Math 11
Calculus-Based Introductory Probability and Statistics

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Today:

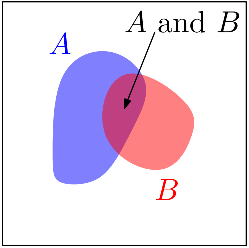
- More Probability Theory

Recap of Last Lecture

- **"or" rule:** If two events A and B are disjoint,

$$P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B).$$

Works for non-disjoint events!



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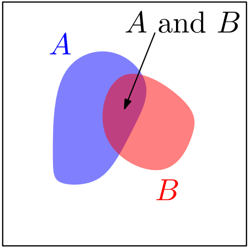

- **“and” rule:** If two events A and B are independent,

$$P(A \text{ and } B) = P(A) \times P(B).$$

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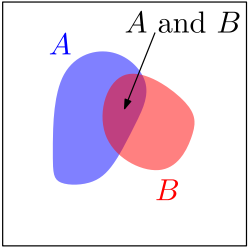

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What if A and B are not independent?

Losing Independence

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Example:

- $A = \{$ It will rain today in San Diego $\}$
- $B = \{$ You see dark storm clouds in the sky $\}$

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Conditional Probability is a tool to handle events that are not independent.

Idea: When computing a probability, you actually have some extra information that you know is true.

Example:

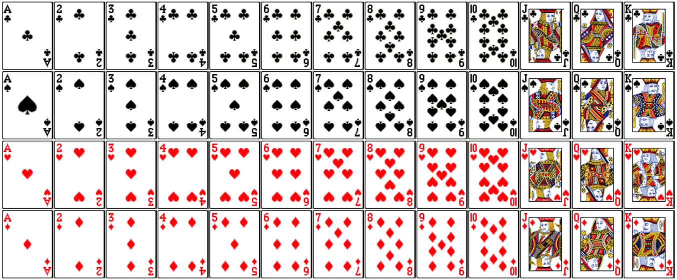
- $A = \{$ It will rain today in San Diego $\}$
- $B = \{$ You see dark storm clouds in the sky $\}$

Although $P(A) \simeq 11.2\%$ is small, you have a much higher chance to see A happen if you know already that B occurred.

Conditional Probability

A card is drawn from a deck. What is the probability that the card is a heart, given that the card is a king?

Intuition says 1/4.



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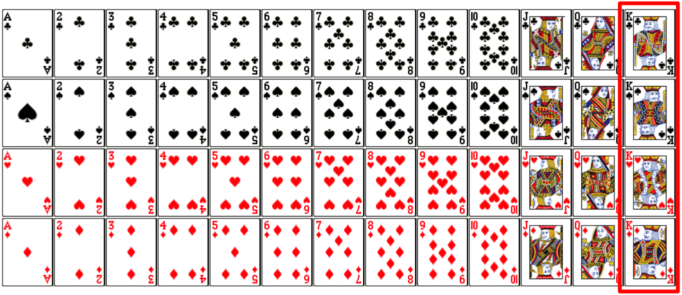

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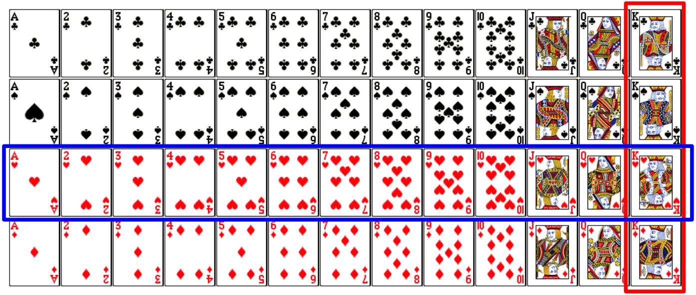

