Math 11
Calculus-Based Introductory Probability and Statistics

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AP&M 5880A

Today:

- Continuous random variables
- The common continous probability models

Beyond Discrete Models

A **discrete random variable** takes on values with spaces in between them.

- Geometric: $Geom(p)$. $X \in \{1, 2, 3, \ldots\}$
- Binomial: $Binom(n, p)$. $X \in \{0, 1, 2, \ldots, n\}$
- Poisson: $Poisson(\lambda)$. $X \in \{0, 1, 2, \ldots\}$.

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A **continuous random variable** is a random quantity that can take on any value on a continuous scale (a smooth interval of possibilities). Examples:

- Amount of water you drink in a day
- How long you wait for a bus
- How far you live from the nearest grocery store

Some Awkward Questions

How do I make a probability table for such a situation?

What is the probability of drinking exactly 1.759823 liters in a given day?

Is the denominator of

$$P(A) = \frac{\#\text{ outcomes in event}}{\#\text{ outcomes in sample space}}$$

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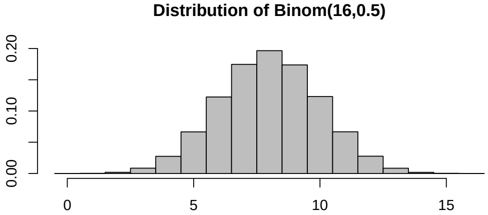
For continuous random variables, we have to:

- change how we present the probability model (no more tables!)
- alter how we ask questions
- generalize our definition of probability
- rethink what a probability of 0 means

From Discrete to Continuous Random Variables

Suppose you flip a fair coin $n = 16$ times and record how many Heads you get. $X = Binom(16, 0.5)$.



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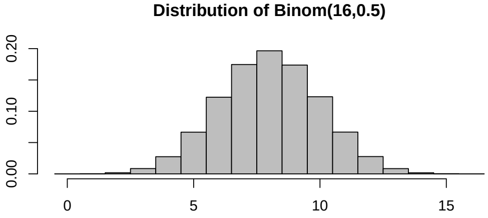
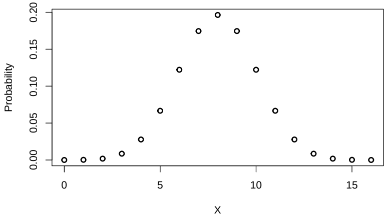


When we visualize the probability table:

- An outcome is <u>more likely if there is more area</u> in the rectangle for that value
- The <u>sum of the areas</u> of the bars <u>must be 1</u>
- The bar <u>heights must be at least 0</u> (no negative heights)

Taking the leaps!

We momentarily use dots insteads of rectangles. The picture suggests how we could generalize: draw curves instead of dots.



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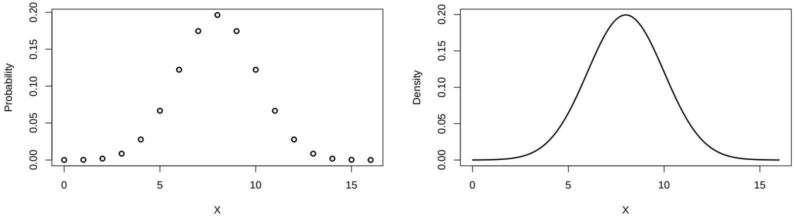

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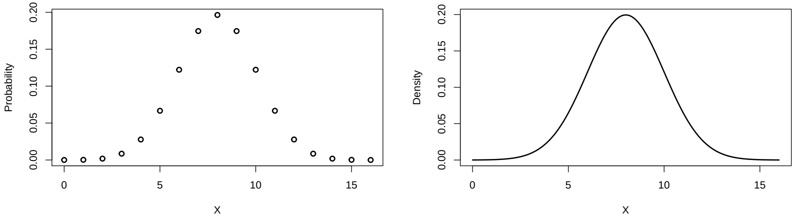

With discrete models, two ideas are linked to probability:

- Height of the y-axis
- Areas of rectangles

With continuous random variables probability is linked to:

- Areas only

Building the New Universe

We model a continuous random variable X through a **density function** $f(x)$ which has two properties:

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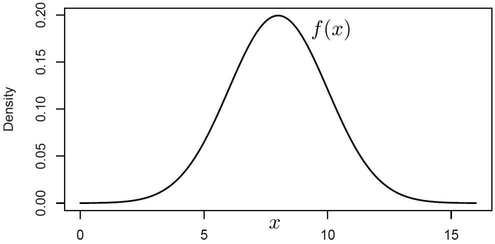

A higher value of $f(x)$ means values <u>nearby</u> x are more likely.

Key Definition

If $f(x)$ is a density function for the continuous random variable X, then we define

$$P(a \leq X \leq b) = \int_a^b f(x)dx.$$

(This formalizes that probability is related to <u>areas</u>)

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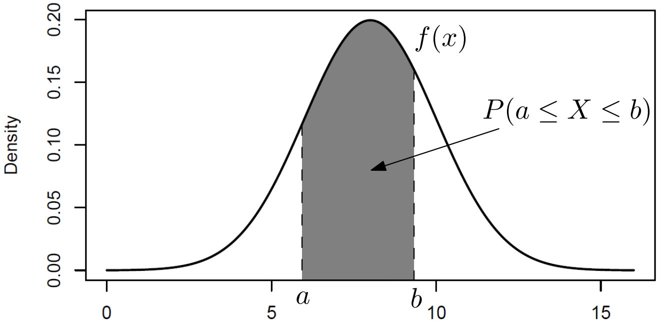

Examples

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To find the probability a person drinks between 2 and 3 liters of water a day, we evaluate

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To find the probability a person drinks less than 4 liters of water a day, we evaluate

$$\int_{-\infty}^4 f(x)dx.$$

The Counter-Intuitive Continuous World

The probability of any particular outcome happening is 0. Indeed,

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The density graph $f(x)$ is <u>not</u> $P(X = x)$! It is a function that helps you figure out probabilities by examining the area underneath it. Its shape suggest that values are more likely (relatively), buth the probability of any particular outcome is still 0.

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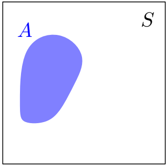
You ask questions about probability of <u>some interval</u> of values occurring since the probability of any individual outcome is always 0.

Probabilities = Areas

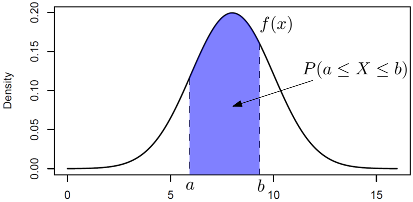
You can reinterpret the discrete world with areas too:

$$P(A) = \frac{\#\text{ outcomes in A}}{\#\text{ outcomes in S}} = \frac{Area(A)}{Area(S)}.$$



Which should sound similar to

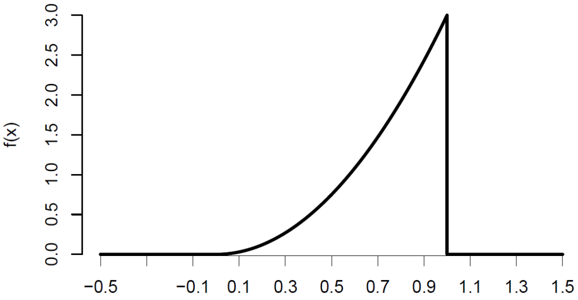
$$P(a \leq X \leq b) = \int_a^b f(x)dx.$$



Example

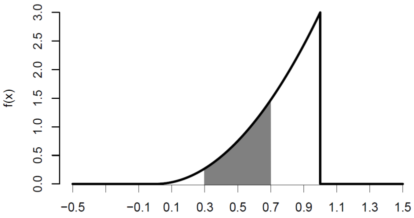
Suppose the concentration of iodine in a chemical sample is modeled by the density function

$$f(x) = \begin{cases} 3x^2 & 0 \leq x \leq 1 \\ 0 & \text{otherwise.} \end{cases}$$



