$-scheme (9.31)

$$\rho : \mathrm{Mor}_{\mathbf{A}^m_{\mathbf{k}}}(\mathscr{C}, X \times \mathbf{A}^m_{\mathbf{k}}; g) \to \mathbf{A}^m_{\mathbf{k}}.$$

We will show that a neighborhood of $[f]$ in that scheme is not contracted by ρ to a point. Since the fiber of ρ at 0 is $\mathrm{Mor}(C, X; f|_{\{p_1,\dots,p_r\}})$, it is enough to show

$$\dim_{[f]} \mathrm{Mor}(C, X; f|_{\{p_1,\dots,p_r\}}) < \dim_{[f]} \mathrm{Mor}_{\mathbf{A}^m_{\mathbf{k}}}(\mathscr{C}, X \times \mathbf{A}^m_{\mathbf{k}}; g). \tag{9.5}$$

By the estimate (9.4), the right-hand side of (9.5) is at least

$$(-K_X \cdot f_* C) + (1-r)\dim(X) + m.$$

The fiber of the restriction

$$\mathrm{Mor}(C, X; f|_{\{p_1,\dots,p_r\}}) \to \mathrm{Mor}(C_0, X; f|_{\{p_1,\dots,p_r\}})$$

is $\prod_{i=1}^m \mathrm{Mor}(C_i, X; f|_{\{q_i\}})$, so the left-hand side of (9.5) is at most

$$\begin{aligned} & \dim_{[f|_{C_0}]} \mathrm{Mor}(C_0, X; f|_{\{p_1,\dots,p_r\}}) + \sum_{i=1}^m \dim_{[f]} \mathrm{Mor}(C_i, X; f|_{\{q_i\}}) \\ = \ & \dim_{[f|_{C_0}]} \mathrm{Mor}(C_0, X; f|_{\{p_1,\dots,p_r\}}) + \sum_{i=1}^m (-K_X \cdot f_* C_i) \\ < \ & m - (K_X \cdot f_* C) - (r-1)\dim(X), \end{aligned}$$

where we used first the local description of $\mathrm{Mor}(C_i, X; f|_{\{q_i\}})$ given in 6.11 and the fact that $f|_{C_i}$ being free, $H^1(C_i, f^*T_X(-q_i)|_{C_i})$ vanishes, and second the hypothesis. So (9.5) is proved.

Let T be the normalization of a 1-dimensional subvariety of $\mathrm{Mor}_{\mathbf{A}^m_{\mathbf{k}}}(\mathscr{C}, X \times \mathbf{A}^m_{\mathbf{k}}; g)$ passing through $[f]$ and not contracted by ρ. The morphism from T to $\mathrm{Mor}_{\mathbf{A}^m_{\mathbf{k}}}(\mathscr{C}, X \times \mathbf{A}^m_{\mathbf{k}}; g)$ corresponds to a morphism

$$\mathscr{C} \times_{\mathbf{A}^m_{\mathbf{k}}} T \to X.$$

After renumbering the coordinates, we may assume that $\{m'+1, \dots, m\}$ is the set of indices i such that y_i vanishes on the image of $T \to \mathbf{A}^m_{\mathbf{k}}$, where m' is a *positive* integer. As we saw above, $\mathscr{C} \times_{\mathbf{A}^m_{\mathbf{k}}} T$ splits as the union of $\mathscr{C}' = \mathscr{C}_{m'} \times_{\mathbf{A}^{m'}_{\mathbf{k}}} T$, which is flat over T, and some other "constant" components $\mathbf{P}^1_{\mathbf{k}} \times T$. The general fiber of $\mathscr{C}' \to T$ is $\mathbf{P}^1_{\mathbf{k}}$, its central fiber is the subcomb C' of C with teeth attached at the points q_i with $1 \le i \le m'$, and $f|_{C'}$ is smoothable keeping $f(p_1), \dots, f(p_r)$ fixed. This proves a).

