A statistical study of the nonideal Rayleigh gas

Cumulants of the Rayleigh gas mixture model

March 2026

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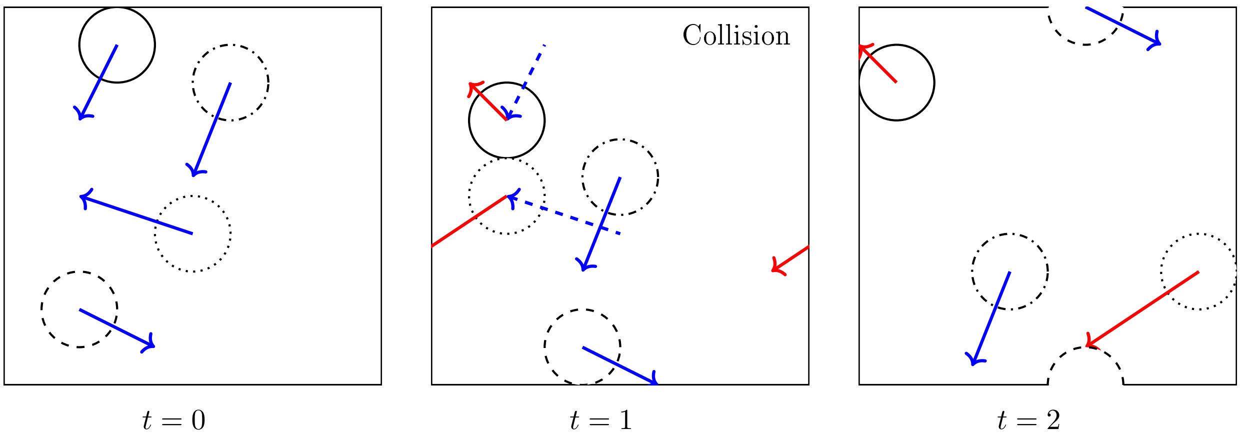

Figure 1: Example of microscopic dynamics on the torus

1 Introduction

Ludwig Eduard Boltzmann introduced in 1872 the famous equation [11] that now bears his name, to provide a statistical model of rarefied gas dynamics. This model is considered a breakthrough, having paved the way to a flourishing litterature studying matter both from an atomistic and statistical point of view. The kinetic representation that Boltzmann constructed may be thought about as an intermediate mesoscopic step between microscopic and macroscopic scales, and might thus be used to derive fluid mechanics equations from Newton classical equations (see for example [4, 5, 23, 31]). This model both allowed numerous quantitative simulations [36, 20] and qualitative understanding of the intrinsic statistical behaviour of fluid matter. Indeed, springing from the Boltzmann equation, the latter introduced the *entropy* as a Lyapunov functional of the system, showing that the density of particles irreversibly converges towards an equilibrium on large time scales. This equilibrium is called the Maxwellian state: the gas occupies uniformly the available space, and the velocities of the particles get distributed according to the Maxwell Gaussian distribution (2.6), only depending on the temperature of the system.

Nevertheless, since the microscopic evolution of the gas, from which is derived the Boltzmann equation, follows classical Newton equations in a completely time reversible way, most of Boltzmann's contemporaries get confused from this seeming paradox, doubting the validity of his model. The rigorous derivation of his equation from microscopic Newton equations has however eventually been proved mathematically in 1975 by Oscar Erasmus Lanford III [25, 26], in the case of hard sphere interactions. The method that Lanford used is very rigid in the tracking of particles' trajectories, hindering to prove his result for time scales larger than very small times, when only about a fifth of particles have collided. The major obstruction to large time scales is the correlation occuring between colliding particles: the quantitative bounds of the system's chaoticity deteriorate over time, making it impossible to deal with recollisions of particles.

Nonetheless, in a setting close to thermodynamic equilibrium (the Maxwellian state), the statistical stability of the dynamics guarantees a certain amount of chaos over large times, allowing a very strong control of the correlations. In the Rayleigh gas model, describing the behaviour of a small fraction of tagged particles near equilibrium [32], a proof relying on the same ideas as Lanford's yielded the derivation *on large time scales* of a linear version of the Boltzmann equation, called the linear Rayleigh–Boltzmann equation. This is the work of Henk van Beijeren, Lanford, Joel Louis Lebowitz and Herbert Spohn in 1980 [34], later completed with applications to color-changing boundary conditions [27].

A few decades later, in 2013, Isabelle Gallagher, Laure Saint-Raymond and Benjamin Texier reopened the work of Lanford and his former student, Francis Gordon King [24], generalizing their proof to the case of compactly supported potentials, and providing in parallel precise estimates on the convergence rate in Lanford's theorem. Eventually, these estimates led in 2016 to an article by Thierry Bodineau, Gallagher and Saint-Raymond on the convergence rate in the linear case, and its dependence on the long time scaling, so as to infer Brownian hydrodynamic limits [7]. These estimates on the convergence rates have been improved in 2024 in our previous paper [17], using the adaptive time cutting that is also used in the present work.

Stemming from their work [7], the authors, joined by Sergio Simonella, also studied the symmetric linearization of the Boltzmann equation on long time scales [8, 10], using similar methods along with a well-chosen time cutting, although precising their study so as to capture rarer dynamics events that are involved in fine correlations between particles. Eventually, they completed their study of the fine scales of the dynamics in 2023 [9], analyzing the finest correlations of the system through its *cumulants* (4.11), exhibiting the equations governing the fluctuations of the empirical measure away from its expected limit, along with its large deviations.

The present work is dedicated to perform this study in the Rayleigh gas setting, where the expected limit of the empirical measure follows the linear Rayleigh–Boltzmann equation. The relevant model to introduce the empirical measure of tagged particles is a gas mixture, in which most of the particles are initially distributed according to the equilibrium state, and in the low-density regime. Meanwhile, some other tagged particles are initially perturbed from this equilibrium, their number going to infinity, yet necessarily in a more dilute regime, to remain in a Rayleigh setting. We choose the grand canonical ensemble to get symmetry and simplify the study of cumulants. This model allows to study the statistical behaviour of the perturbed particles, and permits in particular to explore the phase transition around the low-density scaling. Indeed, looking at the fluctuations of the empirical measure, we show that there is no threshold ahead of this scaling, which is thus the exact regime to consider, to observe correlations at the limit.

On the other hand, we exhibit a linear version of the Hamilton–Jacobi equation that describes the limit cumulants of the tagged particles, and derive a large deviation principle in a tighter functional framework than [9]. These two results are nevertheless still restricted to small time scales, because of the absence of strong *a priori* bounds on the cumulants, due to their non-linear combinatorial structure. The first part of the present paper is hence dedicated to the study of our Rayleigh gas mixture on large time scales, in the sense of correlation functions. We extend the previous results to all the correlation functions, and not only the first one, exhibiting the big combinatorial factors that appear for high correlation functions, stemming from the time cutting method that we use. To perform this study, we harness the adaptive time cutting introduced in [17] to improve the quantitative convergence rates. Performing an analysis on large time scales imposes a condition on the tagged particles' number, which is merely a technical threshold and is not required on small times.

Note that the way we deal with the geometry of recollisions is different in the first part of the study and in the study of cumulants: one may hence appreciate the differences between both methods, and furthermore, in the coherence of the proof, this allows to use the previous method without proving it again, to avoid unnecessary heaviness. We also improve the analysis performed in the most recent geometric method to get a convergence rate in a full power of ε instead of $\varepsilon|\log\varepsilon|$ (see Appendix D).

Recently, Yu Deng, Zaher Hani and Xiao Ma [13] showed that assuming uniform bounds on the solution to the Boltzmann equation, and computing a very fine combinatorial analysis of the dynamics trees, the derivation was valid for long times. Their method might be applied to statistical results like the ones presented in this paper, to extend their time validity, especially since the uniform bounds of the limit can be proved in the linear framework.

The Rayleigh gas mixture model, with the grand canonical ensemble for tagged particles, is exposed in Section 2. Section 3 is dedicated to the convergence in large times of the correlation functions of this system, using the adaptive time cutting method of [7, 17]. We present the fluctuation and large deviation results in Section 4, and introduce the cumulants and their generating function, before giving their equations in terms of dynamics trees and interaction clusters in Section 5. Eventually, we use these equations to compute integrability bounds on the cumulants in Section 6, to find the equations of the limit cumulants, and to study the Hamilton–Jacobi system that governs them in Section 7, finally proving the fluctuation and large deviation theorems in Section 8.

Several appendixes follows to expose technical results, about the change of parametrization of the hard sphere scattering rule (Appendix A), the cumulants and partition function combinatorics (Appendixes B and C), the geometric control of cycles in the particles' dynamics (Appendix D), and finally the functional analysis of the linear Rayleigh–Boltzmann and Boltzmann–Hamilton–Jacobi equations (Appendix E).