Example

$$
\begin{aligned}
P(0.3 \le X \le 0.7) &= \int_{0.3}^{0.7} f(x)dx \\
&= \int_{0.3}^{0.7} 3x^2 dx \\
&= x^3 \Big|_{0.3}^{0.7} \\
&= (0.7)^3 - (0.3)^3 \\
&= 0.316.
\end{aligned}
$$



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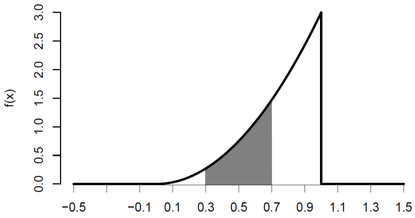
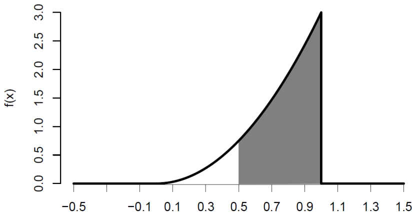


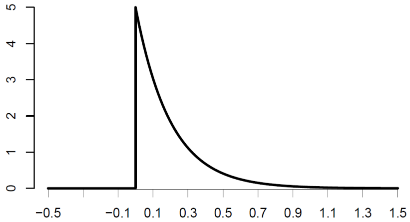
$$
\begin{aligned}
P(X \geq 0.5) &= \int_{0.5}^{\infty} f(x)dx \\
&= \int_{0.5}^{1} 3x^2 dx \\
&= x^3 \Big|_{0.5}^{1} \\
&= 1^3 - (0.5)^3 \\
&= 0.875.
\end{aligned}
$$



Example

Suppose the distance a freshman lives from UCSD (in miles) is modeled by the density

$$g(x) = \begin{cases} 5e^{-5x} & x \geq 0 \\ 0 & \text{otherwise.} \end{cases}$$



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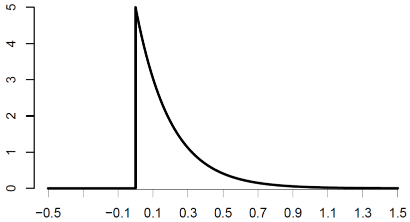

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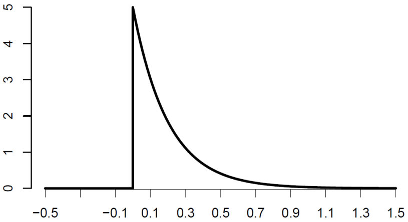

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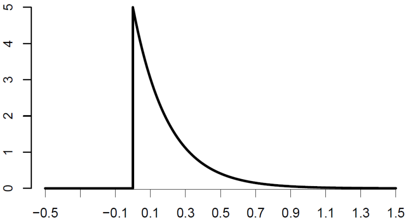

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$$\begin{aligned} \int_{-\infty}^{\infty} g(x)dx = \int_0^{\infty} 5e^{-5x}dx = \lim_{n\to\infty} \int_0^n 5e^{-5x}dx &= \lim_{n\to\infty} -e^{-5x}\Big|_0^n \\ &= \lim_{n\to\infty} (-e^{-5n} - (-e^0)) \\ &= 1. \end{aligned}$$

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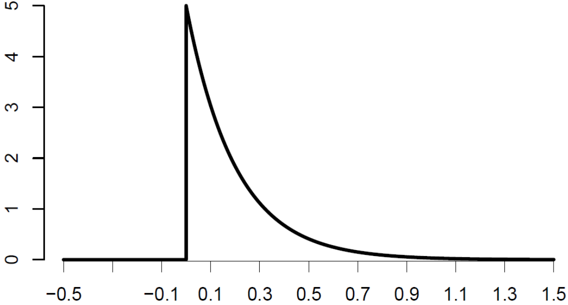
Remark: Here, inclusive or strict inequalities ($\leq$ or $<$) don't make any difference. Indeed, <u>for continuous random variables</u>,

$$\begin{aligned} P(a \leq X \leq b) &= P(a \leq X < b) + P(X = b) \\ &= P(a \leq X < b) + 0 \\ &= P(a \leq X < b) \end{aligned}$$

Example

How far, on average, do we expect a freshman would live from campus?