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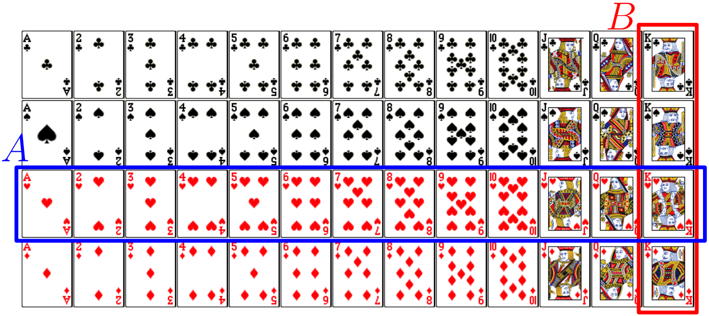

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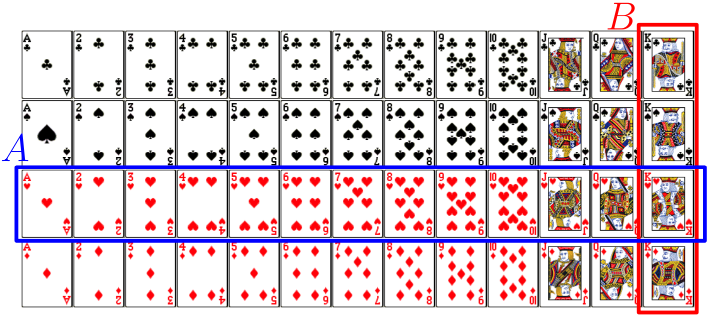
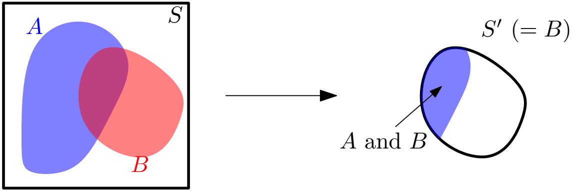


$$P(A \text{ given that } B \text{ occured}) = \frac{P(A \text{ and } B)}{P(B)}$$

Conditional Probability: Definition

For two events A, B the conditional probability of A given B is

$$P(A|B) = \frac{P(A \text{ and } B)}{P(B)}.$$



Computing $P(A|B)$ amounts to do as if B was the sample space.

Understanding Conditional Probability Visually

Go to

http://students.brown.edu/seeing-theory/compound-probability/

and click on Conditional Probability.

Here,

- A is the event "the ball hit ledge A"
- B is the event "the ball hit ledge B"
- (Ignore ledge C)

Note that you can move and stretch any of the ledges.

How can we use the visualization to show $P(A|B) = \dfrac{P(A \text{ and } B)}{P(B)}$?

If you get to assume event B has occurred, you can click the B button (lower right) to make it your whole universe. Then $P(A|B)$ is just the fraction of the B ledge covered by the portion of the A ledge in the picture.

Notion of Independence Revisited

By definition, A and B are independent when

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But for any two events A and B, we have

$$P(A \text{ and } B) = P(A|B) \times P(B).$$

Therefore,

$$\begin{aligned} A \text{ and } B \text{ are independent } &\Leftrightarrow P(A|B) = P(A) \\ &\Leftrightarrow P(B|A) = P(B). \end{aligned}$$

Checking for Independence Revisited

Independent or not?

1. Find $P(A)$
2. Find $P(A$ assuming you know event B has occured)

Do you get the same answer?

- **Yes:** Events are independent
- **No:** Events are NOT independent (= dependent)

Checking for Independence Revisited

Independent or not?

1. Find $P(A)$
2. Find $P(A$ assuming you know event B has occured$) = \mathbf{P(A|B)}$

Do you get the same answer? **= Check if $\mathbf{P(A|B) = P(A)}$**

- **Yes:** Events are independent
- **No:** Events are NOT independent (= dependent)

Social Justice and Independence

Those fighting for social justice often want independence:

$$P(\text{Female}) = P(\text{Female}|\text{Person is a scientist})$$

means the events "Female" and "Person is a scientist" are independent.

$$P(\text{Being pulled over}) = P(\text{Being pulled over}|\text{Being } \textit{insert race here})$$

means that the likelihood to be pulled over should not change based on your race.