In the end, a notation index gathers the various choices of writings at page 72.

2 Modelling a perturbed cloud in a gas of particles at equilibrium

This section is dedicated to present the framework we work with: the classical hard sphere model at microscopic scale, its statistical description and its expected limit density, satisfying the linear Rayleigh–Boltzmann equation. Eventually, we will expose how we model an unbalanced mixture in the grand canonical ensemble, which is the part specific to this paper.

2.1 Microscopic hard sphere model

Microscopically, we consider exactly the hard sphere model, which resembles a perfectly elastic d-dimensional billiards (Fig. 1). The state of a gas of N particles is completely determined by the positions (in the d-dimensional torus $\mathbb{T}^d$) and the velocities of every particle, represented by the vector

$$\underline{z}_N = (z_1, \ldots, z_N) \doteq (\underline{x}_N, \underline{v}_N) \in \mathcal{D}^N \doteq (\mathbb{T}^d \times \mathbb{R}^d)^N.$$

The *hard sphere* model consists in an exclusion condition, which states that two particles cannot get closer than a certain diameter $\varepsilon > 0$: the positions have to belong to the hard sphere exclusion set

$$\mathcal{X}_N^\varepsilon \doteq \{\underline{x}_N \in \mathbb{T}^{dN};\ \forall\, i \neq j,\ d(x_i, x_j) > \varepsilon\}, \tag{2.1}$$

where $d(\cdot,\cdot)$ denotes the distance on the torus; and hence the state of the gas $\underline{z}_N$ must belong to the open domain $\mathcal{D}_N^\varepsilon \doteq \mathcal{X}_N^\varepsilon \times \mathbb{R}^{dN}$.

Within this set, the particles' dynamics is given by the Newton equations for uniform line movement, while on the boundary of $\mathcal{D}_N^\varepsilon$, for at least two particles (let us say i and j) we have that $|x_i - x_j| = \varepsilon$: they collide. Then, if the scalar product $(x_i - x_j)\cdot(v_i - v_j)$ is positive, it means that the particles are exiting the collision in uniform line movement, but otherwise they are entering the collision and must scatter according to the following system giving the post-collisional velocities $({v_i}', {v_j}')$:

$$\begin{cases} {v_i}' = v_i - \left\langle v_i - v_j, \dfrac{x_i - x_j}{\varepsilon} \right\rangle \dfrac{x_i - x_j}{\varepsilon} \\[2ex] {v_j}' = v_j + \left\langle v_i - v_j, \dfrac{x_i - x_j}{\varepsilon} \right\rangle \dfrac{x_i - x_j}{\varepsilon}. \end{cases} \tag{2.2}$$

In the hard sphere case, the interaction is taken instantaneous and elastic: the system (2.2) stems from the preservation of momentum and kinetic energy. Appendix A is dedicated to the parametrization of this scattering system by either one of the following angles.

$$\omega = \frac{x_i - x_j}{\varepsilon} \qquad \text{or} \qquad \sigma = \frac{{v_i}' - {v_j}'}{|{v_i}' - {v_j}'|}. \tag{2.3}$$

This dynamics is well-defined up to a zero measure set of initial configurations, in which infinite amounts of collisions might happen in finite times, along with collisions between more than two particles at a time. This result was proved by Roger Keith Alexander [1], and might also be found in [18]. Some other models use non-instantaneous scattering governed by potentials of interaction that can be short-range [18] or long-range [14, 3]. The review by Cédric Villani [35] gives a global overview of collisional kinetic theory.

2.2 Statistical description and linear Rayleigh–Boltzmann equation

As the number of particles N gets large, this hard sphere dynamics becomes very difficult to compute, especially because it is very chaotic, so that we choose to describe the gas statistically. This paper is dedicated to the study of a gas of identical particles, yet divided in two distinguishable parts, represented by tags $\underline{\ell}_N = (\ell_1, \dots, \ell_N) \in \{0,1\}^N$: the tag $\ell = 0$ will be attributed to particles initially distributed at thermodynamic equilibrium, and the tag $\ell = 1$ to particles initially perturbed from that equilibrium.

At fixed $N \in \mathbb{N}$ and $\varepsilon > 0$, we consider $W_N^\varepsilon(t, \underline{z}_N, \underline{\ell}_N)$ the *canonical* probability density of presence of particles with tags $\underline{\ell}_N$ on the phase space $\mathcal{D}_N^\varepsilon$ at time $t \geq 0$: by exchangeability, it is invariant by permutation among particles with identical tags. The microscopic dynamics provides the Liouville transport equation for W_N^ε within $\mathcal{D}_N^\varepsilon$

$$\partial_t W_N^\varepsilon + \underline{v}_N \cdot \nabla_{\underline{x}_N} W_N^\varepsilon = 0. \tag{2.4}$$

The solutions to this equation are provided by the method of characteristics and expressed in terms of the initial distribution $W_N^\varepsilon(0)$ and of the free transport going *back in time*, in such a way that we need the following boundary condition to pursue transporting back the density when two particles emerge from a collision:

$$d(x_i, x_j) = \varepsilon \text{ and } \langle x_i - x_j, v_i - v_j \rangle > 0 \quad \Rightarrow \quad W_N^\varepsilon(\underline{z}_N) \doteq W_N^\varepsilon(\underline{z}_N^\star), \tag{2.5}$$

where $\underline{z}_N^\star = (z_1, \dots, x_i, v_i{}^\star, \dots, x_j, v_j{}^\star, \dots, z_N)$ denotes the *pre*-collisional state associated to $\underline{z}_N$. Note that by reversibility, the formulas for $(v_i^\star, v_j^\star)$ are the same as for (v_i', v_j').

In the next section, we will present the grand canonical model, which randomizes the number N of particles to a random variable $\mathcal{N}$, the expectancy of which is tuned by a parameter μ called the *chemical potential*. The kinetic limit that we consider is called the *low density limit*, or Boltzmann–Grad limit, and consists in letting this chemical potential μ go to infinity while keeping a constant mean free path $\mu^{-1}\varepsilon^{1-d} = 1$, so that the particles' diameter ε goes to 0. In this limit, assuming initial chaos, the first marginal of the density usually converges to the solution to the Boltzmann equation [25, 18, 13].

This solution to the Boltzmann equation, studied by Boltzmann and Maxwell, when well defined, relaxes in large times to an equilibrium called the *Maxwell state* and defined [11] as

$$M_\beta(x, v) \doteq \left(\frac{\beta}{2\pi}\right)^{d/2} \exp\left(-\frac{\beta}{2}|v|^2\right). \tag{2.6}$$

The parameter β stands for the inverse temperature of the system, tuning its intensive (kinetic) energy. It appears that the density $M_\beta^{\otimes N} \mathbb{1}_{\mathcal{D}_N^\varepsilon}$ is an equilibrium of the microscopic hard sphere dynamics, and in this paper we consider specific initial conditions that are close to this thermodynamic equilibrium, to retrieve a *linear version* of the Boltzmann equation, whose theory is much simpler and hence might be derived for long time scales. More precisely, we consider the *nonideal Rayleigh gas* model [32]: a subset of particles are tagged, breaking the particles' exchangeability. In the case of a single tagged particle, the derivation of this linear equation has been shown [7] when choosing the following perturbation of equilibrium as initial state

$$W_N^\varepsilon(0, \underline{z}_N) = \frac{\mathbb{1}_{\mathcal{X}_N^\varepsilon}(\underline{x}_N)}{\mathcal{Z}_N^{\varepsilon,\mathrm{c}}} \rho(x_1) M_\beta^{\otimes N}(\underline{z}_N),$$

for $\rho \in \mathcal{C}(\mathbb{T}^d)$ a continuous space perturbation on the torus, and $\mathcal{Z}_N^{\varepsilon,\mathrm{c}}$ a normalization constant. Indeed, in the Boltzmann–Grad limit, the first marginal of W_N^ε behaves like the solution $g \doteq M_\beta \varphi$ to the linear *Rayleigh–Boltzmann equation* [7] with initial condition ρ:

$$\begin{cases} \partial_t \varphi + v \cdot \nabla_x \varphi &= \displaystyle\int_{\mathbb{S}^{d-1}} \int_{\mathbb{R}^d} [\varphi(v^\star) - \varphi(v)] M_\beta(v_c) \langle \omega, v_c - v \rangle_+ \mathrm{d}v_c \mathrm{d}\omega, \\ \varphi(0, x, v) &= \rho(x). \end{cases} \tag{2.7}$$

This linear equation (2.7) is globally well-posed in the velocity-weighted space $\mathbb{L}^\infty_x \mathbb{L}^\infty_v(M_{\beta/2})$ (see Appendix E), and allows to derive the linear heat equation in the hydrodynamic limit [7].

