

Math 183

Statistical Methods

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Today: Chapter 2 (beginning)

- Law of Large Numbers
- Vocabulary of probability theory
- Disjointness, Independence
- “or” rule for disjoint events
- “and” rule for independent events

Concepts of Probability

Formalizing everyday life's sentences:

- “You have a 30% chance of surviving this illness.”
- There's a 50/50 chance of getting heads when you flip a fair coin.”

The notion of **probability** covers several related insights.

Bayesian point of view: A probability is a subjective degree of belief. For the same outcome, two persons could have different viewpoints and so assign different probabilities

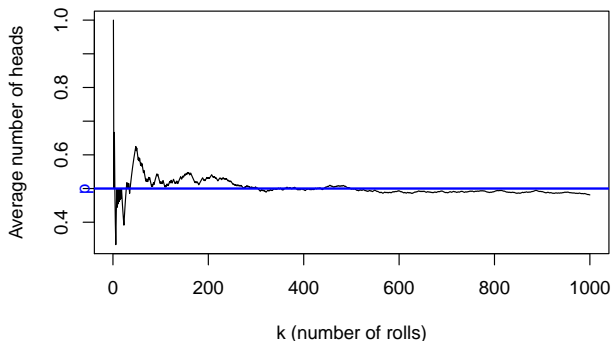
Example: Sports supporters' groups would say their team is more likely to win.

Frequentist approach: The probability of an outcome is the proportion of times the outcome would occur if we observed the random process an infinite number of times.

Law of Large Numbers

The **Law of Large Numbers** (LLN) states the following:

As a random process is repeated more and more, the proportion of times that an event occurs converges to a number (the probability of that event!).



What is Randomness?

Flipping a coin is random...



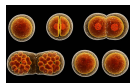
... But by measuring precisely the initial conditions, you could decide whether the result is Heads or Tails.

Stock markets are random...



... Though, knowing all the economic actors' minds would enable to predict trends.

Cell division is random...



... While at the microscopic level, all the processes involved are deterministic (both chemical and physical ones).

What is Randomness?

It can be helpful to model a process as random even if it is not truly random. Random processes often model some lack of information.

Example: coin flip, stock markets, biological processes, ecology,...

Handling randomness is often much more convenient and informative than dozens/thousands parameters.

Going further: introducing randomness artificially can even help in prospective studies

Example: Polls (=Picking people at random in the population and asking them their opinions)

Probability Vocabulary

Let us define some words to talk about theoretical probability:

- **Trial:** Action that creates the data
- **Outcome:** The data created
- **Event:** Some set of outcomes you might care about
- **Sample Space:** The set of all possible outcomes

Let's illustrate this with **Coin flip**:

- **Trial:** Flipping the coin
- **Outcome:** Heads (say)
- **Event:** $A = \{\text{flipping Heads}\}$.
- **Sample Space:** $S = \{\text{Heads, Tails}\}$

Probability Vocabulary

With a **Deck of cards**:

- **Trial:** Drawing a card at random
- **Outcome:** Ace of hearts (say)
- **Event:** $B = \{\text{drawing a King}\}$.
- **Sample Space:** $S = \{\text{the 52 cards}\}$

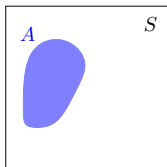
If you are **Gathering info on Facebook usage**:

- **Trial:** Asking three random people if they have a Facebook account
- **Outcome:** NYY (say)
- **Event:** $C = \{\text{all 3 say yes}\}$.
- **Sample Space:**
 $S = \{\text{YYY, YYN, YNY, NYY, YNN, NYN, NNY, NNN}\}$

Defining Probability

If all the outcomes in a (finite) sample space are equally likely, the probability of an event A is defined to be

$$P(A) = \frac{\# \text{ of outcomes in event } A}{\# \text{ of outcomes in sample space } S} = \frac{\# \text{ favorable outcomes}}{\# \text{ possible outcomes}}.$$



From this definition, $0 \leq P(A) \leq 1$.

Different ways to state probabilities numerically:

- Day-to-day life: “This event is 70% likely.”
- Scientific area: “This event has probability .7.”

Probabilities: Example 1

Compute the probability of rolling an odd number on a standard six-sided die

Let A be the event of rolling an odd number: $A = \{1, 3, 5\}$.

The sample space is $S = \{1, 2, 3, 4, 5, 6\}$.

$$P(A) = \frac{\#A}{\#S} = \frac{3}{6} = \frac{1}{2}.$$

Probabilities: Example 2

You flip a fair coin twice. Compute the probability of getting exactly 1 heads.