Under the hypotheses of b), the proof of a) shows that there is a smoothing $\mathscr{C}' \to T$ of a subcomb C' of C with teeth $C'_1, \dots, C'_{m'}$, where $m' > s$, a section $\sigma' : T \to \mathscr{C}'$ passing through a point of C_0, and a morphism $F : \mathscr{C}' \to X$. Assume for simplicity that $\mathscr{C}'$ is smooth[7] and consider the locally free sheaf

$$\mathscr{E} = (F^*T_X)\Big(\sum_{i=1}^{s+1} C'_i - 2\sigma'(T)\Big)$$

on $\mathscr{C}'$. For $i \in \{1, \dots, s+1\}$, we have $((C'_i)^2) = -1$, hence the restriction of $\mathscr{E}$ to C'_i is nef, and so is $\mathscr{E}|_{C_0} \simeq (f^*T_X|_{C_0})(s-1)$. Using the exact sequences

$$0 \to \bigoplus_{i=1}^{m'} (\mathscr{E}|_{C'_i})(-1) \to \mathscr{E}|_{C'} \to \mathscr{E}|_{C_0} \to 0$$

and

$$0 = \bigoplus_{i=1}^{m'} H^1(C'_i, (\mathscr{E}|_{C'_i})(-1)) \to H^1(C', \mathscr{E}|_{C'}) \to H^1(C_0, \mathscr{E}|_{C_0}) = 0,$$

we obtain $H^1(C', \mathscr{E}|_{C'}) = 0$. By semi-continuity, this implies that a nearby smoothing $h : \mathbf{P}^1_{\mathbf{k}} \to X$ (keeping $f(p_1), \dots, f(p_r)$ fixed) of $f|_{C'}$ satisfies $H^1(\mathbf{P}^1_{\mathbf{k}}, (h^*T_X)(-2)) = 0$, hence h is very free. □

We saw in Corollary 9.26 that on a smooth separably rationally connected projective variety X, there is a very free rational curve through a *general* finite subset of X. We now show that we can do better.

Theorem 9.40 *Let X be a smooth separably rationally connected projective variety defined over an algebraically closed field. There is a very free rational curve through any finite subset of X.*

PROOF. We first prove that there is a very free rational curve through any point of X. Proceed by contradiction and assume that the set Y of points of X through which there are no very free rational curves is nonempty. Since X is separably rationally connected, by Corollary 9.26, its complement U is dense in X, and, since it is the image of the smooth morphism

$$\begin{aligned} \mathrm{Mor}^{\mathrm{vfree}}(\mathbf{P}^1_{\mathbf{k}}, X) & \to X \\ [f] & \mapsto f(0), \end{aligned}$$

[7] For the general case, one needs to analyze precisely the singularities of $\mathscr{C}$ and proceed similarly, replacing C'_i by a suitable Cartier multiple.

it is also open in X. By Remark 9.51, any point of Y can be connected by a chain of rational curves to a point of U, hence there is a rational curve $f_0 : \mathbf{P}^1_{\mathbf{k}} \to X$ whose image meets U and a point y of Y. Choose distinct points $t_1, \dots, t_m \in \mathbf{P}^1_{\mathbf{k}}$ such that $f_0(t_i) \in U$ and, for each $i \in \{1, \dots, m\}$, choose a very free rational curve $\mathbf{P}^1_{\mathbf{k}} \to X$ passing through $f_0(t_i)$. We can then assemble a rational comb with handle f_0 and m very free teeth. By choosing m large enough, this comb can by Theorem 9.39.b) be smoothed to a very free rational curve passing through y. This contradicts the definition of Y.

Let now $x_1, \dots, x_r$ be points of X. We proceed by induction on r to show the existence of a very free rational curve through $x_1, \dots, x_r$. Assume $r \geq 2$ and consider such a curve passing through $x_1, \dots, x_{r-1}$. We can assume that it is $(r-1)$-free and, by Proposition 9.25.a), that it passes through a general point of X. Similarly, there is a very free rational curve through x_r and any general point of X. These two curves form a chain that can be smoothed to an $(r-1)$-free rational curve passing through $x_1, \dots, x_r$ by Proposition 9.37.b). ☐

Remark 9.41 By composing it with a morphism $\mathbf{P}^1_{\mathbf{k}} \to \mathbf{P}^1_{\mathbf{k}}$ of degree s, this very free rational curve can be made s-free, with s greater than the number of points. It is then easy to prove that a general deformation of that curve keeping the points fixed is an immersion if $\dim(X) \geq 2$ and an embedding if $\dim(X) \geq 3$.