Some partial results exist for this same model with long-range interactions instead of hard sphere collisions [14, 3], yet the complete derivation of the Rayleigh–Boltzmann equation for general potentials is still an open problem. Other ways to derive the linear Rayleigh–Boltzmann equation for long time scales are the *ideal* Rayleigh gas model, in which the particles at equilibrium don't interact among themselves [16, 28, 29], and the Lorentz gas model, which consists in letting a tagged particle evolve in a frozen random background [33, 12, 22].

2.3 Grand canonical framework for the mixture

As introduced in the previous section, we want to study the behaviour of an arbitrary large subset of tagged particles perturbed away from equilibrium, whose density will follow a linear version of Boltzmann equation. This derivation has been studied in the case of a finite set [7, 17], but to provide extended statistical results on this gas we hereafter take its size to infinity, yet remaining a tiny fraction of the gas. To *preserve a symmetric structure* on the objets that we consider, we will work in the grand canonical ensemble and introduce an additional tagging variable indicating to which set each particle belongs. This approach to describe a gas mixture is different from the canonical one used by Ioakeim Ampatzoglou, Joseph K. Miller and Nataša Pavlović in their article deriving a mixed Boltzmann equation [2]; indeed their description is made in the canonical ensemble, at fixed numbers of particles of each kind.

The particles at equilibrium are taken in the usual low density (Boltzamnn–Grad) limit, whereas the tagged perturbed particles only occupy a tiny fraction of the gas, smaller than the Boltzmann–Grad density: otherwise indeed they would behave like a classical Boltzmann dilute gas, satisfying the non-linear Boltzmann equation. This all boils down to the following scaling,

$$\mu\varepsilon^{d-1} = 1 \quad \text{and} \quad 1 \ll \lambda \ll \mu, \tag{$S_{\varepsilon,\mu,\lambda}$}$$

where $\mu > 0$ corresponds to the chemical potential of the particles at equilibrium, and $\lambda > 0$ to that of tagged particles. The notation $\lambda \ll \mu$ simply means that $\lambda\mu^{-1}$ goes to 0 as λ and μ go to infinity. Formally, **the particles at initial equilibrium will be tagged with a 0, and the initially perturbed particles will be tagged with a 1**. The tags of all particles hence form a vector $\underline{\ell}_n \in \Lambda_n \doteq \{0,1\}^n$, identified to the corresponding subset $\underline{\ell}_n \subset [\![1,n]\!]$, with the following notation

$$|\underline{\ell}_n| = \|\underline{\ell}_n\|_1 = |\{i \leq n \ , \ \ell_i = 1\}| \quad \text{and} \quad \varphi_0^{\otimes \underline{\ell}_n}(\underline{z}_{\underline{\ell}_n}) = \prod_{\substack{i \leqslant n \\ \ell_i = 1}} \varphi_0(z_i).$$

Indeed, instead of ρ a perturbation happening in space only, as in [7], we will hereafter consider an initial perturbation $\varphi_0 \in \mathbb{L}^\infty(\mathbb{T}^d \times \mathbb{R}^d)$ also happening according to the velocities. Now, the mixed grand canonical ensemble consists in relaxing the number of particles, weighting it with a mixed Poisson law depending on the number of tagged particles. More precisely, at fixed $\mu = \varepsilon^{1-d}$, we take the following weighted canonical initial densities

$$W_n^\varepsilon(0, \underline{z}_n, \underline{\ell}_n) \doteq \frac{\lambda^{|\underline{\ell}_n|}\mu^{n-|\underline{\ell}_n|}}{\mathcal{Z}_\mu} M_\beta^{\otimes n}(\underline{v}_n)\varphi_0^{\otimes \underline{\ell}_n}(\underline{z}_{\underline{\ell}_n})\mathbb{1}_{\mathcal{X}_n^\varepsilon}(\underline{x}_n) \tag{2.8}$$

driven by Liouville equation (2.4), for a normalizing constant $\mathcal{Z}_\mu$ that we will adjust soon (2.9). We introduce the following *correlation functions*, based on the marginals of the canonical densities. Note that we also take the marginal according to the tags, which corresponds to a sum over $\underline{\ell}^*_p \in \Lambda_p$, and we denote $\tilde{\underline{\ell}}_{n+p} \doteq (\underline{\ell}_n, \underline{\ell}_p^*)$, $\tilde{\underline{z}}_{n+p} \doteq (\underline{z}_n, \underline{z}_p^*)$:

$$F_n^\varepsilon(t, \underline{z}_n, \underline{\ell}_n) = \sum_{p\geqslant 0} \sum_{\underline{\ell}_p^* \in \Lambda_p} \frac{\lambda^{|\underline{\ell}_p^*|} \mu^{p-|\underline{\ell}_p^*|}}{p!} \int W_{n+p}^\varepsilon(t, \tilde{\underline{z}}_{n+p}, \tilde{\underline{\ell}}_{n+p}) \mathrm{d}\underline{z}_p^*$$
$$\doteq \frac{1}{\mu^{n-|\underline{\ell}_n|}\lambda^{|\underline{\ell}_n|}} \sum_{p\geqslant 0} \frac{1}{p!} W_{n+p}^{\varepsilon,(n)}(t, \underline{z}_n, \underline{\ell}_n),$$

where the normalizing (grand canonical) partition function is hence defined as follows

$$\mathcal{Z}_\mu \doteq \sum_{p\geqslant 0} \sum_{\underline{\ell}_p \in \Lambda_p} \frac{\lambda^{|\underline{\ell}_p|} \mu^{p-|\underline{\ell}_p|}}{p!} \int M_\beta^{\otimes p}(\underline{v}_p) \varphi_0^{\otimes \underline{\ell}_p}(\underline{z}_{\underline{\ell}_p}) \mathbb{1}_{\mathcal{X}_p^\varepsilon}(\underline{x}_p) \mathrm{d}\underline{z}_p. \tag{2.9}$$

In the previous paper [7] presenting the situation where only one particle was perturbed, thanks to the invariance by translation of the system, the partition function was not depending on the perturbation φ_0. Nevertheless, here the perturbed particles are correlated one with another, preventing us from using the same argument, which is why the initial perturbation appears in the formula above. Note that the velocities of the non-perturbed particles (in the complementary of $\underline{\ell}_p$) may be integrated using the fact that the equilibrium M_β is of integral 1. We still keep this formulation for symmetry reasons; a more precise study of this object is made in Appendix C.

Our probabilistic study is based on the random variables $(Z_{\varepsilon,i}^{[t]}, L_i)_{1\leqslant i\leqslant \mathcal{N}}$, giving the states at time t, and the tags of the particles. The density of the initial state $(Z_{\varepsilon,i}^{[0]}, L_i)$ is given above, and the evolution at time t is a deterministic piecewise affine function of the initial state. The correlation functions are in fact defined such that for any observable $H_n \in \mathcal{C}_c^\infty(\mathcal{D}^n \times \Lambda_n)$, we have

$$\begin{aligned}\mathbb{E}\left[\sum_{1\leq i_k\neq i_j\leq \mathcal{N}} H_n(Z_{\varepsilon,i_1}^{[t]}, L_{i_1}, \dots, Z_{\varepsilon,i_n}^{[t]}, L_{i_n})\right] &= \mathbb{E}\left[\delta_{\mathcal{N}\geq n} \frac{\mathcal{N}!}{(\mathcal{N}-n)!} H_n(Z_{\varepsilon,1}^{[t]}, L_1, \dots, Z_{\varepsilon,n}^{[t]}, L_n)\right] \\ &= \sum_{p=n}^\infty \frac{1}{p!} \frac{p!}{(p-n)!} \sum_{\underline{\ell}_p \in \Lambda_p} \int_{\mathcal{D}_p^\varepsilon} W_p^\varepsilon(t, \underline{z}_p, \underline{\ell}_p) H_n(\underline{z}_n, \underline{\ell}_n) \mathrm{d}\underline{z}_p \\ &= \sum_{\underline{\ell}_n \in \Lambda_n} \mu^{n-|\underline{\ell}_n|}\lambda^{|\underline{\ell}_n|} \int_{\mathcal{D}_n^\varepsilon} F_n^\varepsilon(t, \underline{z}_n, \underline{\ell}_n) H_n(\underline{z}_n, \underline{\ell}_n) \mathrm{d}\underline{z}_n.\end{aligned} \tag{2.10}$$