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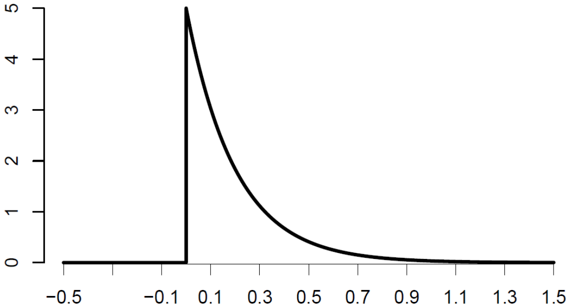

This is not asking for a probability, but for a fact related to the entire model.
We need to know the expected value of a continuous random variable!

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In the previous example, the expected distance is

$$E(X) = \int_0^{\infty} x\left(5e^{-5x}\right)dx = \dots$$

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Easier version:

$$\begin{aligned} Var(X) &= E(X^2) - E(X)^2 \\ &= E(X^2) - \mu^2 \\ &= \int_{-\infty}^{\infty} x^2 f(x)dx - \mu^2. \end{aligned}$$

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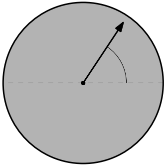
The standard deviation is $SD(X) = \sqrt{Var(X)}$.

Flicking a Spinner

You flick a spinner and measure the angle it makes (from horizontal), in radians, between 0 and 2π. Assuming each angle is equally likely, find the average angle and standard deviation.



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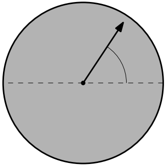

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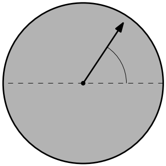

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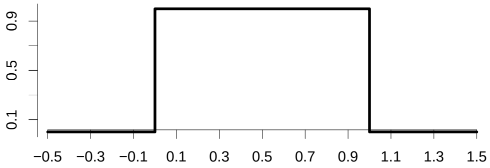

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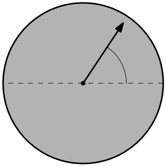

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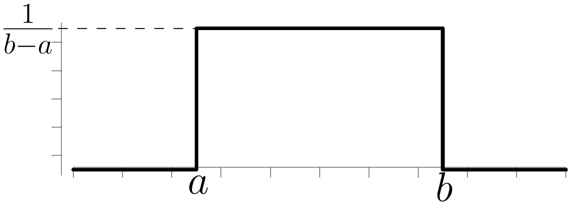
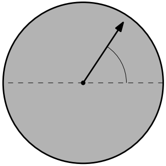


The correct height is $\frac{1}{2\pi}$, since it makes a rectangle of area 1.

Flicking a Spinner: Parameters



Let $f(x) = \begin{cases} \frac{1}{2\pi} & 0 \le x \le 2\pi \\ 0 & \text{otherwise} \end{cases}$

$$E(X) = \int_{-\infty}^{\infty} x f(x) dx$$

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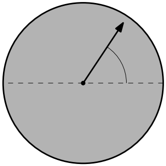

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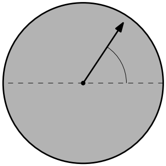

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$$= \frac{1}{2\pi} \left. \frac{x^2}{2} \right|_0^{2\pi} = \frac{1}{2\pi} \frac{4\pi^2}{2} = \pi \text{ rad.}$$

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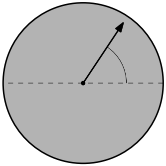