9.8 Separably rationally connected varieties over nonclosed fields

Let $\mathbf{k}$ be a field, let $\bar{\mathbf{k}}$ be an algebraic closure of $\mathbf{k}$, and let X be a smooth projective separably rationally connected $\mathbf{k}$-variety. Given any point of the $\bar{\mathbf{k}}$-variety $X_{\bar{\mathbf{k}}}$, there is a very free rational curve $f : \mathbf{P}^1_{\bar{\mathbf{k}}} \to X_{\bar{\mathbf{k}}}$ passing through that point (Theorem 9.40). One can ask about the existence of such a curve defined over $\mathbf{k}$, passing through a given $\mathbf{k}$-point of X. The answer is unknown in general, but Kollár proved that such a curve does exist over certain fields ([Ko3]).

Definition 9.42 A field $\mathbf{k}$ is *large* if for all smooth connected $\mathbf{k}$-varieties X such that $X(\mathbf{k}) \neq \varnothing$, the set $X(\mathbf{k})$ is Zariski-dense in X.

The field $\mathbf{k}$ is large if and only if, for all smooth $\mathbf{k}$-curve C such that $C(\mathbf{k}) \neq \varnothing$, the set $C(\mathbf{k})$ is infinite.

Examples 9.43 1) Local fields such as $\mathbf{Q}_p$, $\mathbf{F}_p((t))$, $\mathbf{R}$, and their finite extensions, are large (because the implicit function theorem holds for analytic varieties over these fields).

2) For any field $\mathbf{k}$, the field $\mathbf{k}((x_1, \dots, x_n))$ is large for $n \geq 1$.

Theorem 9.44 (Kollár) *Let $\mathbf{k}$ be a large field, let X be a smooth projective separably rationally connected $\mathbf{k}$-variety, and let $x \in X(\mathbf{k})$. There exists a very free $\mathbf{k}$-rational curve $f : \mathbf{P}^1_{\mathbf{k}} \to X$ such that $f(0) = x$.*

Proof. The $\mathbf{k}$-scheme $\mathrm{Mor}^{\mathrm{vfree}}(\mathbf{P}^1_{\mathbf{k}}, X; 0 \mapsto x)$ is smooth and nonempty (because, by Corollary 9.26, it has a point in an algebraic closure of $\mathbf{k}$). It therefore has a point in a finite separable extension ℓ of $\mathbf{k}$, which corresponds to a morphism $f_\ell : \mathbf{P}^1_\ell \to X_\ell$. Let $M \in A^1_{\mathbf{k}}$ be a closed point with residual field ℓ. The curve

$$C = (0 \times \mathbf{P}^1_{\mathbf{k}}) \cup (\mathbf{P}^1_{\mathbf{k}} \times M) \subset \mathbf{P}^1_{\mathbf{k}} \times \mathbf{P}^1_{\mathbf{k}}$$

is a comb over $\mathbf{k}$ with handle $C_0 = 0 \times \mathbf{P}^1_{\mathbf{k}}$, and $\mathrm{Gal}(\ell/\mathbf{k})$ acts simply transitively on the set of teeth of $C_{\bar{\mathbf{k}}}$.

The constant morphism $0 \times \mathbf{P}^1_{\mathbf{k}} \to x$ and $f_\ell : \mathbf{P}^1_{\mathbf{k}} \times M \to X$ coincide on $0 \times M$ hence define a $\mathbf{k}$-morphism $f : C \to X$.

As in §9.33, let $T = \mathbf{P}^1_{\mathbf{k}}$, let $\mathscr{C}$ be the smooth $\mathbf{k}$-surface obtained by blowing-up the closed point $M \times 0$ in $\mathbf{P}^1_{\mathbf{k}} \times T$, and let $\pi : \mathscr{C} \to T$ be the first projection, so that the curve $\mathscr{C}_0 = \pi^{-1}(0)$ is isomorphic to C. We let $\mathscr{X} = X \times T$ and $x_T = x \times T \subset X$, and we consider the inverse image ∞_T in $\mathscr{C}$ of the curve $\infty \times T$. The morphism f then defines $f_0 : \mathscr{C}_0 \to \mathscr{X}_0$, hence a $\mathbf{k}$-point of the T-scheme $\mathrm{Mor}_T(\mathscr{C}, \mathscr{X}; \infty_T \mapsto x_T)$ above $0 \in T(\mathbf{k})$.