This formula between the correlation functions and observables will be used in the combinatorial computations of Section 4.2, when expanding the cumulant generating function. By the usual BBGKY arguments [18, 2] the canonical densities (2.8) satisfy the following hierarchy

$$\begin{aligned}\partial_t W_N^{\varepsilon,(n)} + \underline{v}_n \cdot \nabla_{\underline{x}_n} W_N^{\varepsilon,(n)} &= (N-n)\varepsilon^{d-1} \sum_{\ell=0}^{1} \sum_{i=1}^{n} \int \langle \omega, v_c - v_i \rangle W_N^{\varepsilon,(n+1)}(\underline{z}_n, \underline{\ell}_n, x_i + \varepsilon\omega, v_c, \ell) \mathrm{d}\omega \mathrm{d}v_c \\ &\doteq (N-n)\varepsilon^{d-1} \left(\mathcal{C}_n^{\langle 0\rangle} W_N^{\varepsilon,(n+1)} + \mathcal{C}_n^{\langle 1\rangle} W_N^{\varepsilon,(n+1)}\right).\end{aligned} \tag{2.11}$$

The two operators we hence define are called *collision operators* and represent the action of a $(n+1)$-th particle (tagged 0 or tagged 1) on the average distribution of n particles. To better understand these operators, we rewrite them using the boundary condition (2.5) when $\langle \omega, v_{n+1} - v_i \rangle > 0$, and the change of variable $\omega \mapsto -\omega$ otherwise:

$$\begin{aligned}\mathcal{C}_n^\ell W_N^{\varepsilon,(n+1)} = \sum_{i=1}^n \int &\mathrm{d}\omega \mathrm{d}v_{n+1} \langle \omega, v_{n+1} - v_i \rangle_+ \times \\ &\left[W_N^{\varepsilon,(n+1)}(\underline{z}_n^\star, \underline{\ell}_n, x_i + \varepsilon\omega, v_{n+1}^\star, \ell) - W_N^{\varepsilon,(n+1)}(\underline{z}_n, \underline{\ell}_n, x_i - \varepsilon\omega, v_{n+1}, \ell)\right].\end{aligned} \tag{2.12}$$

Morally, we look at the influence of a $(n+1)$-th particle—with tag ℓ—on the dynamics, colliding with one of the n existing ones with angle ω and velocity v_{n+1}, whence the name collision operators. The *cross section* $\langle \omega, v_{n+1} - v_i \rangle_+$ weights the likelihood of such a collision. As a consequence, the correlation functions satisfy

$$\begin{aligned} \partial_t F_n^\varepsilon + \underline{v}_n \cdot \nabla_{\underline{x}_n} F_n^\varepsilon &= \frac{1}{\mu^{n-|\underline{\ell}_n|}\lambda^{|\underline{\ell}_n|}} \sum_{k\geqslant 0} \frac{k\varepsilon^{d-1}}{k!} \left(\mathcal{C}_n^{\langle 0\rangle} W_{n+k}^{\varepsilon,(n+1)} + \mathcal{C}_n^{\langle 1\rangle} W_{n+k}^{\varepsilon,(n+1)} \right) \\ &= \frac{\mu\varepsilon^{d-1}}{\mu^{n+1-|\underline{\ell}_n|}\lambda^{|\underline{\ell}_n|}} \sum_{k\geqslant 0} \frac{1}{k!} \mathcal{C}_n^{\langle 0\rangle} W_{n+k+1}^{\varepsilon,(n+1)} + \frac{\lambda\varepsilon^{d-1}}{\mu^{n-|\underline{\ell}_n|}\lambda^{|\underline{\ell}_n|+1}} \sum_{k\geqslant 0} \frac{\varepsilon^{d-1}}{k!} \mathcal{C}_n^{\langle 1\rangle} W_{n+k+1}^{\varepsilon,(n+1)}, \end{aligned}$$

which eventually leads to the following hierarchy in our mixed Boltzmann–Grad scaling ($S_{\varepsilon,\mu,\lambda}$)

$$\partial_t F_n^\varepsilon + \underline{v}_n \cdot \nabla_{\underline{x}_n} F_n^\varepsilon = \mathcal{C}_n^{\langle 0\rangle} F_{n+1}^\varepsilon + \frac{\lambda}{\mu} \mathcal{C}_n^{\langle 1\rangle} F_{n+1}^\varepsilon. \tag{2.13}$$

3 Long-time convergence of the correlation functions

The formal limit of the hierarchy above (2.13) with initial conditions (2.8), presented in Section 3.2, is satisfied by the family $(M_\beta^{\otimes n} \varphi^{\otimes \underline{\ell}_n})_{n\geqslant 1}$, with φ the solution to the Rayleigh–Boltzmann equation (2.7) with initial data φ_0. This result is formalized in the following section, with a quantitative convergence rate.

3.1 Convergence result, a law of large numbers

The following theorem provides a convergence rate of the mixed correlation functions to the solutions of the Rayleigh–Boltzmann equation, which is a generalization of [17, 7] with a time scale of validity improved by a power $1/4$ thanks to a more precise computation (see the proof of Proposition 3.6.1). We extend the result to all the correlation functions (not only the first one), yet at the cost of a bad constant n^{cn} stemming from the time cutting method we use. This constant, which did not appear in the convergence of the first marginal, is due to an accumulation of errors at each time step of our cutting. Note finally that we use here the adaptive time cutting introduced in [17], improving greatly the convergence rate compared with [7].

Theorem 3.1 (Convergence of the correlation functions). *For some sets $\Delta_n^\varepsilon \subset \mathcal{D}^n$ whose measure goes to 0 with ε, there exists a constant c_β depending only on the temperature and the dimension such that, for any $\alpha \in (0, 3/4)$, as long as*

$$t \lesssim (\log|c_\beta \log\varepsilon|)^{\frac{3}{4}-\alpha} \quad \text{and} \quad \lambda \lesssim |\log\varepsilon|^{1-\alpha}, \tag{3.1}$$

and for ε small enough, one has the following convergence rate of the correlation functions to the linear Rayleigh–Boltzmann solutions in our mixed low density scaling ($S_{\varepsilon,\mu,\lambda}$), for a constant $c > 0$;

$$\|F_n^\varepsilon - M_\beta^{\otimes n} \varphi^{\otimes \underline{\ell}_n}\|_{\mathbb{L}^\infty([0,t]\times\mathcal{D}^n\setminus\Delta_n^\varepsilon)} \leq n^{cn} \exp\left(-c_\beta |\log\varepsilon|^{1-\alpha}\right).$$

The notation $t \lesssim (\log|c_\beta \log\varepsilon|)^{\frac{3}{4}-\alpha}$ means that, for a good constant $c > 0$ depending only on the dimension d and the inverse temperature β, one has

$$t \leq c\,(\log|c_\beta \log\varepsilon|)^{\frac{3}{4}-\alpha}.$$

The proof of this theorem is the subject of Sections 3.2 to 3.8. It is close to the proof presented in [17], but in the mixed grand canonical framework and for all the correlation functions instead of the first marginal only.

This theorem provides a first corollary on the statistical behaviour of the gas. To do so, for an observable $H \in \mathcal{C}_c^\infty(\mathcal{D} \times \Lambda_1)$ we define the random variables

$$\pi_t^\varepsilon[H] \doteq \frac{1}{\mu} \sum_{i=1}^{\mathcal{N}} H(Z_{\varepsilon,i}^{[t]}, L_i) \tag{3.2}$$

the empirical measure of all particles, and

$$\tilde{\pi}_t^\varepsilon[H] \doteq \frac{1}{\lambda} \sum_{i=1}^{\mathcal{N}} H(Z_{\varepsilon,i}^{[t]}, L_i) \mathbb{1}_{L_i=1} \tag{3.3}$$

the empirical measure of the tagged particles. Thanks to Theorem 3.1, one can deduce the convergence of these empirical measures, hence providing a law of large numbers for the hard sphere dynamics. This result is given in the space $\mathbf{L}^2$ of *square-integrable random variables*, embedded with the norm $\mathbb{E}\left[|\cdot|^2\right]$.

