Homework 7
Math 181B, UCSD, Spring 2018
Do not turn in

Complete the following exercises:

Exercise 1

Observations (X_i, Y_i), $1 \leq i \leq n$ are made according to the model

$$Y_i = a + bX_i + \varepsilon_i,$$

where X_i's are fixed constants and $\varepsilon_1, \dots, \varepsilon_n \sim_{iid} N(0, \sigma^2)$. The model is then reparametrized as

$$Y_i = \alpha + \beta(X_i - \bar{X}_n) + \varepsilon_i.$$

Let $\hat{a}$ and $\hat{b}$ be MLE estimators of a and b, and similarly $\hat{\alpha}$ and $\hat{\beta}$ the MLE's of α and β.

1. Show that $\hat{b} = \hat{\beta}$.

2. Show that $\hat{a} \neq \hat{\alpha}$. In fact, show that $\hat{\alpha} = \bar{Y}_n$.

3. Show that $\hat{\alpha}$ and $\hat{\beta}$ are independent.

(Hint: Use the fact that, from Cochran, empirical mean and empirical variance are independent for Gaussian data)

4. Explain in words what is the meaning of this exercise. Does centering the explanatory variable matter?

Exericse 2

Observations (X_i, Y_i), $1 \leq i \leq n$ are made according to the model

$$Y_i = \beta X_i + \varepsilon_i,$$

where the X_i's are fixed constants and $\varepsilon_1, \dots, \varepsilon_n$ are i.i.d. random variables with uniform distribution over the interval $[0, 1]$.

1. Propose one suitable MLE estimator of β.

2. Recall that we write $X_{(1)} = \min_i X_i$, $X_{(n)} = \max_i X_i$ and similarly for the Y_i's. Prove or disprove that

$$\hat{\beta} = \frac{1}{3}\frac{Y_{(1)}}{X_{(n)}} + \frac{2}{3}\frac{Y_{(n)}}{X_{(1)}}$$

is a MLE estimator of β.

Exercise 3

We consider the model

$$Y_i = m + \sigma\varepsilon_i,$$

where the $\varepsilon_1, \dots, \varepsilon_n \sim_{iid} N(0,1)$, and parameters $(m, \sigma) \in \mathbb{R} \times \mathbb{R}_+$.

1. We assume that σ is known.

(a) Derive a confidence interval of level $1 - \alpha$ for m.

(b) For $\sigma = 3$, how many observations must we have to have the length of the 95% confidence interval smaller than 2? Give the form of this 95% confidence interval for $\sigma = 3$, $n = 25$ and $\bar{y}_{25} = 20$.

(c) Let $\alpha \in (0,1)$. Propose a test of level α for H_0: $m = m_0$ vs H_1: $m \neq m_0$. For $\sigma = 3$, $n = 25$ and $\bar{y}_{25} = 20$ and $m_0 = 18.9$, compute the p-value of this test. Can we accept H_0 at the levels 1%? 5%? 10%?

2. We do not assume that σ is known anymore. We write $\hat{\sigma}_n = \sqrt{\frac{1}{n-1}\sum_{i=1}^n (Y_i - \bar{Y}_n)^2}$.

(a) Write the linear regression model associated to the assumptions made above (on the for $Y = X\beta + \sigma\varepsilon$), and give the least squares estimator.

(b) Apply Cochran's theorem in this case.

(c) Give the exact distribution of $\sqrt{n}\frac{\bar{Y}_n - m}{\hat{\sigma}_n}$.

(d) Test H_0: $\sigma^2 = 3$ against H_1: $\sigma^2 \neq 3$ at the level α.

(e) Derive a confidence interval with level $1 - \alpha$ for m. Deduce a test for H_0: $m = m_0$ against H_1: $m \neq m_0$.