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PRANDTL–BATCHELOR FLOW IN A CYLINDRICAL DOMAIN*

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Abstract. In this paper, the classical problem of two-dimensional flow in a cylindrical domain, driven by a nonuniform tangential velocity imposed at the boundary, is reconsidered in straightforward manner. When the boundary velocity is a pure rotation Ω plus a small perturbation $\eta\Omega f(\theta)$ and when the Reynolds number based on Ω is large (Re $\gg 1$), this flow is of "Prandtl–Batchelor" type, namely, a flow of uniform vorticity ω_c in a core region inside a viscous boundary layer of thickness $\mathrm{O(Re)}^{-1/2}$. The $\mathrm{O}(\eta^2)$ contribution to ω_c is determined here by asymptotic analysis up to $\mathrm{O(Re^{-1})}$. The result is in good agreement with numerical computation for Re $\gtrsim 400$.

Key words. vortex flows, boundary layers, vorticity, Prandtl–Batchelor theorem

MSC code. 76D17

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1. Introduction.

We consider two-dimensional flow inside a cylinder $r = a$, on which a tangential velocity $u_\theta = \Omega a[1 + \eta f(\theta)]$ is prescribed. The Reynolds number of the flow is $\mathrm{Re} \equiv \Omega a^2/\nu$, and we assume that $\delta^2 \equiv \mathrm{Re}^{-1} \ll 1$; under this condition, we anticipate a "Prandtl–Batchelor flow" for which the vorticity is uniform in a core region bounded by a Prandtl-type boundary layer of thickness $\mathrm{O}(\delta)$ on $r = a-$ (Prandtl [9] and Batchelor [1]). This problem has been previously treated using the formal machinery of matched asymptotic expansions (Kim [5]) (and we note that the formal existence of a solution has been recently established by Fei et al. [3] for $\eta \ll 1$). In this paper, we analyze the problem afresh, first by linearized analysis at $\mathrm{O}(\eta)$ for arbitrary $\delta > 0$ (following Burggraf [2]), and then by asymptotic analysis at $\mathrm{O}(\eta^2)$ for $\delta \ll 1$. We confirm Batchelor's prediction concerning the level of the uniform-core vorticity (as reported almost simultaneously by Feynman and Lagerstrom [4]; see also section 7.6 of Lagerstrom and Casten [6]), and for the particular choice $f(\theta) = \cos\theta$, we obtain the following asymptotic dependence of this core vorticity on Reynolds number for Re $\gg 1$ (see (4.24)):

$$\omega_{core} = \Omega\left[2 + \frac{1}{2}\eta^2 + \eta^2\left\{\frac{3}{2}\sqrt{2}\,\mathrm{Re}^{-1/2} + 6\,\mathrm{Re}^{-1} + \mathrm{O}\left(\mathrm{Re}^{-3/2}\right)\right\}\right]. \tag{1.1}$$

Kim [5] obtained this result up to $\mathrm{O(Re^{-1/2})}$; this is accurate for $\delta \lesssim 0.01$, i.e., Re $\gtrsim 10^4$ (see Figure 4). We show that inclusion of the additional term 6 Re^{-1} in this expansion allows accurate description of the behavior over the greatly extended range $\delta \lesssim 0.05$, i.e., Re $\gtrsim 400$.

The Prandtl–Batchelor problem provides a prototype of behavior that is encountered in a wide range of situations in geophysical and astrophysical contexts. For example, the phenomenon of "homgenization of vorticity" is well known in the dynamics

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of ocean and atmosphere (Rhines and Young [10]), where it is "potential vorticity" that is homogenized in regions of closed streamlines by the slow expulsion of gradients of this quantity when no internal forcing or heating is present. Similarly, magnetic flux is expelled from regions of closed streamlines in highly conducting fluids permeated by a magnetic field (Parker [8] and Moffatt and Dormy [7]), a process of fundamental importance in astrophysical fluid dynamics.

The results originally obtained by Prandtl and Batchelor apply only in the limit $\mathrm{Re} \to \infty$, whereas applications may well be in the finite range of Re from $\sim 10^2$ to 10^4. The aim of the present paper is to provide an improved theory for large but finite Re, and in so doing to determine how large Re must be for the boundary-layer approach pioneered by Prandtl and Batchelor to be legitimate.