Lemma 9.45 *The T-scheme $\mathrm{Mor}_T(\mathscr{C},\mathscr{X};\infty_T\mapsto x_T)$ is smooth at $[f_0]$.*

Proof. It is enough to check $H^1(C,(f^*T_X)(-\infty))=0$. The restriction of $(f^*T_X)(-\infty)$ to the handle C_0 is isomorphic to $\mathscr{O}_{C_0}(-1)^{\oplus\dim(X)}$, and its restriction to each tooth is f^*T_X, hence is ample. We conclude with Lemma 9.36. □

Lemma 9.45 already implies, since $\mathbf{k}$ is large, that $\mathrm{Mor}_T(\mathscr{C},\mathscr{X};\infty_T\mapsto x_T)$ has a $\mathbf{k}$-point whose image in T is not 0. It corresponds to a morphism $\mathbf{P}^1_{\mathbf{k}}\to X$ sending ∞ to x. However, there is no reason why this morphism should be very free, and we will need to work a little bit more for that. By Lemma 9.45, there exists a smooth connected $\mathbf{k}$-curve

$$T'\subset\mathrm{Mor}_T(\mathscr{C},\mathscr{X};\infty_T\mapsto x_T)$$

passing through $[f_0]$ and dominating T. It induces a $\mathbf{k}$-morphism

$$F:\mathscr{C}\times_T T'\to X$$

such that $F(T'\times_T\infty_T)=\{x\}$. Since $T'(\mathbf{k})$ is nonempty (it contains $[f_0]$), it is dense in T' because $\mathbf{k}$ is large. Let $T'_0=T'\times_T(T-\{0\})$ and let $t\in T'_0(\mathbf{k})$. The restriction of F to $C\times_T t$ is a $\mathbf{k}$-morphism $F_t:\mathbf{P}^1_{\mathbf{k}}\to X$ sending ∞ to x.

For F_t to be very free, we need to check $H^1(\mathbf{P}^1_{\mathbf{k}},(F_t^*T_X)(-2))=0$. By semi-continuity and density of $T'_0(\mathbf{k})$, it is enough to find an effective relative $\mathbf{k}$-divisor $D\subset\mathscr{C}$, of degree ≥ 2 on the fibers of π, such that

$$H^1(\mathscr{C}\times_T[f_0],(F^*T_X)(-D')|_{C\times_T[f_0]})=0,$$

where $D'=D\times_T T'$. Take for $D\subset\mathscr{C}$ the union of ∞_T and of the strict transform of $M\times T$ in $\mathscr{C}$. The divisor $(D_0)_{\bar{\mathbf{k}}}$ on the comb $(\mathscr{C}\times_T[f_0])_{\bar{\mathbf{k}}}$ has degree 1 on the handle and degree 1 on each tooth. We conclude with Lemma 9.36 again. □

9.9 R-equivalence

Definition 9.46 Let X be a proper variety defined over a field $\mathbf{k}$. Two points x and y in $X(\mathbf{k})$ are *directly R-equivalent* if there exists a morphism $f:\mathbf{P}^1_{\mathbf{k}}\to X$ such that $f(0)=x$ and $f(\infty)=y$.

They are *R-equivalent* if there are points $x_0,\dots,x_m\in X(\mathbf{k})$ such that $x_0=x$ and $x_m=y$ and x_i and x_{i+1} are directly R-equivalent for all $i\in\{0,\dots,m-1\}$. This is an equivalence relation on $X(\mathbf{k})$ called *R-equivalence.*

Theorem 9.47 *Let X be a smooth projective rationally connected real variety. The R-equivalence classes are the connected components of $X(\mathbf{R})$.*

Proof. Let $x\in X(\mathbf{R})$ and let $f:\mathbf{P}^1_{\mathbf{R}}\to X$ be a very free curve such that $f(0)=x$ (Theorem 9.44). The $\mathbf{R}$-scheme $M=\mathrm{Mor}^{\mathrm{vfree}}(\mathbf{P}^1_{\mathbf{R}},X;\infty\mapsto f(\infty))$ is locally of finite type and the evaluation morphism $M\times\mathbf{P}^1_{\mathbf{R}}\to X$ is smooth on $M\times A^1_{\mathbf{R}}$ (Proposition 9.25.a)). By the local inversion theorem, the induced map $M(\mathbf{R})\times A^1(\mathbf{R})\to X(\mathbf{R})$ is therefore open. Its image contains x, hence a neighborhood of x, which is contained in the R-equivalence class of x (any point in the image is directly R-equivalent to $f(\infty)$, hence R-equivalent to x).