Conditional Probability in a Contingency Table

	Junior	Other Levels	Margin totals
Biology Major	41	20	61
Other Major	44	87	131
Margin Totals	85	107	192

What is $P(\text{Junior})$?

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What is $P(\text{Junior})$?

$$p = \frac{85}{192} \simeq 44.2\%.$$

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What is $P(\text{Junior})$? $p = \dfrac{85}{192} \simeq 44.2\%.$

What is $P(\text{Junior}|\text{Biology Major})$?

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What is $P(\text{Junior})$? $$p = \frac{85}{192} \simeq 44.2\%.$$

What is $P(\text{Junior}|\text{Biology Major})$? $$p = \frac{41}{61} \simeq 67.2\%.$$

So the *Level* and *Major* are NOT independent.

Things Work The Same With Probabilities, Not Counts

	Like Video Games	Dislike Video Games	Margin Totals
Like Learning	0.62	0.18	0.8
Dislike Learning	0.08	0.12	0.2
Margin Totals	0.7	0.3	1

Find $P(\text{Dislike Video Games}|\text{Like Learning})$.

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In general,

$$P(A|B) = \frac{P(A \text{ and } B)}{P(B)}.$$

Natural Language and Probabilities

We study data from 3921 emails sent to one user over 3 months, and describe each email with:

- Spam/Not Spam: Indicator for whether the email was spam
- None/Small/Big: Saying whether there was no number, a small number (under 1 million), or a big number

	None	Small	Big
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$$P(\text{ spam and no number }) = \frac{149}{3921} \simeq 3.8\%$$

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- What percent of messages do not contain a small number, ignoring the categorization of spam?

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$$P(\text{ not spam and not none }) = \frac{2659 + 495}{3921} \simeq 80.4\%$$

Dependent VS Independent Events

A box contains 2 red balls and 3 green ones.
You pick two balls **with replacement** (= you pick a ball, note its color, put it back in the box, and then pick a second one).
What is the probability that both balls are red?

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By independence, $P(A \text{ and } B) = P(A) \times P(B) = \frac{2}{5} \times \frac{2}{5} = \frac{4}{25}$.

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$$P(A \text{ and } B) = P(B|A) \times P(A) = \frac{1}{4} \times \frac{2}{5} = \frac{1}{10}.$$

Tree Diagram

A box contains 2 red balls and 3 green ones.
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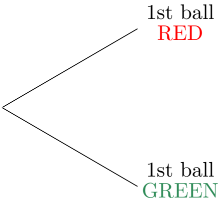

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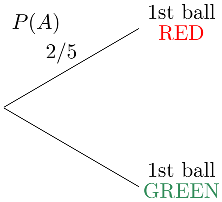

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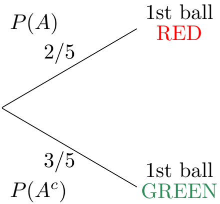

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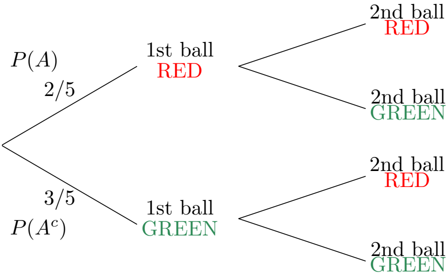

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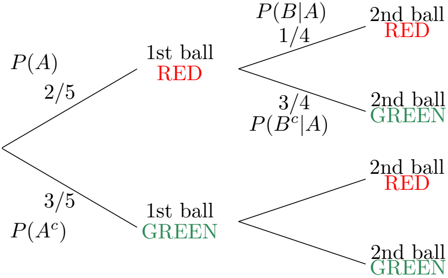