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$$\begin{aligned} E(X) = \int_{-\infty}^{\infty} xf(x)dx &= \frac{1}{2\pi}\int_0^{2\pi} xdx \\ &= \frac{1}{2\pi}\left.\frac{x^2}{2}\right|_0^{2\pi} = \frac{1}{2\pi}\frac{4\pi^2}{2} = \pi \text{ rad.} \end{aligned}$$

$Var(X) = E(X^2) - E(X)^2$, and

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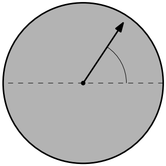

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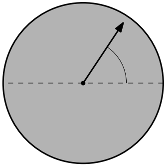

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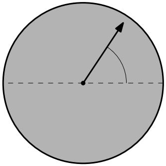

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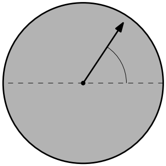

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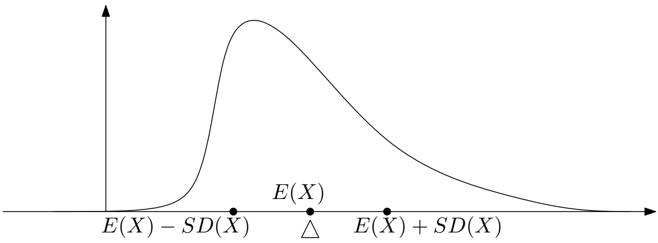
$$Var(X) = \frac{4\pi^2}{3} - \pi^2 = \frac{\pi^2}{3} \text{ , and } SD(X) = \sqrt{Var(X)} = \frac{\pi}{\sqrt{3}} \text{ rad.}$$

Visual Interpretation of the Mean and Standard Deviation



The mean $E(X)$ is the balance point of the density function.

The standard deviation $SD(X) = \sqrt{Var(X)}$ is the distance you must go from the mean to embrace a large chunk of the most likely outcomes. It gives a sense of how compressed the possibilities are around the mean.

Practice

After serious investigations, a TA claims that the free time she has before a student arrives (in minutes) in a given office hour is given by the density:

$$f(x) = \begin{cases} \dfrac{60-x}{1800} & 0 \leq x \leq 60 \\ 0 & \text{otherwise.} \end{cases}$$

Is it a valid density function ?

Find the mean arrival time of that first student, and the standard deviation.

Common Density Function #1

Sometimes, a finite interval of possibilities are all equally likely:

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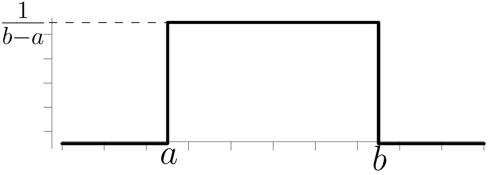
- The random number generator of Minitab, that outputs random real numbers in the interval $[0, 1]$.

Uniform Distribution

The density of the **uniform distribution** over $[a, b]$ is

$$f(x) = \begin{cases} \dfrac{1}{b-a} & a \leq x \leq b \\ 0 & \text{otherwise.} \end{cases}$$



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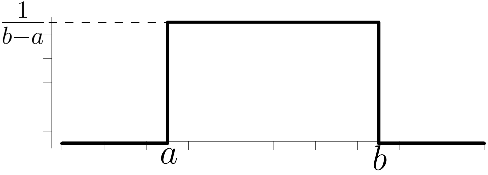

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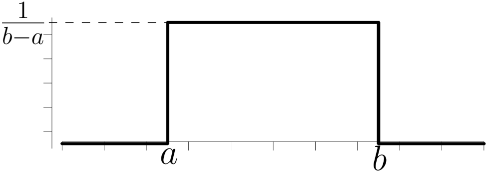

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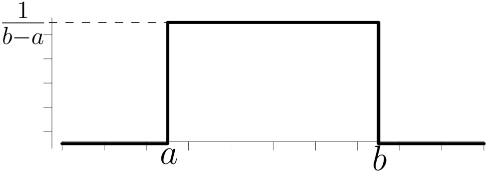

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$$Var(X) = \frac{(b-a)^2}{12} \qquad SD(X) = \frac{b-a}{\sqrt{12}}$$

(Try deriving these yourself!)