2. Formulation of the problem. Consider then the steady flow of an incompressible fluid in the cylinder $r \le a$, forced by an imposed tangential "squirming" velocity with components

$$u_r = 0, \quad u_\theta = \Omega a[1 + \eta f(\theta)] \quad \text{on } r = a. \tag{2.1}$$

Here we suppose that $f(\theta)$ is a smooth 2π-periodic function with zero mean, and $|\eta| \ll 1$; this is rigid-body rotation with angular velocity Ω, plus a small perturbation. If $\eta = 0$, the flow is then simple rigid-body rotation $(0, \Omega r)$, with stream-function $-\frac{1}{2}\Omega r^2$. When $\eta > 0$, the flow then has a stream-function $-\frac{1}{2}\Omega r^2 + \psi(r,\theta)$, with velocity components

$$\mathbf{u} = \{u_r, u_\theta\} = (0, \Omega r) + (r^{-1}\partial\psi/\partial\theta, -\partial\psi/\partial r) \tag{2.2}$$

and vorticity

$$\omega(r,\theta) = 2\Omega - \nabla^2\psi. \tag{2.3}$$

To simplify notation, we nondimensionalize length and time with respect to a and Ω^{-1} (thus in effect setting $a = 1, \Omega = 1$, and $\nu = \mathrm{Re}^{-1}$). In the steady state considered, the exact vorticity equation then reduces to[1]

$$\frac{\partial\omega}{\partial\theta} - \frac{1}{r}\frac{\partial(\psi, \omega)}{\partial(r, \theta)} = \mathrm{Re}^{-1}\nabla^2\omega, \tag{2.4}$$

with boundary conditions (2.1), together with the requirement that the velocity and vorticity be finite at $r = 0$.

3. Fourier series solution at $\mathbf{O}(\boldsymbol{\eta})$. We may expand $\psi, \mathbf{u}$, and ω in powers of η:

$$\psi(r,\theta) = \eta\left[\psi_1(r,\theta) + \eta\,\psi_2(r,\theta) + \cdots\right], \tag{3.1}$$

$$\mathbf{u}(r,\theta) = \eta\left[\mathbf{u}_1(r,\theta) + \eta\,\mathbf{u}_2(r,\theta) + \cdots\right], \tag{3.2}$$

$$\omega(r,\theta) = \eta\left[\omega_1(r,\theta) + \eta\,\omega_2(r,\theta) + \cdots\right]. \tag{3.3}$$

We focus first on the term $\psi_1(r,\theta)$, which may be obtained by linearized analysis as follows. The $\mathrm{O}(\eta)$ vorticity satisfies

$$\partial\omega_1/\partial\theta = \mathrm{Re}^{-1}\nabla^2\omega_1, \quad \nabla^2\psi_1 = -\omega_1. \tag{3.4}$$

[1]The problem may be formulated alternatively in a frame of reference rotating with angular velocity Ω; see the appendix.

The boundary conditions are that ψ_1 and ω_1 are finite at $r=0$ and that

$$\psi_1 = 0, \quad \partial\psi_1/\partial r = -f(\theta) \quad \text{on } r=1. \tag{3.5}$$

With $\int_0^{2\pi} f(\theta)\,\mathrm{d}\theta = 0$, we may express $f(\theta)$ as a Fourier series

$$f(\theta) = \sum_{n=1}^{\infty} a_n \mathrm{e}^{\mathrm{i}n\theta}, \tag{3.6}$$

with real parts understood throughout, and correspondingly

$$\psi_1 = \sum_{n=1}^{\infty} a_n \hat{\psi}_{1n}(r)\mathrm{e}^{\mathrm{i}n\theta}, \quad \omega_1 = \sum_{n=1}^{\infty} a_n \hat{\omega}_{1n}(r)\mathrm{e}^{\mathrm{i}n\theta}. \tag{3.7}$$

From (3.4), we then have

$$\mathrm{i}n\,\hat{\omega}_{1n} = \mathrm{Re}^{-1}\left[\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\left(r\frac{\mathrm{d}\hat{\omega}_{1n}}{\mathrm{d}r}\right) - \frac{n^2}{r^2}\hat{\omega}_{1n}\right], \tag{3.8}$$

or, defining $\delta^2 \equiv \mathrm{Re}^{-1}$,

$$L_n\,\hat{\omega}_{1n} = \left(\mathrm{i}n/\delta^2\right)\hat{\omega}_{1n}, \quad \text{where} \quad L_n \equiv \left[\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\left(r\frac{\mathrm{d}}{\mathrm{d}r}\right) - \frac{n^2}{r^2}\right]. \tag{3.9}$$

The stream-function $\hat{\psi}_{1n}(r,\delta)$ then satisfies

$$L_n\,\hat{\psi}_{1n}(r,\delta) = -\hat{\omega}_{1n}(r,\delta), \tag{3.10}$$

and, from (3.5) and (3.6), we have to satisfy the boundary conditions

$$\hat{\psi}_{1n} = 0, \quad \partial\hat{\psi}_{1n}/\partial r = -1 \quad \text{on } r=1. \tag{3.11}$$