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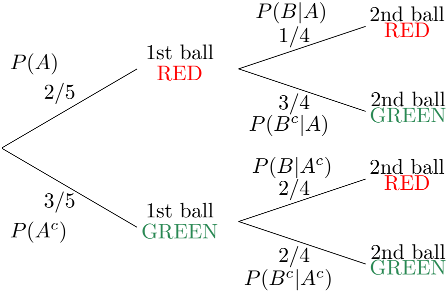

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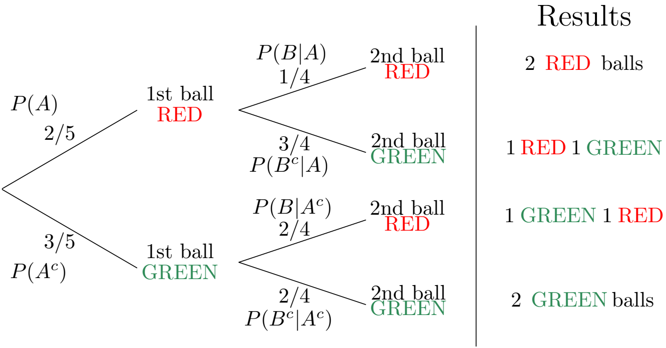

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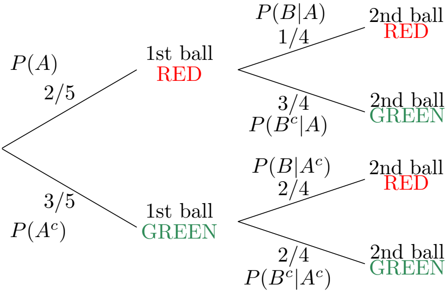

Results	Probability
2 RED balls	$\frac{2}{5} \times \frac{1}{4}$
1 RED 1 GREEN	$\frac{2}{5} \times \frac{3}{4}$
1 GREEN 1 RED	$\frac{3}{5} \times \frac{2}{4}$
2 GREEN balls	$\frac{3}{5} \times \frac{2}{4}$

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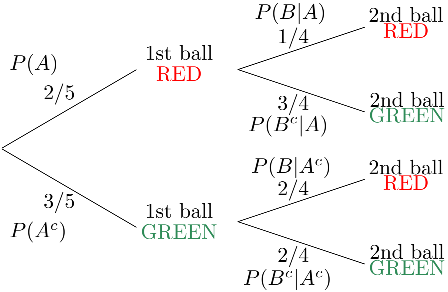

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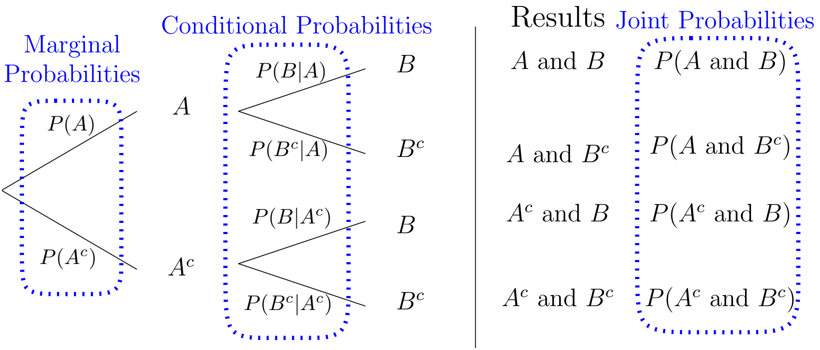
1 GREEN 1 RED	$\frac{3}{5} \times \frac{2}{4}$
2 GREEN balls	$\frac{3}{5} \times \frac{2}{4}$