It follows that R-equivalence classes are open and connected in $X(\mathbf{R})$. Since they form a partition of this topological space, they are its connected components. □

Let X be a smooth projective separably rationally connected $\mathbf{k}$-variety. When $\mathbf{k}$ is large, there is a very free curve through any point of $X(\mathbf{k})$. When $\mathbf{k}$ is algebraically closed, there is such a curve through any finite subset of $X(\mathbf{k})$ (Theorem 9.40). This cannot hold in general, even when $\mathbf{k}$ is large (when $\mathbf{k}=\mathbf{R}$, two points belonging to different connected components of $X(\mathbf{R})$ cannot be on the same rational curve defined over $\mathbf{R}$). We have however the following result, which we will not prove here (see [Ko4]).

Theorem 9.48 (Kollár) *Let X be a smooth projective separably rationally connected variety defined over a large field* $\mathbf{k}$. *Let $x_1, \dots, x_r \in X(\mathbf{k})$ be R-equivalent points. There exists a very free rational curve passing through*
$x_1, \dots, x_r$.

In particular, $x_1, \dots, x_r$ are all mutually directly R-equivalent.

9.10 Rationally chain connected varieties

We know study varieties for which two general points can be connected by a chain of rational curves (so this is a property weaker than rational connectedness). For the same reasons as in §9.3, we have to modify slightly this geometric definition. We will eventually show that rational chain connectedness implies rational connectedness for *smooth* projective varieties in *characteristic zero* (this will be proved in Theorem 9.53).

Definition 9.49 Let $\mathbf{k}$ be a field and let $\mathbf{K}$ be an algebraically closed extension of $\mathbf{k}$. A $\mathbf{k}$-variety X is *rationally chain connected* if it is *proper* and if there exist a $\mathbf{K}$-variety M and a closed subscheme $\mathscr{C}$ of $M \times X_{\mathbf{K}}$ such that:

- the fibers of the projection $\mathscr{C} \to M$ are connected proper curves with only rational components;
- the projection $\mathscr{C} \times_M \mathscr{C} \to X_{\mathbf{K}} \times X_{\mathbf{K}}$ is dominant.

This definition does not depend on the choice of the algebraically closed extension $\mathbf{K}$.

Remark 9.50 Rational chain connectedness is *not* a birational property: the projective cone over an elliptic curve E is rationally chain connected (pass through the vertex to connect any two points by a rational chain of length 2), but its canonical desingularization (a $\mathbf{P}^1_{\mathbf{k}}$-bundle over E) is not. However, it is a birational property among *smooth* projective varieties in characteristic zero, because it is then equivalent to rational connectedness (Theorem 9.53).

Remark 9.51 If X is a rationally chain connected variety, two general points of $X_{\mathbf{K}}$ can be connected by a chain of rational curves (and the converse is true when $\mathbf{K}$ is uncountable); actually *any* two points of $X_{\mathbf{K}}$ can be connected by a chain of rational curves (this follows from "general principles"; see [Ko1], Corollary 3.5.1).

Remark 9.52 Let $X \to T$ be a proper and equidimensional morphism with normal fibers defined over a field of characteristic zero. The set

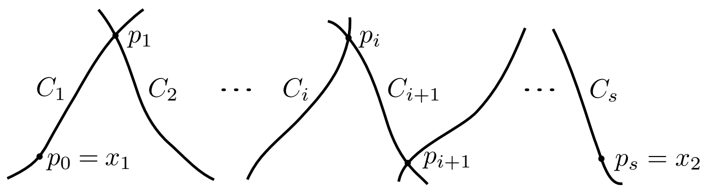
$$\{t \in T \mid X_t \text{ is rationally chain connected}\}$$

is closed (this is difficult; see [Ko1], Theorem 3.5.3). If the morphism is moreover smooth and projective, this set is also open (Theorem 9.53 and Exercise 9.32).

In characteristic zero, we prove that a *smooth* rationally chain connected variety is rationally connected (recall that this is false for singular varieties by Remark 9.50). The basic idea of the proof is to use Proposition 9.37 to smooth a rational chain connecting two points. The problem is to make *each* link free; this is achieved by adding lots of free teeth to each link and by deforming the resulting comb into a free rational curve, keeping the two endpoints fixed, in order not to lose connectedness of the chain.