The solution of this linear problem, regular at $r=0$, was obtained by Burggraf [2]; in the present notation, this solution is

$$\hat{\psi}_{1n}(r,\delta) = \frac{\mathrm{i}^{1/2}\delta\left[r^n J_n\left(\mathrm{i}^{3/2}n^{1/2}/\delta\right) - J_n\left(\mathrm{i}^{3/2}n^{1/2}r/\delta\right)\right]}{n^{1/2}J_{n+1}\left(\mathrm{i}^{3/2}n^{1/2}/\delta\right)}, \tag{3.12}$$

where $J_n(z)$ is the Bessel function of order n.

The corresponding vorticity is

$$\hat{\omega}_{1n}(r,\delta) = -L_n\,\hat{\psi}_{1n}(r,\delta) = \frac{\mathrm{i}^{3/2}n^{1/2}J_n\left(\mathrm{i}^{3/2}n^{1/2}r/\delta\right)}{\delta\,J_{n+1}\left(\mathrm{i}^{3/2}n^{1/2}/\delta\right)}. \tag{3.13}$$

The real and imaginary parts of this function are shown in Figures 1(a)–(c) for $n=1$ and $\delta = 0.04$, 0.03, and 0.02. The position $r-1 = 10\,\delta$ is shown in each case by the black dotted line; this may conveniently be regarded as the edge of the boundary layer, whose structure is evident. This is confirmed in Figure 1(d), in which the abscissa $(X = 1-(1-r)/10\,\delta)$ is stretched by a factor $(10\,\delta)^{-1}$ and the ordinate is squeezed by a factor δ; the limiting situation as $\delta \to 0$ is shown in black. It is evident that in this limit the boundary-layer thickness is proportional to δ, and the vorticity at the boundary scales as δ^{-1}.

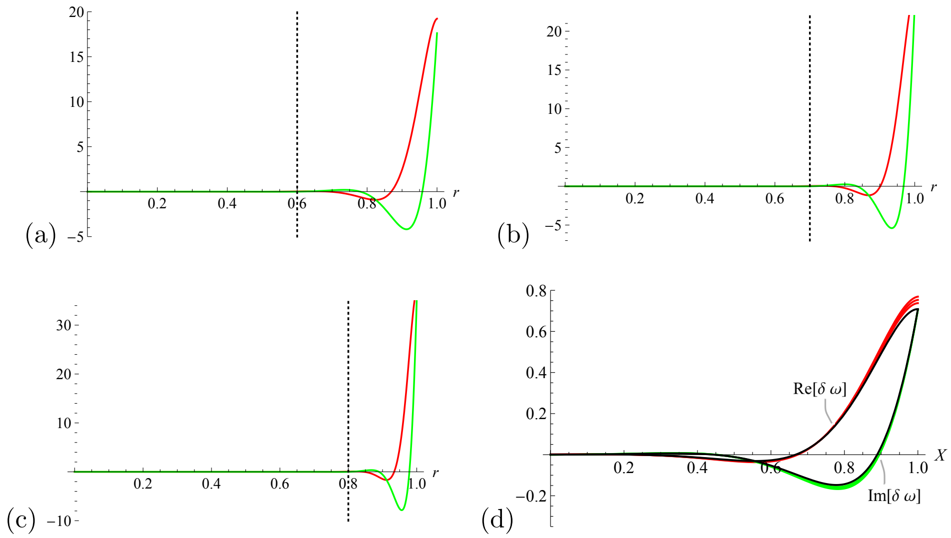

FIG. 1. *Real and imaginary parts (red and green, respectively) of* $\hat{\omega}_{1n}(r,\delta)$ *for* $n=1$ *and* (a) $\delta=0.04$, (b) $\delta=0.03$, (c) $\delta=0.02$*; the black dotted lines are at the edge of the boundary layer, taken to be at* $r-1=10\delta$ *in each case.* (d) *Vorticity curves for the same three values of* δ *with boundary-layer scaling; the limiting forms as* $\delta\to 0$ *are superposed in black; the abscissa,* $X=1-(1-r)/10\delta$, *runs from* $X=0$ *(i.e.,* $r=1-10\delta$*) to* $X=1$ *(*$r=1$*). (Color figure available online.)*

Figure 2 shows streamline patterns $Re[\hat{\psi}_{1n}(r,\delta)\mathrm{e}^{\mathrm{i}n\theta}]=$ const. for $n=1,2,3,$ and 4, and with the choice $\delta=0.04$, sufficiently small to reveal the boundary-layer character, but still large enough for the boundary-layer structure to be visible. In each case, except in the boundary layer, the core flow is the potential flow (zero vorticity) with stream-function proportional to $r^n\mathrm{e}^{\mathrm{i}n\theta}$. Of course, since the problem at $\mathrm{O}(\eta)$ is linear, any linear combination of the solutions $[\hat{\psi}_{1n}(r,\delta)\mathrm{e}^{\mathrm{i}n\theta}]$ is also a solution, so that the general solution for ψ_1 in (3.7) is in effect determined.