Let B be the event of getting exactly 1 heads: $B = \{HT, TH\}$.

The sample space is $S = \{HH, HT, TH, TT\}$.

$$P(B) = \frac{\#B}{\#S} = \frac{2}{4} = \frac{1}{2}.$$

Probabilities: Example 3

You roll two four-sided dice. Compute the probability that the number of the first is lower or equal to the number on the second.

Let C be the event $C = \{D_1 \leq D_2\}$.

The sample space is $S = \{\text{all pairs } (D_1, D_2), \text{ with } 1 \leq D_1, D_2 \leq 4\}$.

$$S = \left\{ \begin{array}{cccc} (1, 1) & (1, 2) & (1, 3) & (1, 4) \\ (2, 1) & (2, 2) & (2, 3) & (2, 4) \\ (3, 1) & (3, 2) & (3, 3) & (3, 4) \\ (4, 1) & (4, 2) & (4, 3) & (4, 4) \end{array} \right\}$$

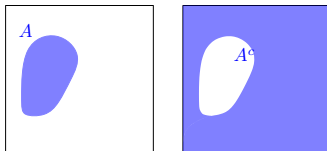
$$C = \left\{ \begin{array}{cccc} (1, 1) & (1, 2) & (1, 3) & (1, 4) \\ & (2, 2) & (2, 3) & (2, 4) \\ & & (3, 3) & (3, 4) \\ & & & (4, 4) \end{array} \right\}$$

$$P(C) = \frac{\#C}{\#S} = \frac{10}{16} = \frac{5}{8}.$$

Further Examining Probabilities

- If an event cannot occur, then $P(A) = 0$.
Example: $A = \{\text{getting heads and tails in one coin flip}\}$
- If an event must occur, then $P(A) = 1$.
Example: $A = \{\text{getting heads or tails}\}$.
- Given an event A , its **complement** A^c is the set of all outcomes not in A but in the sample space ($A^c = S \setminus A$). In particular,

$$P(A^c) = 1 - P(A).$$



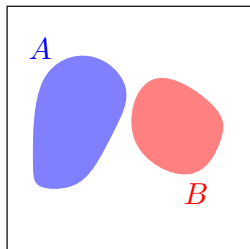
Example: If $A = \{\text{rolling an even number on a die}\}$, then

$$\begin{aligned} A^c &= \{\text{not rolling an even number}\} \\ &= \{\text{rolling an odd number on a die}\} \end{aligned}$$

The “or” Rule

If two events A and B are disjoint, then

$$P(A \text{ or } B) = P(A) + P(B).$$



Generalization: If A_1, A_2, \dots, A_n are mutually disjoint events, then

$$P(A_1 \text{ or } A_2 \text{ or } \dots \text{ or } A_n) = P(A_1) + P(A_2) + \dots + P(A_n).$$

The “or” Rule: Example

You flip a coin 5 times. Compute the probability to get five heads or exactly one tail.

Let A be the event of five Heads:

$$A = \{HHHHH\}.$$

Let B be the event of exactly one Tail:

$$B = \{THHH, HTHHH, HHTHH, HHHTH, HHHHT\}.$$

S is the list of all words of length 5, formed with letters $\{H, T\}$. Hence,

$$\#S = 2 \times 2 \times 2 \times 2 \times 2 = 32.$$

A and B have no outcome in common, therefore,

$$P(A \text{ or } B) = P(A) + P(B) = \frac{1}{32} + \frac{5}{32} = \frac{6}{32} = \frac{3}{16}.$$

It's All Gotta Add To 1

If you break up the sample space into disjoint events, the probabilities of these events must add to 1 (=100%)

Example: Suppose the weather in SD is either Sunny, Cloudy, Rainy, or Snowy. If the first three have probabilities 0.85, 0.08 and 0.06, what is the probability of a snowy day? We have

$$P(\text{Sunny}) + P(\text{Cloudy}) + P(\text{Rainy}) + P(\text{Snowy}) = 1,$$

So

$$0.85 + 0.08 + 0.06 + P(\text{Snowy}) = 1,$$

which yields $P(\text{Snowy}) = 0.01$.

Independence

Two events A and B are said to be **independent** if knowing the outcome of one provides no useful information about the outcome of the other.