Theorem 9.53 *A smooth rationally chain connected projective variety defined over a field of characteristic zero is rationally connected.*

Proof. Let X be a smooth rationally chain connected projective variety defined over a field $\mathbf{k}$ of characteristic zero. We may assume that $\mathbf{k}$ is algebraically closed and uncountable. We need to prove that there is a rational curve through two general points x_1 and x_2 of X. There exists a rational chain connecting x_1 and x_2, which can be described as the union of rational curves $f_i : \mathbf{P}^1_{\mathbf{k}} \to C_i \subset X$, for $i \in \{1, \dots, s\}$, with $f_1(0) = x_1$, $f_i(\infty) = f_{i+1}(0)$, $f_s(\infty) = x_2$.



The rational chain connecting x_1 and x_2

We may assume that x_1 is in the subset X^{free} of X defined in Proposition 9.16, so that f_1 is free. We will construct by induction on i rational curves $g_i : \mathbf{P}^1_{\mathbf{k}} \to X$ with $g_i(0) = f_i(0)$ and $g_i(\infty) = f_i(\infty)$, whose image meets X^{free}.

When $i = 1$, take $g_1 = f_1$. Assume that g_i is constructed with the required properties; it is free, so the evaluation map

$$\begin{array}{rccc} \text{ev}: & \operatorname{Mor}(\mathbf{P}^1_{\mathbf{k}}, X) & \longrightarrow & X \\ & g & \longmapsto & g(\infty) \end{array}$$

is smooth at $[g_i]$ (this is not exactly Proposition 9.12, but follows from its proof). Let T be an irreducible component of $\text{ev}^{-1}(C_{i+1})$ that passes through $[g_i]$; it dominates C_{i+1}.

We want to apply the following principle to the family of rational curves on X parametrized by T: *a very general deformation of a curve which meets X^{free} has the same property.* More precisely, given a flat family of curves on X

$$\begin{array}{ccc} \mathscr{C} & \xrightarrow{F} & X \\ \Big\downarrow{\scriptstyle \pi} & & \\ T & & \end{array}$$

parametrized by a variety T, if one of these curves meets X^{free}, the same is true for a very general curve in the family.

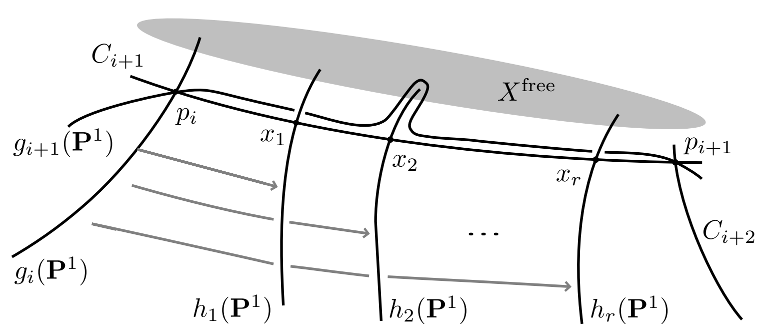
Indeed, X^{free} is the intersection of a countable *nonincreasing* family $(U_i)_{i \in \mathbf{N}}$ of open subsets of X. Let $\mathscr{C}_t$ be the curve $\pi^{-1}(t)$. The curve $F(\mathscr{C}_t)$ meets X^{free} if and only if $\mathscr{C}_t$ meets $\bigcap_{i \in \mathbf{N}} F^{-1}(U_i)$. We have

$$\pi\Big(\bigcap_{i \in \mathbf{N}} F^{-1}(U_i)\Big) = \bigcap_{i \in \mathbf{N}} \pi(F^{-1}(U_i)).$$

Let us prove this equality. The right-hand side contains the left-hand side. If t is in the right-hand side, the $\mathscr{C}_t \cap F^{-1}(U_i)$ form a nonincreasing family of nonempty open subsets of $\mathscr{C}_t$. Since the base field is uncountable, their intersection is nonempty. This means exactly that t is in the left-hand side.