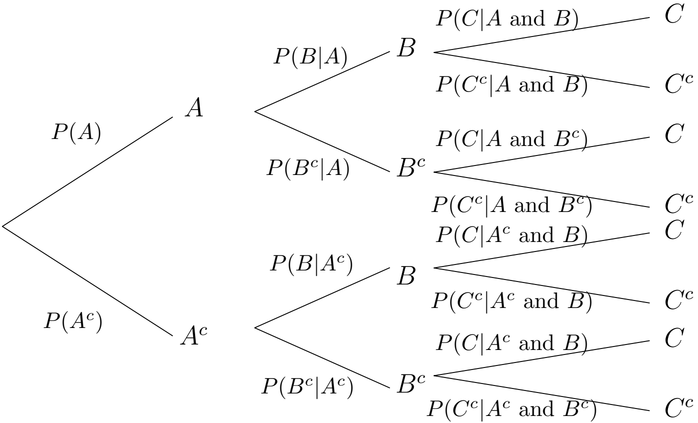
$$P(2 \text{ red}) = \frac{2}{20}, \quad P(1 \text{ red and } 1 \text{ green}) = \frac{12}{20}, \quad P(2 \text{ green}) = \frac{6}{20}.$$

Tree Diagram: General Case



Bigger Tree Diagrams?

If you deal with tree events A, B and C, you can still make a tree.

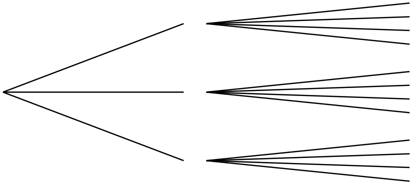


Examples:

- Draw 3 balls/cards without replacement
- Studying the weather of 3 consecutive days
- ...

Richer Tree Diagrams?

If you deal with variables having more than 2 outcomes, you can branch more broadly.



Examples:

- Outcome of a die roll (6 branches)
- Color of a ball in a box containing red,green and blue ones (3 branches)
- ...

Exercise

Consider a board game in which you need to kill an enemy that has 3 health points. You begin by spinning a dial that comes up A with probability 2/3, and B with probability 1/3. In the event of A, you roll a four-sided die (1—4); In the event of B, you roll a six-sided die (1—6). Your roll is the amount of damage you deal the opponent. What is the probability the enemy dies?

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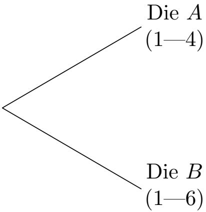

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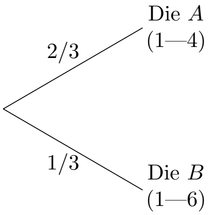

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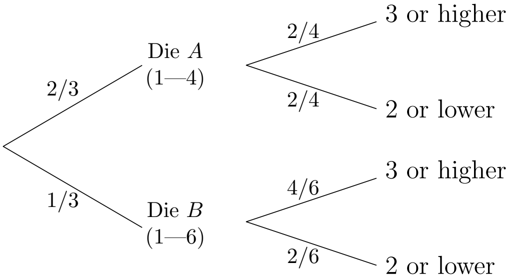

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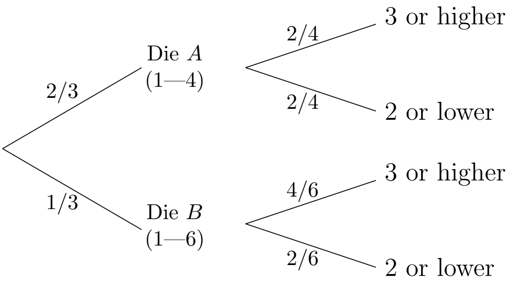

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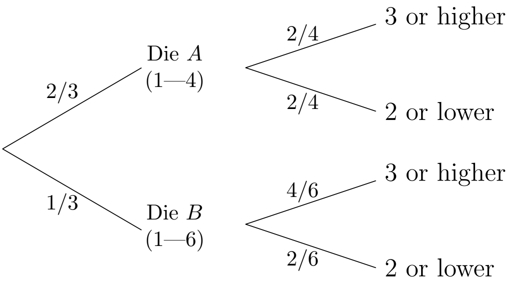

$$\frac{2}{3} \times \frac{2}{4} = \frac{4}{12}$$

$$\frac{1}{3} \times \frac{4}{6} = \frac{4}{18}$$

$$P(\text{enemy dies}) = \frac{4}{12} + \frac{4}{18} = \frac{5}{9}.$$

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Find the probability of pulling two red cards from a standard deck of 52 cards on your first two pulls.