Corolary 3.1.1 (Law of large numbers of the dynamics)**.** *The empirical measures converge as random variables in* $\mathbf{L}^2$*, the non-tagged particles towards an equilibrium state, and the tagged ones to a state described by the linear Rayleigh–Boltzmann equation* (2.7)*, in the following way*

$$\pi_t^\varepsilon[H] \xrightarrow[\varepsilon \to 0]{\mathbf{L}^2} \int M_\beta(v) H(z,0) \mathrm{d}z, \tag{3.4}$$

and

$$\tilde{\pi}_t^\varepsilon[H] \xrightarrow[\varepsilon \to 0]{\mathbf{L}^2} \int M_\beta(v) \varphi(t,z) H(z,1) \mathrm{d}z. \tag{3.5}$$

Proof. To show that the random variable $\pi_t^\varepsilon[H]$ converges in $\mathbf{L}^2$ to a deterministic limit $a \in \mathbb{R}$, writing

$$\pi_t^\varepsilon[H] - a = \pi_t^\varepsilon[H] - \mathbb{E}\big[\pi_t^\varepsilon[H]\big] + \mathbb{E}\big[\pi_t^\varepsilon[H]\big] - a,$$

it is enough to show that $\mathbb{E}\big[\pi_t^\varepsilon[H]\big] \xrightarrow[\varepsilon \to 0]{} a$ and $\mathbb{E}\left[\left|\pi_t^\varepsilon[H] - \mathbb{E}\big[\pi_t^\varepsilon[H]\big]\right|^2\right] \xrightarrow[\varepsilon \to 0]{} 0$. Using formula (2.10), one can write

$$\mathbb{E}\left[\frac{1}{\mu} \sum_{i=1}^{\mathcal{N}} h(Z_{\varepsilon,i}^{[t]}, L_i)\right] = \int F_1^\varepsilon(t,z_1,0) H(z_1,0) \mathrm{d}z_1 + \frac{\lambda}{\mu} \int F_1^\varepsilon(t,z_1,1) H(z_1,1) \mathrm{d}z_1,$$

so that the expectancies converge by Theorem 3.1 above. For concision, we show the fact that the variance vanishes in the tagged case, denoting $h \doteq H(\cdot,1)$. The equilibrium case is treated in a similar8 though simpler way. Let us compute, once again by formula (2.10),

$$\begin{aligned}
\mathbb{E}\left[\left|\tilde{\pi}_t^\varepsilon - \mathbb{E}\big[\tilde{\pi}_t^\varepsilon[H]\big]\right|^2\right] &= \frac{1}{\lambda^2} \mathbb{E}\left[\sum_{i,j=1}^{n} h(Z_{\varepsilon,i}^{[t]}) h(Z_{\varepsilon,j}^{[t]})\right] - \frac{2}{\lambda} \mathbb{E}\left[\sum_{i=1}^{n} h(Z_{\varepsilon,i}^{[t]}) \int F_1^\varepsilon(1) h\right] + \left(\int F_1^\varepsilon(1) h\right)^2 \\
&= \int F_2^\varepsilon(1,1) h^{\otimes 2} + \frac{1}{\lambda} \int F_1^\varepsilon(1) h^2 - 2\left(\int F_1^\varepsilon(1) h\right)^2 + \left(\int F_1^\varepsilon(1) h\right)^2 \\
&\xrightarrow[\varepsilon \to 0]{} \int M_\beta^{\otimes 2} \varphi^{\otimes 2} + 0 - \left(\int M_\beta \varphi\right)^2 = 0,
\end{aligned}$$

thanks to the convergence of the correlation functions (Theorem 3.1). The other convergence (3.4) follows similarly in the scaling $\frac{\lambda}{\mu}$ goes to 0. ∎

3.2 Pseudo-trajectories and strategy of proof

We now choose to denote $p_\mu \doteq \dfrac{\lambda}{\mu}$ the fraction of initially perturbed particles, so that iterating Duhamel formula as in [18] or [7], we can write the Dyson expansion

$$F_n^\varepsilon(t) = \sum_{k\geqslant 0} \sum_{\underline{\ell}_k^* \in \Lambda_k} p_\mu^{|\underline{\ell}_k^*|} Q_{n,\underline{\ell}_k^*}(t) F_{n+k}^\varepsilon(0), \tag{3.6}$$

developing the choice of the encountered tags $\underline{\ell}_k^* \doteq (\tilde{\ell}_{n+1}, \dots, \tilde{\ell}_{n+k})$, with the successive-collision operators defined as

$$Q_{n,\underline{\ell}_k^*}(t) \doteq \int_{T_k(t)} \Theta_n(t-t_1)\mathcal{C}_n^{\tilde{\ell}_{n+1}}\Theta_{n+1}(t_1-t_2)\dots\mathcal{C}_{n+k-1}^{\tilde{\ell}_{n+k}}\Theta_{n+k}(t_k)\mathrm{d}\underline{t}_k, \tag{3.7}$$

where $\Theta_n(\tau)$ denotes the transport semi-group operator in $\mathcal{D}_n^\varepsilon$ with specular reflections, for a time τ. The collision times are integrated over

$$T_k(t) \doteq \left\{ \underline{t}_k \;\middle|\; 0 \doteq t_{k+1} \leq t_k \leq \dots \leq t_1 \leq t_0 \doteq t \right\}. \tag{3.8}$$

The main idea of the proof, coming from Lanford's original paper [25], is to use a coupling between this expansion and its limit version, implying imaginary histories of the particles, that eventually lead to the state $\underline{z}_n$ at time t. These histories, called *pseudo-trajectories*, are non-physical trajectories that—in a way—allow to extend the method of characteristics for the successive-collision operators.

Indeed, the transport operators appearing in (3.7) correspond to following the characteristics of free transport, with specular reflections: taking the first operator $\Theta_n(t-t_1)$ of a functional amounts to consider this functional at time t_1, in a state $\underline{z}_n^{[t_1]}$ given by the hard sphere dynamics.

Then, the first collision operator (2.12) writes

$$\mathcal{C}_n^\ell F_{n+1}^\varepsilon = \sum_{i=1}^n \sum_{s_1=\pm 1} s_1 \int \mathrm{d}\omega_1 \mathrm{d}v_{n+1} \langle \omega_1, v_{n+1} - v_i \rangle_+ F_{n+1}^\varepsilon(\underline{z}_n^{\langle s_1 \rangle}, \underline{\ell}_n, x_i + s_1\varepsilon\omega_1, v_{n+1}^{\langle s_1 \rangle}, \ell),$$

where $\underline{z}_n^{\langle +1 \rangle} = \underline{z}_n^\star$ and $\underline{z}_n^{\langle -1 \rangle} = \underline{z}_n$, scattered for the gain term, and let unchanged for the loss term, so that the collision is always incoming, allowing to pursue the backwards method of characteristics with the next transport operator. Hence, for given collision parameters $(i, s_1, \omega_1, v_{n+1})$, this operator can be seen as a weighted adjunction of a particle to the characteristics—or pseudo-trajectory—which scatters (or not, according to s_1) with particle i, creating a new state $\underline{z}_{n+1}^{[t_1]} \doteq (\underline{z}_n^{[t_1]}, x_i + s_1\varepsilon\omega_1, v_{n+1}^{\langle s_1 \rangle})$. The integration and sum over these collision parameters will yield an integral over pseudo-trajectories. Iterating this extended method of characteristics and tracking the pseudo-trajectories $(\underline{z}_{n+j}^{[t_j]})$ thus constructed, we bring the analysis back to the value of the functional at time $\tau = 0$, in the state $\underline{z}_{n+k}^{[0]}$.

We will have to record the labels of the existing particles meeting the new one, the velocities of the particles that spring up, the angles at which the encounter happens, and whether they scatter or not. The pseudo-trajectories will also keep track of the tags of the encountered particles.

Here is precisely how we construct the pseudo-trajectories. The choice of the successive encountered tags is registered in $\underline{\ell}_k^* = (\tilde{\ell}_{n+1}, \dots, \tilde{\ell}_{n+k})$, and expanding all the sums in all the collision operators (3.7), we can sum it up to the history $(m_1, \dots, m_k)$ of which particle encountered the $(n+i)$-th new one. These particles naturally belong to the following set

$$\mathcal{M}_{n,k} \doteq \left\{ (m_1, \dots, m_k) \;\middle|\; \forall i \leq k, m_i \leq n+i-1 \right\}.$$

We consider the scattering labels $(s_1,\dots,s_k)\in\{\pm1\}^k$. The fact that some encounters do not scatter, along with the fact that particles are artificially added, is why the pseudo-trajectories are not physical trajectories. Once the total history

$$\chi_k \doteq (\underline{m}_k,\underline{\ell}_k^*,\underline{s}_k)\in\mathcal{H}_{n,k}\doteq\mathcal{M}_{n,k}\times\Lambda_k\times\{\pm1\}^k \tag{3.9}$$

is fixed, for given collision parameters $(\underline{\omega}_k,v_{n+1},\dots,v_{n+k})$, we can construct the pseudo-trajectories for every $\underline{z}_n=\underline{z}_n^{[t]}$, backwards in time to a configuration $\underline{z}_{n+m}^{[0]}$, following the inductive procedure below:

$$\begin{cases} \underline{z}_n^{[t]}\doteq\underline{z}_n \\ \forall\ i\in[\![0,k]\!],\ \forall\tau\in(t_{i+1},t_i),\ \underline{z}_{n+i}^{[\tau]}\text{ follows (backwards) the physical hard sphere dynamics} \\ \forall\ i\in[\![1,k]\!],\ \underline{z}_{n+i}^{[t_i]}=\left(\underline{z}_{n+i-1}^{[t_i^+],\langle s_i\rangle},x_{m_i}+s_i\varepsilon\omega_i,v_{n+1}^{\langle s_i\rangle}\right). \end{cases} \tag{3.10}$$