The core vorticity at this level of approximation is evidently just that of the rigid-body rotation Ω, i.e., 2Ω. There is no contribution at order η to the core vorticity, consistent with the observation of Fei et al. [3] and in accord with Batchelor [1] and Kim [5] that the core vorticity is perturbed only at order η^2, an effect that is considered in the following section.

4. Nonlinear effect at order $\boldsymbol{\eta^2}$. We have in effect determined the stream-function $\psi_1(r,\theta)$ through the linearized analysis of the preceding section. We now seek to determine $\psi_2(r,\theta)$. Note that if we average (3.3) over θ, we have

$$\langle\omega\rangle(r)=\eta^2\,\langle\omega_2\rangle(r)+\cdots, \tag{4.1}$$

and particular interest focuses on the term $\langle\omega_2\rangle(r)$ and on the asymptotic level of this function when $\delta\ll 1$ in the core region $\{r\geq 0,\, 1-r\gg\delta\}$.

At order η^2, the vorticity equation (2.4) gives

$$\frac{\partial\omega_2}{\partial\theta}-\frac{1}{r}\frac{\partial(\psi_1,\omega_1)}{\partial(r,\theta)}=\delta^2\nabla^2\omega_2,\quad\text{with}\quad\omega_2=-\nabla^2\psi_2. \tag{4.2}$$

It is sufficient here to consider the special situation when $f(\theta)=a_1\cos n\theta$; the constant a_1 can be absorbed in the definition of η in (2.1) so that, in effect, $a_1=1$ and

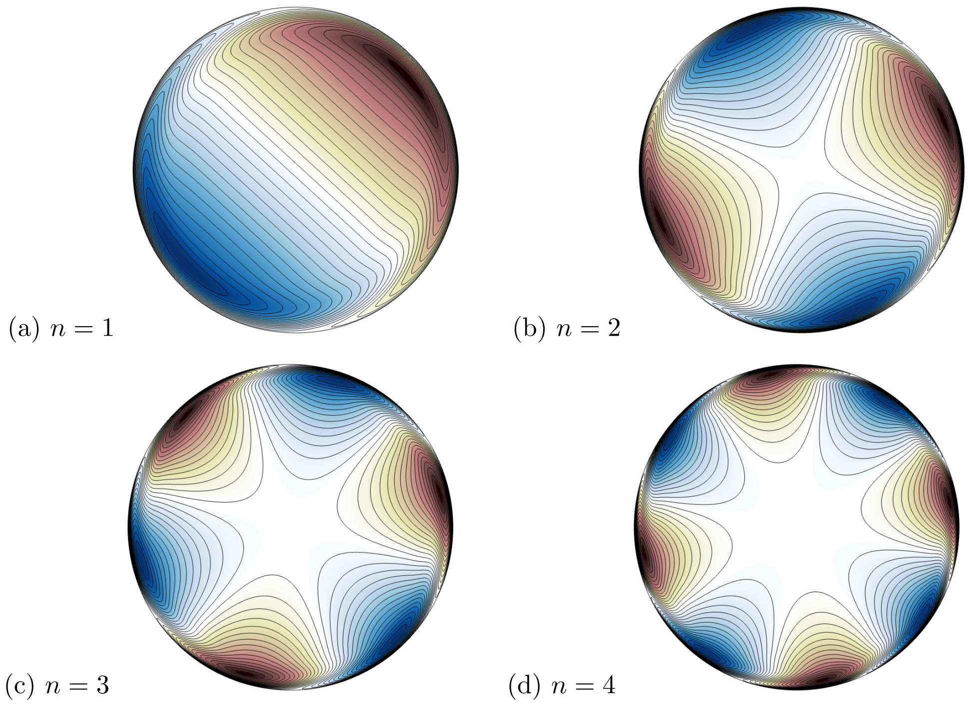

FIG. 2. *Instantaneous streamlines* $Re[\hat{\psi}_{1n}(r,\delta)\mathrm{e}^{\mathrm{i}n\theta}] =$ *const. for* $\delta = \mathrm{Re}^{-1/2} = 0.04$ *(*$Re = 625$*) as given by* (3.12).