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$$\begin{aligned} P(A \text{ and } B) &= P(A) \times P(B|A) \\ &= \frac{26}{52} \times \frac{25}{51} \simeq 24.5\% \end{aligned}$$

“and” rule: General Case

- If two events A and B are independent,

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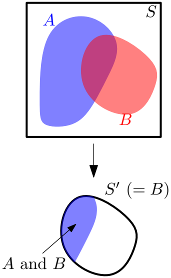

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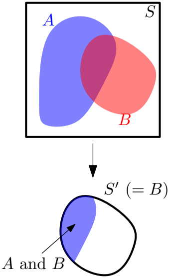

Remark: The second formula coincides with the first one when A and B are independent, since $P(A|B) = P(A)$ and $P(B|A) = P(B)$.

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Dividing by $P(B)$ gives **Bayes' rule**:

$$\boxed{P(A|B) = \frac{P(B|A) \times P(A)}{P(B)}.}$$

Bayes' Rule: Example

In a Math 11 class, 30% of students got an A on Midterm I. 40% of students studied a long time. Suppose the probability that you got an A given that you studied a long time is 70%.
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$$P(S|A) = \frac{P(A|S) \times P(S)}{P(A)} = \frac{0.7 \times 0.4}{0.3} \simeq 0.93.$$

Bayes' Rule: Level 2

One cannot use only Bayes' rule if $P(B)$ is unknown:

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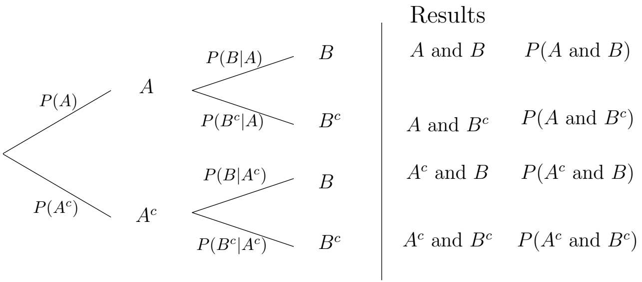

Looking at the tree, we see that:

$$P(B) = P(B|A) \times P(A) + P(B|A^c) \times P(A^c).$$

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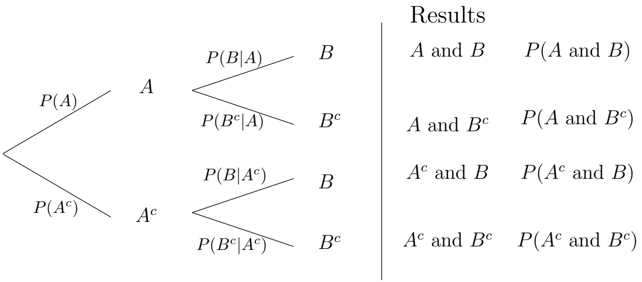

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(Exercise for CS/Math majors: prove this rigorously!)

Bayes' Rule: Level 2

We just derived the **advanced Bayes' rule**:

$$\boxed{P(A|B) = \frac{P(B|A) \times P(A)}{P(B|A) \times P(A) + P(B|A^c) \times P(A^c)}.}$$

Another Example

Doctors wanted to study the effectiveness of a new HIV test. When they used the test on people known to have HIV, it gave a positive result 99.9% of the time. On those known to not have HIV, it gave negative result 99% of the time. Supposing someone from San Diego (HIV rate: 0.0172%) tests positive on this new test, what is the probability the person actually has HIV?

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Name your events:

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The questions amounts to find $P(yes|+)$.