Since π, being flat, is open ([G3], th. 2.4.6), this proves that the set of $t \in T$ such that $f_t(\mathbf{P}^1_{\mathbf{k}})$ meets X^{free} is the intersection of a countable family of dense open subsets of T.

We go back to the proof of the theorem: since the curve g_i meets X^{free}, so do very general members of the family T. Since they also meet C_{i+1} by construction, it follows that given a very general point q of C_{i+1}, there exists a deformation $h_q : \mathbf{P}^1_{\mathbf{k}} \to X$ of g_i which meets X^{free} and x.



Replacing a link with a free link

Picking distinct very general points $q_1, \dots, q_m$ in $C_{i+1} - \{p_i, p_{i+1}\}$, we get free rational curves $h_{q_1}, \dots, h_{q_m}$ which, together with the handle C_{i+1}, form a rational comb C with m teeth (as defined in Definition 9.38) with a morphism $f : C \to X$ whose restriction to the teeth is free. By Theorem 9.39.a), for m large enough, there exists a subcomb $C' \subset C$ with at least one tooth such that $f|_{C'}$ can be smoothed leaving p_i and p_{i+1} fixed. Since C' meets X^{free}, so does a very general smooth deformation by the above principle again. So we managed to construct a rational curve $g_{i+1} : \mathbf{P}^1_{\mathbf{k}} \to X$ through $f_{i+1}(0)$ and $f_{i+1}(\infty)$ which meets X^{free}.

In the end, we get a chain of free rational curves connecting x_1 and x_2. By Proposition 9.37, this chain can be smoothed leaving x_2 fixed. This means that x_1 is in the closure of the image of the evaluation map $\text{ev} : \mathbf{P}^1_{\mathbf{k}} \times \text{Mor}(\mathbf{P}^1_{\mathbf{k}}, X; 0 \mapsto x_2) \to X$. Since x_1 is any point in X^{free}, and the latter is dense in X because the ground field is uncountable, ev is dominant. In particular, its image meets the dense subset $X^{\text{vfree}}_{x_2}$ defined in Proposition 9.30, hence there is a very free rational curve on X, which is therefore rationally connected (Corollary 9.26.a)). □

Corollary 9.54 *A smooth projective rationally chain connected complex variety is simply connected.*

Proof. A smooth projective rationally chain connected complex variety is rationally connected by the theorem, hence simply connected by Corollary 9.28.b). □

9.11 Exercises

1) Let X^d_N be the hypersurface in $\mathbf{P}^N_{\mathbf{k}}$ defined by the equation

$$x_0^d + \cdots + x_N^d = 0.$$

Assume that the field $\mathbf{k}$ has characteristic $p > 0$. Assume also $N \geq 3$.

a) Let r be a positive integer, set $q = p^r$, take $d = p^r + 1$, and assume that $\mathbf{k}$ contains an element ω such that $\omega^d = -1$. The hypersurface X^d_N then contains the line ℓ joining the points $(1, \omega, 0, 0, \dots, 0)$ and $(0, 0, 1, \omega, 0, \dots, 0)$. The pencil

$$-t\omega x_0 + t x_1 - \omega x_2 + x_3 = 0$$

of hyperplanes containing ℓ induces a rational map $\pi : X^d_N \dashrightarrow A^1_{\mathbf{k}}$ which makes $k(X^d_N)$ an extension of $\mathbf{k}(t)$. Show that the generic fiber of π is isomorphic over $\mathbf{k}(t^{1/q})$ to

- if $N = 3$, the rational plane curve with equation

$$y_2^{q-1} y_3 + y_1^q = 0;$$

- if $N \geq 4$, the singular rational hypersurface with equation

$$y_2^q y_3 + y_2 y_1^q + y_4^{q+1} + \cdots + y_n^{q+1} = 0$$

in $\mathbf{P}^{N-1}_{\mathbf{k}}$.

Deduce that X^d_N has a purely inseparable cover of degree q which is rational.

b) Show that X^d_N is unirational whenever d divides $p^r + 1$ for some positive integer r.

2) Let X be a smooth projective variety, let C be a smooth projective curve, and let $f : C \to X$ be a morphism, birational onto its image. Let $g : \mathbf{P}^1 \to X$ be a free rational curve whose image meets $f(C)$. Show that there exists a morphism $f' : C \to X$, birational onto its image, such that $(K_X \cdot f'(C)) < 0$ (*Hint:* form a comb.)

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