$$\psi_1 = Re\left[\hat{\psi}_{1n}(r)\mathrm{e}^{\mathrm{i}n\theta}\right], \quad \omega_1 = Re\left[\hat{\omega}_{1n}(r)\mathrm{e}^{\mathrm{i}n\theta}\right]. \tag{4.3}$$

We then have

$$\delta^2\nabla^2\omega_2 - \partial\omega_2/\partial\theta = -j_2(r,\theta,\delta), \tag{4.4}$$

where $j_2(r,\theta,\delta)$ is the Jacobian

$$j_2(r,\theta,\delta) = \frac{1}{r}\frac{\partial\left(Re\left[\hat{\psi}_{1n}(r)\mathrm{e}^{\mathrm{i}n\theta}\right], Re\left[\hat{\omega}_{1n}(r)\mathrm{e}^{\mathrm{i}n\theta}\right]\right)}{\partial(r,\theta)}. \tag{4.5}$$

It is evident that this Jacobian must take the form

$$j_2(r,\theta,\delta) = j_{20}(r,\delta) + j_{2nc}(r,\delta)\cos 2n\theta + j_{2ns}(r,\delta)\sin 2n\theta, \tag{4.6}$$

where $j_{20}(r,\delta)$ is the average of $j_2(r,\theta,\delta)$ over θ, and that correspondingly,

$$\hat{\omega}_2(r,\theta,\delta) = \hat{\omega}_{20}(r,\delta) + \hat{\omega}_{2nc}(r,\delta)\cos 2n\theta + \hat{\omega}_{2ns}(r,\delta)\sin 2n\theta. \tag{4.7}$$

From (4.6) we note that

$$j_{20}(r,\delta) = \frac{1}{2}[j_2(r,0,\delta) + j_2(r,\pi/2n,\delta)] \tag{4.8}$$

and that

$$j_{2nc}(r,\delta) = j_2(r,0,\delta) - j_{20}(r,\delta), \quad j_{2ns}(r,\delta) = j_2(r,\pi/4n,\delta) - j_{20}(r,\delta). \tag{4.9}$$

As indicated above, our interest focuses on the term $\hat{\omega}_{20}(r,\delta)$ which provides the $\mathrm{O}(\eta^2)$ contribution to the core vorticity. We may obtain this by averaging (4.4) over θ, giving

$$\nabla^2\hat{\omega}_{20}(r,\delta) \equiv \frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}r\frac{\mathrm{d}\hat{\omega}_{20}(r,\delta)}{\mathrm{d}r} = -\frac{1}{\delta^2}j_{20}(r,\delta), \tag{4.10}$$

with

$$\hat{\omega}_{20}(r,\delta) = -\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}r\frac{\mathrm{d}\hat{\psi}_{20}(r,\delta)}{\mathrm{d}r}. \tag{4.11}$$

Equations (4.10) and (4.11) must be integrated subject to boundary conditions that

$$\hat{\psi}_{20}(r,\delta) \;\text{ and }\; \hat{\omega}_{20}(r,\delta) \;\text{ are finite at } r=0 \tag{4.12}$$

and

$$\hat{\psi}_{20}=0, \quad \partial\hat{\psi}_{20}/\partial r=0 \quad \text{at } r=1, \tag{4.13}$$

(the no-slip condition on $r=1$ being already accommodated at the $\mathrm{O}(\eta)$ level).

Before proceeding to the asymptotic (small δ) analysis of this system, we first show numerical results obtained for several small values of δ. For this purpose, anticipating zero $\mathrm{O}(\eta^2)$ contribution to velocity at $r=0$ and a constant core vorticity for $\delta \ll 1$, we replace the finiteness conditions (4.12) by the zero-gradient conditions:

$$\partial\hat{\psi}_{20}/\partial r=0, \quad \partial\hat{\omega}_{20}/\partial r=0 \quad \text{at } r=0. \tag{4.14}$$

Figure 3(a) shows resulting computed curves of vorticity $\hat{\omega}_{20}(r,\delta)$ for $n=1$ and for values of δ increasing from 0.01 to 0.05. For these curves, a clear "core" region of uniform vorticity $0 < r \lesssim 1 - 10\,\delta$ is evident. Figure 3(b) shows the same curves, augmented by additional dashed curves in the range $0.06 \le \delta \le 0.1$; for these, the boundary layer becomes so thick that the concept of a uniform vorticity core becomes increasingly suspect.

