Final Exam
Math 181B, UCSD, Spring 2018
Monday, June 11th, 3pm–6pm
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Exercise I

A subtle form of racial discrimination in housing is "racial steering." Racial steering occurs when real estate agents show prospective buyers only homes in neighborhoods already dominated by that family's race. This violates the Fair Housing Act of 1968. According to an article in Chance magazine (Vol. 14, no. 2 [2001]), tenants at a large apartment complex recently filed a lawsuit alleging racial steering. The complex is divided into two parts: Section A and Section B. The plaintiffs claimed that white potential renters were steered to Section A, while African-Americans were steered to Section B. The table displays the data that were presented in court to show the locations of recently rented apartments.

	New Renters		
	White	Black	Total
Section A	87	8	95
Section B	83	34	117
Total	170	42	212

Do you think there is evidence of racial steering? After identifying the test to use, compute the p-value (or give bound(s) on it from the tables) and draw your conclusion at the level $\alpha = 5\%$.

Exercise II

Recall that for all $a, b > 0$, the Gamma distribution with parameters (a, b) has the following density

$$f_{(a,b)}(x) = \frac{b^a}{\Gamma(a)} x^{a-1} e^{-bx},$$

where $\Gamma(a) = \int_0^\infty x^{a-1} e^{-x} dx$. Derive the asymptotic likelihood ratio test for the two sample problem

$$H_0 : \lambda_1 = \lambda_2 \text{ vs } H_1 : \lambda_1 \neq \lambda_2,$$

where the two samples $X_1, \dots, X_n$ and $Y_1, \dots, Y_m$ came from Gamma distribution with parameters $(2, \lambda_1)$ and $(2, \lambda_2)$ respectively. For your answer to be complete, you must show Λ, $\dim \Theta_0$, $\dim \Theta$, and the rejection region of level $\alpha \in (0, 1)$.

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Exercise III

We consider the regression model

$$Y_i = a + bt_i + \varepsilon_i, \quad 1 \le i \le n,$$

where $\varepsilon_1, \dots, \varepsilon_n \sim_{iid} N(0, \sigma^2)$, the real numbers $(t_i)_{1 \le i \le n}$ are known and the parameters a, b and σ^2 are unknown. We assume that $\sum_{i=1}^n t_i = 0$, and we write

$$\bar{Y}_n = \frac{1}{n}\sum_{i=1}^n Y_i, \qquad v_t = \frac{1}{n}\sum_{i=1}^n t_i^2, \qquad v_Y = \frac{1}{n}\sum_{i=1}^n Y_i^2 - \bar{Y}_n, \qquad \rho = \frac{1}{n}\sum_{i=1}^n Y_i t_i.$$

0. Write the model under the matrix form $Y = X\beta + \varepsilon$, where you'll specify $Y \in \mathbb{R}^{n\times 1}$, $X \in \mathbb{R}^{n \times p}$, $\beta \in \mathbb{R}^{p\times 1}$ and $\varepsilon \in \mathbb{R}^{n \times 1}$ explicitly.

1. Give the identifiability conditions of the model. From now on, we assume this condition holds.

2. Compute the least squares estimators $\hat{a}$, $\hat{b}$, and $\hat{\sigma}^2$ of a, b, and σ^2 as a function of $\bar{Y}_n, v_t, v_Y$ and ρ.

3. Describe the joint distribution of $(\hat{a}, \hat{b}, \hat{\sigma}^2)$.

4. Let $\alpha \in (0,1)$. Give confidence intervals of levels $1-\alpha$ for a and b separately.

5. Build a confidence rectangle of level 95% for the parameter (a, b).

6. Build a confidence ellipsoid of level 95% for the parameter (a, b).

7. Derive the exact distribution of $\hat{a} - \hat{b}$.

(Hint: notice that $\hat{a} - \hat{b} = B\begin{pmatrix}\hat{a}\\ \hat{b}\end{pmatrix}$*, where* B *is the row matrix* $(1\ -1)$*.)*

8. Deduce a non-asymptotic rejection region with level α for the test

$$H_0\colon\ a - b = 0 \text{ against } H_1\colon\ a - b \neq 0.$$

Exercise IV

Let $X_1, \dots, X_n$ be a i.i.d. sample of size n having common density f, and cumulative distribution function F. Recall that for all $t \in \mathbb{R}$, $F_n(t) = \frac{1}{n}\sum_{i=1}^n \mathbf{1}_{X_i \le t}$ stands for the empirical distribution function of the sample.

1. For $t \in \mathbb{R}$ fixed, what is the exact distribution of $nF_n(t)$?

2. Derive the limiting distribution of $F_n(t)$.

We assume furthermore that $F(t) \in (0,1)$.

3. Deduce an asymptotic confidence interval for $F(t)$ of level $1-\alpha$.

4. (Bonus) Let $q \in (0,1)$ be given. Find an asymptotic confidence interval of level $1-\alpha$ for the quantile $F^{-1}(q)$. *(Hint:* $F_n(x) \ge u \Leftrightarrow x \ge F_n^{-1}(u)$ *and* $F_n(x) < v \Rightarrow x \le F_n^{-1}(v)$*)*