One may observe that the change of velocities in the last step is automatic by the hard sphere dynamics' boundary condition, but it will not be for the limit version of pseudo-trajectories, since the limit particles are formally pointwise. In the end, one can write the *pseudo-trajectory formulation* of the Dyson expansion

$$F_n^\varepsilon(t)=\sum_{k\geqslant0}\sum_{\chi_k}p_\mu^{|\underline{\ell}_k^*|}\int_{T_k(t)}\mathrm{d}\underline{t}_k\int\mathrm{d}\underline{\omega}_k\mathrm{d}v_{n+1}\dots\mathrm{d}v_{n+k}\prod_{i=1}^k s_i\langle\omega_i,v_{n+i}-v_{m_i}^{[t_i^+]}\rangle_+F_{n+k}^\varepsilon\big(0,\underline{z}_{n+k}^{[0]},\underline{\tilde{\ell}}_{n+k}\big). \tag{3.11}$$

A small technical detail lies in the fact that the added particles must satisfy the exclusion condition. A way to deal with it may be to impose a condition on the domain on integration of the collision angles [9], yet here to simplify we merely change the definition of the pseudo-trajectories: if at any moment the exclusion condition is violated by the adjunction of a particle, then the trajectories are frozen in this state until time $\tau=0$, so that the integral formally vanishes thanks to the initial distribution $F_{n+k}^\varepsilon(0)$ being 0 outside of $\mathcal{D}_{n+k}^\varepsilon$.

Our goal is now to prove the convergence of the correlation functions to the **limit densities**

$$G_n(t,\underline{z}_n,\underline{\ell}_n)\doteq M_\beta^{\otimes n}(\underline{v}_n)\varphi^{\otimes\underline{\ell}_n}(t,\underline{z}_{\underline{\ell}_n}),$$

where φ is the solution to the linear Rayleigh–Boltzmann equation (2.7). Because of the structure of this equation, this family satisfies the following hierarchy

$$(\partial_t+\underline{v}\cdot\nabla_{\underline{x}})G_n=\sum_{i=1}^n\sum_{s_c=\pm1}s_c\int\mathrm{d}\omega\mathrm{d}v_{n+1}\langle\omega,v_{n+1}-v_i\rangle_+G_{n+1}(\underline{z}_n^{\langle s_c\rangle},\underline{\ell}_n,x_i,v_{n+1}^{\langle s_c\rangle},0), \tag{3.12}$$

noticing that the terms vanish when the scattering occur between two particles distributed according to the equilibrium $M_\beta^{\otimes n}$. This equation is the formal limit of the BBGKY hierarchy (2.13) in the mixed low density regime ($S_{\varepsilon,\mu,\lambda}$): it makes only appear the collision operator linked to equilibrum particles, tagged 0, since the other one has a factor λ/μ which vanishes at the limit. It leads to the following limit version of the Dyson expansion (3.6)

$$G_n(t)=\sum_{k\geqslant0}Q_{n,\underline{0}_k}^{\lim}(t)G_{n+k}(0), \tag{3.13}$$

where the following successive-collision operators contain only collisions with particles at equilibrium, and limit free transport of pointwise particles, without scattering:

$$Q_{n,\underline{0}_k}^{\lim}(t)\doteq\int_{T_k(t)}\Theta_n^{\lim}(t-t_1)\mathcal{C}_n^{\langle0\rangle,\lim}\Theta_{n+1}^{\lim}(t_1-t_2)\dots\mathcal{C}_{n+k-1}^{\langle0\rangle,\lim}\Theta_{n+k}^{\lim}(t_k)\mathrm{d}\underline{t}_k. \tag{3.14}$$

The limit collision operators $\mathcal{C}_n^{\langle 0\rangle,\lim}$ are the formal limit of the collision operators (2.12), for $\varepsilon = 0$. The same computation as below for this limit hierarchy leads to a similar writing in terms of pseudo-trajectories

$$G_n(t) = \sum_{k\geqslant 0}\sum_{\chi_k} \mathbb{1}_{\underline{\ell}_k^*=\underline{0}_k} \int_{T_k(t)} \mathrm{d}\underline{t}_k \int \mathrm{d}\underline{\omega}_k \mathrm{d}v_{n+1}\ldots\mathrm{d}v_{n+k} \prod_{i=1}^k s_i\langle\omega_i, v_{n+i} - v_{m_i}^{[t_i^+]}\rangle_+ G_{n+k}\big(0, \underline{\zeta}_{n+k}^{[0]}, \underline{\tilde{\ell}}_{n+k}\big), \tag{3.15}$$

where the limit pseudo-trajectories $(\underline{\zeta}_{n+i}^{[t_i]})$ are defined as their hard sphere versions (3.10) for $\varepsilon = 0$, with the noticeable difference that in the dynamics followed on each time interval (t_{i+1}, t_i), the particles are pointwise and hence follow the free flow without any scattering.

Strategy of proof We will couple both pseudo-trajectory formulations (3.11) and (3.15), bringing down the difference at a certain time $(F_n^\varepsilon - G_n)(t, \underline{z}_n)$ to the difference at time 0 of higher correlation functions $\Big[F_{n+k}^\varepsilon(t, \underline{z}_{n+k}^{[0]}) - G_{n+k}(t, \underline{\zeta}_{n+k}^{[0]})\Big]$. For these hierarchies to be well coupled, the two pseudo-trajectories must be close; the classical argument is to show that the transport operators Θ_k, appearing between the collision operators, do not imply additional recollisions, since the trajectories would diverge from the limit transport operator $\Theta_k^{\lim}$, defined on the whole space $\mathcal{D}^k$ without recollisions.

Indeed with this method, it will be enough to use **continuity estimates** on the operators to bring the convergence back to time $\tau = 0$ where we can use explicit **initial proximity**. Nevertheless, these continuity estimates demand to work with trajectories that do not contain too many particles, so that we will first compute a **tree pruning**, and control the pruned-out term using some **a priori estimates** on the densities. The last step before proving the **convergence** will be to **discard the pseudo-trajectories** in which some perturbed particles are encountered, as this situation does not happen in the limit pseudo-trajectories. This strategy will be followed in the following sections.

3.3 Initial proximity

The total kinetic energy, preserved by the transport and by elastic collisions, will be denoted

$$\|\underline{v}_k\|^2 \doteq \sum_{i=1}^k |v_i|^2,$$

where $|v_i|$ is the Euclidean norm of the velocity $v_i \in \mathbb{R}^d$ of particle i. For an inverse temperature $\beta > 0$ and $k \in \mathbb{N}^*$, we consider the space $\mathcal{F}_{k,\beta}$ of measurable functions defined almost everywhere on the domain $\mathcal{D}^k$ such that

$$\|f_k\|_{k,\beta} \doteq \operatorname*{supess}_{\underline{z}_k\in\mathcal{D}^k} \big|f_k(\underline{z}_k)\exp(\beta\|\underline{v}_k\|^2)\big| < \infty, \tag{3.16}$$

hence decreasing at least as the Gaussian equilibrium $M_\beta^{\otimes k}$ in velocities. We denote

$$C_0 \doteq \max\Big[\|M_\beta\|_{1,\beta}\ ;\ \|M_\beta\varphi_0\|_{1,\beta}\ ;\ \|\varphi_0\|_{\mathbb{L}^\infty(\mathcal{D})}\Big]. \tag{3.17}$$

The initial error between the microscopic densities and the limit ones is mainly due to the exclusion condition, of which we can compute an explicit control. Some technical difficulties, dealt with in Appendix C, appear because of the structure of the grand canonical mixture and its partition function.