The integration of the exact system (4.10)–(4.13) cannot be carried out analytically. However, for $\delta \ll 1$, we can make analytical progress as follows.

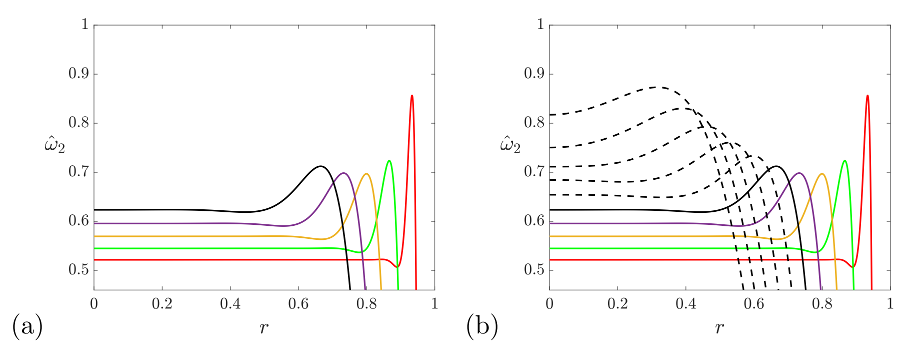

FIG. 3. (a) *Vorticity curves* $\hat{\omega}_2(r,\delta)$ *(for* $n=1$*) from numerical integration of* (4.10)–(4.11)*;* $\delta=0.01$ *(red),* 0.02 *(green),* 0.03 *(orange),* 0.04 *(purple) and* 0.05 *(black);* (b) *the same, augmented by dashed curves for* $\delta=0.06, 0.07, 0.08, 0.09,$ *and* 0.1.

4.1. Asymptotic analysis for $\delta \ll 1$. For simplicity, and by way of illustration, we restrict our attention to the case $n=1$, so that the boundary conditions on $r=1$ now become $\psi=0$, $\partial\psi/\partial r=-\eta\cos\theta$, and, from (3.12) and (3.13),

$$\hat{\psi}_1(r,\delta)=\frac{\mathrm{i}^{1/2}\delta\left[r\,J_1\left(\mathrm{i}^{3/2}\delta\right)-J_1\left(\mathrm{i}^{3/2}r/\delta\right)\right]}{J_2\left(\mathrm{i}^{3/2}/\delta\right)},\quad \hat{\omega}_1(r,\delta)=\frac{\mathrm{i}^{3/2}J_1\left(\mathrm{i}^{3/2}r/\delta\right)}{\delta\,J_2\left(\mathrm{i}^{3/2}/\delta\right)}. \tag{4.15}$$

For $0<\delta\ll r<1$, and with $z\propto 1/\delta$, we shall adopt the large z asymptotic expansion

$$J_m(z)\sim\frac{(-\mathrm{i})^m\mathrm{e}^{\mathrm{i}z}}{\sqrt{2\pi\mathrm{i}z}}\left[1-\frac{4m^2-1}{8\mathrm{i}z}+\frac{(4m^2-1)(4m^2-9)}{2!(8\mathrm{i}z)^2}+\mathrm{O}\left(z^{-3}\right)\right] \qquad (\pi/2<\arg z<\pi). \tag{4.16}$$

Retaining terms in the square bracket in this expansion up to order z^{-2} for $m=1$ and 2, we obtain from (4.15) the following asymptotic expressions for $\hat{\psi}_1(r,\delta)$ and $\hat{\omega}_1(r,\delta)$:

$$\begin{aligned}\hat{\psi}_{1a}(r,\delta)\sim\mathrm{i}^{3/2}\delta\Bigg\{r^{-1/2}\exp\left[\frac{-\mathrm{i}^{1/2}(1-r)}{\delta}\right]\Bigg[1+\frac{3\,\mathrm{i}^{3/2}\delta}{8}\left(r^{-1}-5\right)\\ +\frac{15\,\mathrm{i}\,\delta^2}{128}\left(r^{-2}+6r^{-1}-23\right)\Bigg]-r\left(1-\frac{3\,\mathrm{i}^{3/2}\delta}{2}-\frac{15\,\mathrm{i}\,\delta^2}{8}\right)\Bigg\}\end{aligned} \tag{4.17}$$

and

$$\hat{\omega}_{1a}(r,\delta)\sim\frac{\mathrm{i}^{1/2}}{r^{1/2}\delta}\exp\left[\frac{-\mathrm{i}^{1/2}(1-r)}{\delta}\right]\left[1+\frac{3\,\mathrm{i}^{3/2}\delta}{8}\left(r^{-1}-5\right)+\frac{15\,\mathrm{i}\,\delta^2}{128}\left(r^{-2}+6r^{-1}-23\right)\right]. \tag{4.18}$$

Note that, since we have applied the asymptotic expansion (4.16) to $J_1(\mathrm{i}^{3/2}r/\delta)$ as well as to $J_1(\mathrm{i}^{3/2}/\delta)$ and $J_2(\mathrm{i}^{3/2}/\delta)$, the expressions (4.17) and (4.18) are not uniformly valid down to $r=0$; they are valid only for $0<\delta\ll r<1$.