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The exponential distribution models the amount of time we have to wait for an event that occurs with frequency λ (units/time interval)

Exponential Distribution

The density of the **exponential distribution** is

$$f(x) = \begin{cases} \lambda e^{-\lambda x} & x \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

where $\lambda > 0$ describes a time-based rate.

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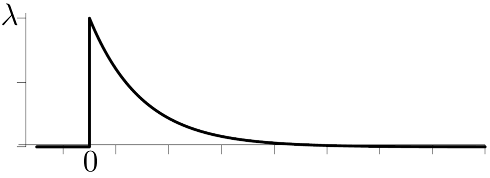

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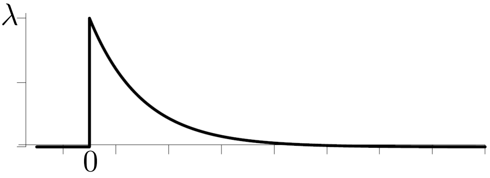

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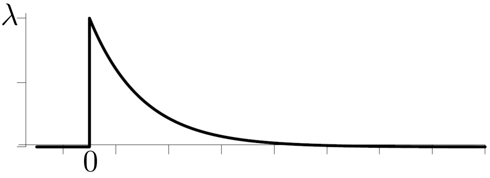

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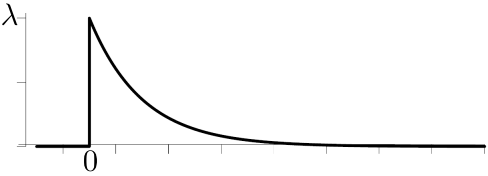

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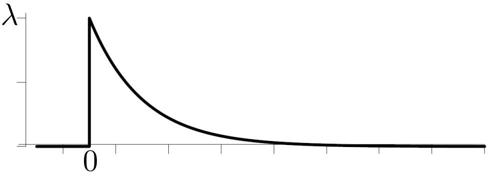

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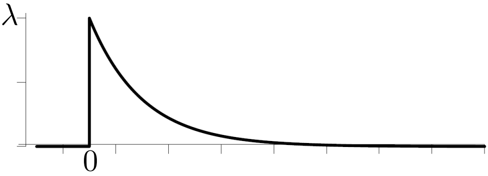

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- $P(X \geq s + t | X \geq s)$: the probability that you will have to wait extra time t or more for something to occur, given that you have already waited time s.
- $P(X \geq t)$: the probability you will have to wait time t or more for something to occur.

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That is to say,

> the probability we can make it a whole day without a lie given that we've already made it 3 days without a lie

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Remark: The exponential distributions $Exp(\lambda)$ are the only distributions having this property.

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Write your question in notation: $P(X \geq 8) = ?$.

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Be sure to:

- Cut out spots where the density has height 0 (this did not happen here)
- Change infinite bounds to variables and use limits to head there
- Line up equals sign if you work vertically
- Box your answer and give an exact answer and approximation

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In our dishwasher example,

$$E(X) = \frac{1}{1/5} = 5 \text{ years} \qquad\qquad SD(X) = \frac{1}{1/5} = 5 \text{ years}$$

Best Dishwasher Ever

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We want $P(X \geq 30 + 8 | X \geq 30)$.

Best Dishwasher Ever

You have the best dishwasher ever. It hasn't broken down in 30 years. What is the probability you get at least 8 more years before the first breakdown?

We want $P(X \geq 30 + 8 | X \geq 30)$.

By the memoryless property,

$$P(X \geq 30 + 8 | X \geq 30) = P(X \geq 8) = \boxed{e^{-8/5} \simeq 0.202.}.$$

The dishwasher has no memory of its amazingness. It wakes up today as if the last 30 years did not even exist.