Proposition 3.3.1 (Initial proximity). *For all $n \in \mathbb{N}$ and tags $\underline{\ell}_n \in \Lambda_n$, for any $\lambda, \mu > 0$ in the scaling $(S_{\varepsilon,\mu,\lambda})$, one has*

$$\Big\|\mathbb{1}_{\mathcal{X}_n} M_\beta^{\otimes n}\varphi_0^{\otimes\underline{\ell}_n} - F_n^\varepsilon(0, \underline{\ell}_n)\Big\|_{n,\beta} \leq C_0^n\varepsilon. \tag{3.18}$$

Proof of the proposition. We denote $\mathbb{1}_{i\not\sim j} \doteq \mathbb{1}_{d(x_i,x_j)>\varepsilon}$ and $\mathbb{1}_{i\sim j} \doteq \mathbb{1}_{d(x_i,x_j)\leq\varepsilon}$ the indicator of exclusion between i and j and its complementary. Recalling that $\mathbb{1}_{\mathcal{X}_N^\varepsilon}$ denotes the exclusion condition (2.1) and by definition (2.9) of the partition function $\mathcal{Z}_\mu$, we can write, once again denoting $\underline{\tilde{\ell}}_{n+p} \doteq (\underline{\ell}_n, \underline{\ell}_p^*)$:

$$\mathbb{1}_{\mathcal{X}_n^\varepsilon} M_\beta^{\otimes n}\varphi_0^{\otimes \underline{\ell}_n} - F_n^\varepsilon(0)$$
$$= \frac{1}{\mathcal{Z}_\mu}\sum_{p\geqslant 0}\frac{1}{p!}\left[\sum_{\underline{\ell}_p^*\in\Lambda_p}\lambda^{|\underline{\ell}_p^*|}\mu^{p-|\underline{\ell}_p^*|}\mathbb{1}_{\mathcal{X}_n^\varepsilon}M_\beta^{\otimes n}\varphi_0^{\otimes\underline{\ell}_n}\int\varphi_0^{\otimes\underline{\ell}_p^*}M_\beta^{\otimes p}\mathbb{1}_{\mathcal{X}_p^\varepsilon} - \left(\varphi_0^{\otimes\underline{\tilde{\ell}}_{n+p}}M_\beta^{\otimes n+p}\mathbb{1}_{\mathcal{X}_{n+p}^\varepsilon}\right)^{(n)}\right],$$

and expanding the marginals' formula, the exclusion indicator might be decomposed as

$$\mathbb{1}_{\mathcal{X}_n^\varepsilon}M_\beta^{\otimes n}\varphi_0^{\otimes\underline{\ell}_n}\int\varphi_0^{\otimes\underline{\ell}_p^*}M_\beta^{\otimes p}\mathbb{1}_{\mathcal{X}_p^\varepsilon} - \int\varphi_0^{\otimes\underline{\tilde{\ell}}_{n+p}}M_\beta^{\otimes n+p}\mathbb{1}_{\mathcal{X}_{n+p}^\varepsilon}\mathrm{d}\underline{z}_p^*$$
$$= \mathbb{1}_{\mathcal{X}_n^\varepsilon}M_\beta^{\otimes n}\varphi_0^{\otimes\underline{\ell}_n}\int(M_\beta\varphi_0)^{\otimes\underline{\ell}_p^*}(\underline{z}_{\underline{\ell}_p^*}^*)\mathbb{1}_{\mathcal{X}_p^\varepsilon}(\underline{x}_p^*)\left(1-\prod_{\substack{1\leqslant i\leqslant n\\1\leqslant j\leqslant p}}\mathbb{1}_{x_i\not\sim x_j^*}\right)\mathrm{d}\underline{x}_p^*\underline{v}_{\underline{\ell}_p^*}^*,$$

using that M_β is of integral 1. Then, we will harness the following basic set property,

$$1-\prod_{\substack{1\leqslant i\leqslant n\\1\leqslant j\leqslant p}}\mathbb{1}_{x_i\not\sim x_j^*} \leq \sum_{\substack{1\leqslant i\leqslant n\\1\leqslant j\leqslant p}}\mathbb{1}_{x_i\sim x_j^*},$$

yielding

$$\mathbb{1}_{\mathcal{X}_n^\varepsilon}M_\beta^{\otimes n}\varphi_0^{\otimes\underline{\ell}_n} - F_n^\varepsilon(0) \leq \frac{\mathbb{1}_{\mathcal{X}_n^\varepsilon}M_\beta^{\otimes n}\varphi_0^{\otimes\underline{\ell}_n}}{\mathcal{Z}_\mu}\sum_{p\geqslant 0}\sum_{\underline{\ell}_p^*\in\Lambda_p}\frac{\lambda^{|\underline{\ell}_p^*|}\mu^{p-|\underline{\ell}_p^*|}}{p!}\sum_{\substack{1\leqslant i\leqslant n\\1\leqslant j\leqslant p}}\int(M_\beta\varphi_0)^{\otimes\underline{\ell}_p^*}\mathbb{1}_{\mathcal{X}_p^\varepsilon}\mathbb{1}_{x_i\sim x_j^*}\mathrm{d}\underline{x}_p^*.$$

To be able to integrate over x_j^*, we denote $\underline{\check{\ell}}_p^{(j)} \doteq (\ell_1^*,\ldots,\ell_{j-1}^*,\ell_{j+1}^*,\ldots,\ell_p^*)$ the vector of all tags apart from j, so that (using definition (3.17) of C_0)

$$\left\|\mathbb{1}_{\mathcal{X}_n^\varepsilon}M_\beta^{\otimes n}\varphi_0^{\otimes\underline{\ell}_n} - F_n^\varepsilon(0)\right\|_{n,\beta} \leq \frac{C_0^n}{\mathcal{Z}_\mu}\sum_{p\geqslant 0}\sum_{\substack{1\leqslant i\leqslant n\\1\leqslant j\leqslant p}}\sum_{\underline{\check{\ell}}_p^{(j)}\in\Lambda_{p-1}}\frac{\lambda^{|\underline{\check{\ell}}_p^{(j)}|}\mu^{p-|\underline{\check{\ell}}_p^{(j)}|}}{p!}(\lambda C_0+\mu)\int(M_\beta\varphi_0)^{\otimes\underline{\check{\ell}}_p^{(j)}}\mathbb{1}_{\mathcal{X}_p^\varepsilon}\mathbb{1}_{x_i\sim x_j^*}$$
$$\leq \frac{C_0^n}{\mathcal{Z}_\mu}\sum_{p\geqslant 0}np\sum_{\underline{\ell}_{p-1}^*\in\Lambda_{p-1}}\frac{\lambda^{|\underline{\ell}_{p-1}^*|}\mu^{p-1-|\underline{\ell}_{p-1}^*|}}{p!}2\mu|\mathbb{S}^{d-1}|\varepsilon^d\int(M_\beta\varphi_0)^{\otimes\underline{\ell}_{p-1}^*}\mathbb{1}_{\mathcal{X}_{p-1}^\varepsilon}$$

using the exchangeability of identical particles, and we get in the mixed low density scaling ($S_{\varepsilon,\mu,\lambda}$)

$$\left\|\mathbb{1}_{\mathcal{X}_n^\varepsilon}M_\beta^{\otimes n}\varphi_0^{\otimes\underline{\ell}_n} - F_n^\varepsilon(0)\right\|_{n,\beta} \leq 2|\mathbb{S}^{d-1}|\varepsilon\frac{C_0^n}{\mathcal{Z}_\mu}n\sum_{p\geqslant 1}\sum_{\underline{\ell}_{p-1}^*\in\Lambda_{p-1}}\frac{\lambda^{|\underline{\ell}_{p-1}^*|}\mu^{p-1-|\underline{\ell}_{p-1}^*|}}{(p-1)!}\int(M_\beta\varphi_0)^{\otimes\underline{\ell}_{p-1}^*}\mathbb{1}_{\mathcal{X}_{p-1}^\varepsilon},$$

which concludes the proof recognizing the partition function $\mathcal{Z}_\mu$ (2.9) after an index shift. □

3.4 A priori estimates

As in the previous works [7, 17] about the Rayleigh gas, the long-time derivation is allowed thanks to a priori estimates yielded by the rigid structure of equilibrium. Indeed, the canonical densities initially defined in (2.8) satisfy for all $(\underline{z}_n,\underline{\ell}_n)\in\mathcal{D}^n\times\Lambda_n$ that

$$W_n(0,\underline{z}_n,\underline{\ell}_n) \leq \frac{\lambda^{|\underline{\ell}_n|}\mu^{n-|\underline{\ell}_n|}}{\mathcal{Z}_\mu}\|\varphi_0\|_{\mathbb{L}^\infty}^{|\underline{\ell}_n|}M_\beta^{\otimes n}(\underline{v}_n)\mathbb{1}_{\mathcal{X}_n^\varepsilon}(\underline{x}_n), \tag{3.19}$$

but their evolution is simply given by the global transport of n particles, by which the equilibrium $M_\beta^{\otimes}\mathbb{1}_{\mathcal{X}_n^\varepsilon}$ is invariant, so that for all times $t \geq 0$ the bound (3.19) is propagated by the transport and remains true. Hence, taking the marginals we get

$$\begin{aligned} W_n^{(k)}(t, \underline{z}_k, \underline{\ell}_k) &\leq \sum_{\underline{\ell}_{n-k}^* \in \Lambda_{n-k}} \frac{(\lambda C_0)^{|\underline{\ell}_k| + |\underline{\ell}_{n-k}^*|} \mu^{n-|\underline{\ell}_k| - |\underline{\ell}_{n-k}^*|}}{\mathcal{Z}_\mu} M_\beta^{\otimes k}(\underline{v}_k) \left(\mathbb{1}_{\mathcal{X}_n^\varepsilon}\right)^{(k)}(\underline{x}_k) \\ &\leq (\lambda C_0 + \mu)^{n-k} \frac{(\lambda C_0)^{|\underline{\ell}_k|} \mu^{k-|\underline{\ell}_k|}}{\mathcal{Z}_\mu} M_\beta^{\otimes k}(\underline{v}_k) \mathbb{1}_{\mathcal{X}_n^\varepsilon}^{(k)}(\underline{x}_k), \end{aligned}$$

where the factor $(\lambda C_0 + \mu)^{n-k}$ stems from the sum over $\underline{\ell}_{n-k}^*$, using the binomial theorem. The main difference in the linear case, compared to the general non-linear one, is that the bound (3.19) on the canonical densities uses the invariant density $M^{\otimes n}$, that passes to the k-th marginals to become $M^{\otimes k}$, contrary to the constant C^n in the general case.