From these expressions, we may construct a corresponding asymptotic expression for the mean Jacobian (cf. (4.5)),

$$j_{0a}(r,\delta)=\frac{1}{r}\left\langle\frac{\partial\left(Re\left[\hat{\psi}_{1a}(r)\mathrm{e}^{\mathrm{i}\theta}\right],Re\left[\hat{\omega}_{1a}(r)\mathrm{e}^{\mathrm{i}\theta}\right]\right)}{\partial(r,\theta)}\right\rangle. \tag{4.19}$$

The explicit expanded expression for $j_{0a}(r,\delta)$ is very involved and need not be shown here; it is sufficient to note that, in view of its dependence on $\hat{\omega}_{1a}(r)$, $j_{0a}(r,\delta)$ is exponentially small outside the boundary layer.

The mean velocity at $\mathrm{O}(\eta^2)$ has only the θ-component $u_{\theta 2}(r,\delta)=-\mathrm{d}\langle\psi_2\rangle(r,\delta)/\mathrm{d}r$, and this satisfies the third-order equation

$$\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\left(r\frac{\mathrm{d}}{\mathrm{d}r}\right)\left(\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\left(r\,u_{\theta 2}\right)\right)=-\frac{1}{\delta^2}j_{0a}(r,\delta). \tag{4.20}$$

Remarkably, despite the complicated form of $j_{0a}(r,\delta)$, this equation can be integrated to provide an explicit particular integral $u^P_{\theta 2}(r,\delta)$ that decreases exponentially in passing from the boundary layer to the core region. The solution finite at $r=0$ is then given by

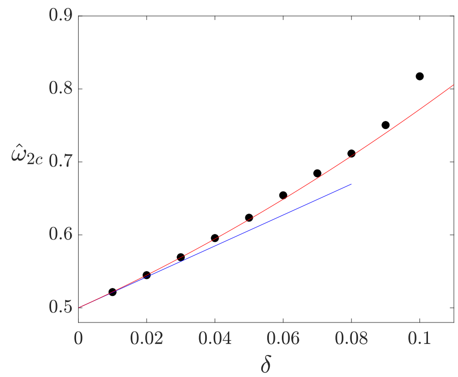

FIG. 4. *Core vorticity as a function of* δ*; the red curve is the asymptotic function* $\hat{\omega}_{2c} \sim (1/2) + (3/2)\sqrt{2}\delta + 6\delta^2$*; the blue line is the linear approximation* $\hat{\omega}_{2c} \sim (1/2) + (3/2)\sqrt{2}\delta$*, as determined by Kim* [5]*; and the points are vorticity levels at* $r = 0$*, corresponding to the computed curves of Figure* 3(b). *(Color figure available online.)*

$$u_{\theta 2}(r, \delta) = u_{\theta 2}^P(r, \delta) + \frac{1}{2}\hat{\omega}_{2c}(\delta)\, r, \tag{4.21}$$

where $\hat{\omega}_{2c}(\delta)$ is the contribution to core vorticity at $\mathrm{O}(\eta^2)$; this is determined by the requirement that $u_{\theta 2}(1, \delta) = 0$, i.e.,

$$\hat{\omega}_{2c}(\delta) = -2\, u_{\theta 2}^P(1, \delta). \tag{4.22}$$

In this way, with the help of Mathematica, the following asymptotic formula for $\hat{\omega}_{2c}(\delta)$ is obtained:

$$\hat{\omega}_{2c}(\delta) = \frac{1}{2} + \frac{3}{2}\left(\sqrt{2}\,\delta\right) + 3\left(\sqrt{2}\,\delta\right)^2 + \mathrm{O}\left(\delta^3\right). \tag{4.23}$$

The term $\frac{3}{2}\sqrt{2}\,\delta$ was obtained by Kim [5] by the matched-asymptotic-expansion procedure; the additional term at order δ^2 is new.