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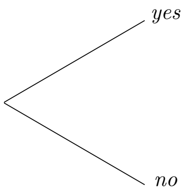
$$P(yes|+) = \frac{P(+|yes) \times P(yes)}{P(+)} = \frac{0.999 \times 0.000172}{??}$$

Another Example



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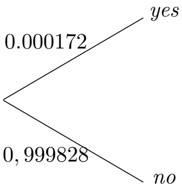

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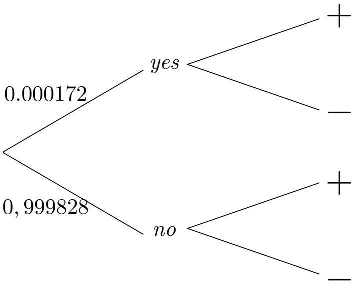

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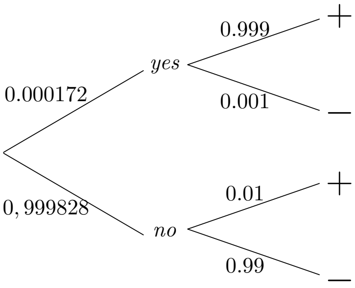

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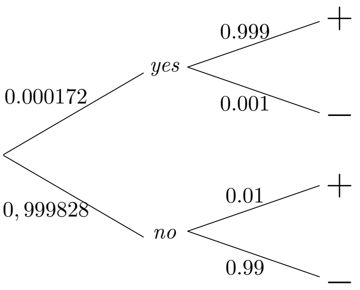

$$0.000172 \times 0.999 = 0,000171828$$

$$0.999828 \times 0.01 = 0.00999828$$

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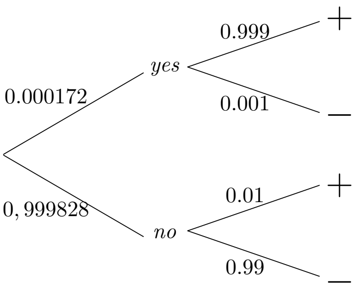

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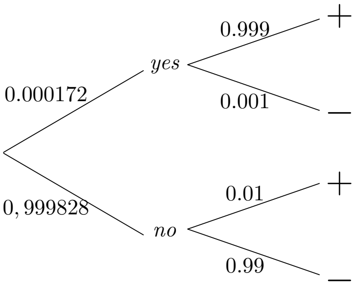

$0.000172 \times 0.999 = 0,000171828$

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$$P(yes|+) = \frac{P(+|yes) \times P(yes)}{P(+)} = \frac{0.999 \times 0.000172}{0.000171828 + 0.00999828} \simeq 1.7\%.$$

What's Happening There?

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Here, we have a great mismatch between

- The accuracy of the test (only 99.9% and 99%)
- The extreme rareness of the disease (0.0172%) in the population.

For positive test results to be useful (that is, for $\mathrm{P}(\text{yes}|+)$ to be high), you need the orders of magnitude of "test accuracy" and "disease prevalence" to be better matched.

Dependence/Independence in Small/Large Populations

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You pick 5 balls in a box containing green and red balls balls.
Study the event $A =$ "all 5 balls are green".

- 3 red 7 green (10 total)
  - **With replacement:** $P(A) = \left(\frac{7}{10}\right)^5 \simeq 0.168.$
  - **Without replacement:** $P(A) = \frac{7}{10}\frac{6}{9}\frac{5}{8}\frac{4}{7}\frac{3}{6} \simeq 0.083.$

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- 30 red 70 green (100 total)
  - **With replacement:** $P(A) = \left(\frac{70}{100}\right)^5 \simeq 0.168.$
  - **Without replacement:** $P(A) = \frac{70}{100}\frac{69}{99}\frac{68}{98}\frac{67}{97}\frac{66}{96} \simeq 0.161.$

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When you draw a sample without replacement, your choices are always dependent.
<u>Although dependence is still present</u>, it tends to disappear when population is large.

Lesson About Sampling

Moral (to be used later in the course):

If your sample size is less than 10% of the population you are sampling from, you may assume the choices are independent (even though they aren't).

Said differently: for samples $< 10\%$ population size, the distinction between "with replacement" and "without replacement" is unnecessary.