Examples of independent events:

- A = “Getting into UCSD”
 B = “Getting Tails on a coin flip”
- A = “Getting heads on flip 1 of a coin”
 B = “Getting heads on flip 2 of a coin”

Examples of dependent events:

- A = “Getting into UCSD”
 B = “Getting into UCLA”
- A = “Getting a red card on top of a deck”
 B = “Getting a red card as the next card in a deck”

The “and” Rule

If two events A and B are independent, then

$$P(A \text{ and } B) = P(A) \times P(B).$$

Example: You toss a coin twice. What is the probability of getting two Heads?

A = “Getting heads on flip 1 of a coin”

B = “Getting heads on flip 2 of a coin”

Then,

$$\begin{aligned} P(\text{Two Heads}) &= P(\text{Heads on 1st toss and Heads on 2nd toss}) \\ &= P(\text{Heads on 1st toss}) \times P(\text{Heads on 2nd toss}) \\ &= \frac{1}{2} \times \frac{1}{2} = \frac{1}{4}. \end{aligned}$$

Deciding If Events are Disjoint or Not

Disjoint or not? Ask yourself

Is there an outcome common to both events?

- **Yes:** Events are NOT disjoint
- **No:** Events are disjoint

Example: $A = \{\text{Person uses Facebook}\}, B = \{\text{Person uses Twitter}\}.$

There are people on both services, so the events A and B are NOT disjoint.

Deciding If Events are Independent or Not

Independent or not?

1. Find $P(A)$
2. Find $P(A \text{ assuming you know event } B \text{ has occurred})$

Do you get the same answer?

- **Yes:** Events are independent
- **No:** Events are NOT independent (= dependent)

Example:

$$P(\text{Facebook}) = 0.64$$

$$P(\text{Facebook given they use Twitter}) = 0.93$$

These event are NOT independent (= dependent).

Practice

Let

$A = \{\text{Someone is a student at UCSD}\}$

$B = \{\text{Someone is part of Muir college}\}$

Events A and B are:

1. Disjoint, independent
2. Disjoint, dependent
3. Not disjoint, independent
4. Not disjoint, dependent **Not disjoint, dependent**

Are there any outcomes in both events? Yes: a student at UCSD in Muir college. Not disjoint events.

$P(\text{student at UCSD})$ is small,

$P(\text{student at UCSD one you know he/she is in Muir college}) = 1.$

Different results means dependent events.

Practice

Let

$A = \{\text{A student's first semester of undergrad is at UCSD}\}$

$B = \{\text{A student's first semester of undergrad is at UCLA}\}$

Events A and B are:

1. Disjoint, independent
2. Disjoint, dependent **Disjoint, dependent**
3. Not disjoint, independent
4. Not disjoint, dependent

Your first semester can be at both schools: we have disjoint events.

$P(A) > 0$, $P(B) > 0$ and $P(A \text{ and } B) = 0$, so

$$P(A \text{ and } B) \neq P(A)P(B).$$

We have dependent events.

Remark: Disjoint \Rightarrow Dependent (The converse is not true!)

Practice

The probability that a batter hits the ball is 0.6. Assuming batting attempts are independent, what is the probability of 3 straight misses?

The probability of a miss is $1 - 0.6 = 0.4$. As a consequence,

$$\begin{aligned} P(3 \text{ misses in a row}) &= P(\text{Miss 1st and Miss 2nd and Miss 3rd}) \\ \text{using independence} \rightarrow &= P(\text{Miss 1st}) \times P(\text{Miss 2nd}) \times P(\text{Miss 3rd}) \\ &= 0.4 \times 0.4 \times 0.4 = 0.064. \end{aligned}$$

What is the probability that first hit is on 3rd attempt?

$$\begin{aligned} P(3 \text{ misses in a row}) &= P(\text{Miss 1st and Miss 2nd and Hits 3rd}) \\ &= P(\text{Miss 1st}) \times P(\text{Miss 2nd}) \times P(\text{Miss 3rd}) \\ &= 0.4 \times 0.4 \times 0.6 = 0.096. \end{aligned}$$

Practice

You roll a green four-sided die (numbered 1-4) and note the outcome G . Do the same with a red four-sided die and note the outcome R . What is the probability that G/R is an integer?

The values (G, R) that turn G/R integer are the 8 outcomes

1/1, 2/1, 2/2, 3/1, 3/3, 4/1, 4/2 and 4/4.

The sample space has size $\#S = 4 \times 4 = 16$. Hence,

$$P(G/R \text{ integer}) = \frac{\#\{(G, R) \text{ such that } G/R \text{ integer}\}}{\#S} = 0.5.$$