From (3.3), the total core vorticity up to $\mathrm{O}(\eta^2)$ is now given by

$$\omega_{core} = 2 + \eta^2\hat{\omega}_{2c}(\delta) = 2 + \frac{1}{2}\eta^2 + \eta^2\left(\frac{3}{2}\sqrt{2}\,\delta + 6\,\delta^2 + \mathrm{O}\left(\delta^3\right)\right). \tag{4.24}$$

We note here that, according to Batchelor [1], with $f(\theta) = \cos n\theta$, the core vorticity in the limit $\delta \to 0$ is given by

$$\omega_{core}^2 \sim 4\left\langle [1 + f(\theta)]^2 \right\rangle = 4\left(1 + \frac{1}{2}\eta^2\right), \tag{4.25}$$

and hence

$$\omega_{core} \sim 2 + \frac{1}{2}\eta^2 - \frac{1}{16}\eta^4 + \mathrm{O}\left(\eta^6\right). \tag{4.26}$$

The expression (4.24) therefore correctly captures the term $\frac{1}{2}\eta^2$ at order η^2. However, the Batchelor result is evidently correct only in the strict asymptotic limit $\delta = 0$; corrections at $\mathrm{O}(\eta^2)$ when $0 < \delta \ll 1$ are given by (4.24). With $\delta \equiv \mathrm{Re}^{-1/2}$, this result establishes the Reynolds-number dependence of the core vorticity when $\mathrm{Re} \gg 1$ (as recorded in (1.1) in the introduction).

Figure 4 shows the asymptotic curve (4.23) (in red), and the linear approximation of Kim [5] (in blue). The points represent the core vorticity (at $r = 0$) for the ten values of δ corresponding to the curves of Figure 3(b). The linear result fits well for $\delta \lesssim 0.01$ (i.e., Re $\gtrsim 10^4$). For $\delta \lesssim 0.05$, (i.e., for Re $\gtrsim 400$), a uniform vorticity core is reasonably well defined, and the full asymptotic result (4.23) is remarkably accurate. For larger values of δ, the boundary layer is so thick that a uniform vorticity core is not well defined, and departures from the asymptotic formula, as evident in Figure 4, are as might be expected.

Appendix A. In the alternative frame of reference suggested in section 1, $\mathcal{F}_\Omega$ rotating with angular velocity Ω, the perturbation rotates with angular velocity $-\Omega$, and the boundary conditions (2.1) become

$$u_r = 0, \quad u_\theta = \eta\,\Omega a f(\theta + \Omega t) \quad \text{in } \mathcal{F}_\Omega. \tag{A.1}$$

In this frame, the flow is forced by the boundary condition (A.1) on a fluid otherwise at rest. The Coriolis force in $\mathcal{F}_\Omega$, $2\boldsymbol{\Omega}\wedge\mathbf{u} = -2\Omega\,\nabla\psi$, is irrotational and therefore has no effect on the flow. The exact Navier–Stokes equation in $\mathcal{F}_\Omega$ gives

$$\frac{\partial\omega}{\partial t} + \mathbf{u}\cdot\nabla\omega \equiv \frac{\partial\omega}{\partial t} - \frac{1}{r}\frac{\partial(\psi,\omega)}{\partial(r,\theta)} = Re^{-1}\nabla^2\omega, \quad \nabla^2\psi = -\omega_(r,\theta,t). \tag{A.2}$$

In view of (A.1), it is natural to write $\psi = \eta\tilde{\psi}$ and $\omega = \eta\tilde{\omega}$; then (A.2) takes the form of a mixed parabolic/elliptic system, with weak nonlinearity:

$$\frac{\partial\tilde{\omega}}{\partial t} - \mathrm{Re}^{-1}\nabla^2\tilde{\omega} = \eta\,\frac{1}{r}\frac{\partial(\tilde{\psi},\tilde{\omega})}{\partial(r,\theta)}, \quad \nabla^2\tilde{\psi} = -\tilde{\omega}(r,\theta,t). \tag{A.3}$$

This equivalent formulation is in effect what guarantees the existence and uniqueness in the case considered by Fei et al. [3].

We may note in conclusion that omission of the nonlinear term $\mathbf{u}\cdot\nabla\omega$ in the Navier–Stokes equation, while retaining the term $\partial\omega_1/\partial t$ in (A.2), is a standard approximation first used by Stokes [11] in consideration of the behavior of a sphere executing small oscillations in air. In that seminal paper, Stokes also considered the oscillations of a cylinder and showed that, in the limit as the frequency of oscillation decreased to zero, the resulting steady equations have no solution satisfying the no-slip condition on the cylinder and zero velocity at infinity (Stokes' paradox). Our present investigation, being concerned with the flow *inside* a cylinder, does not run into this difficulty.

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