Eventually, these bounds over the marginals of the canonical densities leaves the following a priori estimate for the correlation functions

$$\begin{aligned} F_n^\varepsilon(t, \underline{z}_n, \underline{\ell}_n) &\leq \frac{M_\beta^{\otimes n}}{\mu^{n-|\underline{\ell}_n|} \lambda^{|\underline{\ell}_n|}} \sum_{p \geqslant 0} \frac{1}{p!} \frac{(\lambda C_0 + \mu)^p (\lambda C_0)^{|\underline{\ell}_n|} \mu^{n-|\underline{\ell}_n|}}{\mathcal{Z}_\mu} \mathbb{1}_{\mathcal{X}_{n+p}^\varepsilon}^{(n)} \\ &\leq \frac{C_0^{|\underline{\ell}_n|} M_\beta^{\otimes n}}{\mathcal{Z}_\mu} \sum_{p \geqslant 0} \frac{(\lambda C_0 + \mu)^p}{p!} \int \mathbb{1}_{\mathcal{X}_p^\varepsilon} \\ &\leq \frac{C_0^{|\underline{\ell}_n|} M_\beta^{\otimes n}}{\mathcal{Z}_\mu} \sum_{q,r \geqslant 0} \frac{(\lambda C_0)^q \mu^r}{q! r!} \int \mathbb{1}_{\mathcal{X}_{q+r}^\varepsilon} \end{aligned} \tag{3.20}$$

where at line (3.20) we computed a direct binomial theorem. The key point to end our a priori estimate is now to control the remaining quotient implying the partition function $\mathcal{Z}_\mu$ and the slightly modified version of it, which is the following proposition.

Proposition 3.4.1. *There exists a constant C_d depending only on the dimension such that for μ large enough, in our mixed Boltzmann-Grad scaling ($S_{\varepsilon,\mu,\lambda}$), we have*

$$\frac{1}{\mathcal{Z}_\mu} \sum_{q,r \geqslant 0} \frac{(\lambda C_0)^q \mu^r}{q! r!} \int \mathbb{1}_{\mathcal{X}_{q+r}^\varepsilon} \leq C_d^{C_0 \lambda}.$$

The proof of this technical result is given in Appendix C, using an explicit expansion of the partition function according to the cumulants of the exclusion. Eventually, this leads to the following proposition, which is the main argument of our long time analysis.

Proposition 3.4.2 (A priori estimates for the correlation functions)**.** *For any $n \in \mathbb{N}$ and $\varepsilon > 0$ in the mixed scaling ($S_{\varepsilon,\mu,\lambda}$), one has*

$$F_n^\varepsilon(t, \underline{z}_n, \underline{\ell}_n) \leq C_0^{|\underline{\ell}_n|} M_\beta^{\otimes n} \times C^{C_0 \lambda}. \tag{3.21}$$

3.5 Continuity estimates

Thanks to the a priori estimates (3.21) given in the previous Section 3.4, all the correlation functions (F_n^ε) extended by 0 out of $\mathcal{D}_n^\varepsilon$ belong to the space $\mathcal{F}_{n,\beta}$ defined in (3.16), with

$$\sup_{t \geqslant 0} \|F_n^\varepsilon(t)\|_{n,\beta} \leq C_0^n C^{C_0 \lambda}. \tag{3.22}$$

Our whole long time derivation is allowed thanks to the fact that these bounds are valid for every time with the same inverse temperature β. Indeed, in the non-linear case, this parameter is downgraded over time until vanishing in finite time [18, Section 5].

Technically, the downgrading of this norm parameter stems from the fact that to control the collision operators, the sum over the velocities is resorbed thanks to a fraction of the sub-Gaussian decreasing, eventually providing the following continuity estimates.

Proposition 3.5.1 ([17]; Continuity of the successive-collision operators)**.**
There exists a constant C_d depending only on the dimension such that for all $n, k \in \mathbb{N}^*$, and all times $t > 0$, fixing tags $\underline{\ell}_{n+k} \in \Lambda_k$ and choosing two inverse temperature $\beta' < \beta$, we have

$$f_{n+k} \in \mathcal{F}_{n+k,\beta} \Rightarrow Q_{n,\underline{\ell}_k}(t) f_{n+k} \in \mathcal{F}_{n,\beta'}, \text{ with}$$

$$\left\| Q_{n,\underline{\ell}_k}(t) f_{n+k} \right\|_{n,\beta'} \leq e^n \left(\frac{C_d \ t}{\sqrt{\beta^d (\beta - \beta')}} \right)^k \|f_{n+k}\|_{n+k,\beta}. \tag{3.23}$$

For fixed tags, the proof of this proposition is exactly the same as in the paper [17] about non-ideal Rayleigh gas, adapted from the article by Bodineau, Gallagher and Saint-Raymond, which derived the hydrodynamic limit of this gas [7]. Compared to this proof, we merely chose to write $\beta' = \beta\left(1 - \frac{1}{b}\right)$ for better clarity in what follows.

3.6 Tree pruning of the Dyson expansion for long times

The continuity estimate presented in the previous section justifies the well-posedness of the Dyson series (3.6) for short times. To perform a derivation for long times, we will iterate the Dyson series, but each iteration will make appear a factor e^n stemming from the estimate of Proposition 3.5.1. These factors stack, so that at each iteration the successive times would decrease extremely fast and their sum would eventually be summable, leaving the derivation on finite short times. The method we use here, introduced in [7], consists in putting aside these stacking factors e^n and to keep it controlled by bounding the number of collisions appearing in the Dyson series.

More precisely, for a fixed time $t > 1$ we will split the time interval $[0, t]$ into K pieces and impose a piece-dependant amount of collisions on each small interval of this cutting. The number K will be tuned in the end of the proof, and henceforth we write

$$t = \sum_{i=1}^{K} h_i, \text{ with time steps } t_k^{\mathrm{p}} \doteq t - \sum_{j=1}^{k} h_i, \tag{3.24}$$

so that $t_K^{\mathrm{p}} = 0$. Like in [17], and contrary to [7], we will not choose a uniform cutting, but an adaptive one. At the k-th time quantum of length h_k, we want at most 2^k collisions to have happened; we prune the collision tree every time it becomes more than exponentially big. Explicitly, between t and $t_1^{\mathrm{p}} = t - h_1$, we first truncate the Dyson series (3.6) to 2 collisions, then expand it again between $t - h_1$ and $t - h_2$ truncated to 2^2 collisions, and iterate this process K times. Denoting the step number of

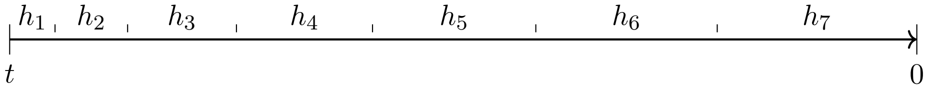

Figure 2: Backwards division of the time interval under study

additional tagged particles and the step total number of particles

$$L_k \doteq \sum_{i=1}^{k} |\underline{\ell}_{j_i}^*| \quad \text{and} \quad N_k \doteq n + j_1 + \cdots + j_k, \tag{3.25}$$

this yields, similarly as in [7, 17], the following pruned expansion

$$F_n^\varepsilon(t) = \sum_{\left(\substack{j_i \leqslant 2^i \\ \underline{\ell}^*_{j_i} \in \Lambda_{j_i}}\right)_{1\leqslant i \leqslant K}} p_\mu^{L_K} Q_{n,\underline{\ell}^*_{j_1}}(h_1)\dots Q_{N_{K-1},\underline{\ell}^*_{j_K}}(h_K) F_{N_K}(0) + R_n^{[K]}(t) \tag{3.26}$$

where the remainder is defined as

$$R_n^{[K]}(t) \doteq \sum_{k=1}^K \sum_{\left(\substack{j_i \leqslant 2^i \\ \underline{\ell}^*_{j_i} \in \Lambda_{j_i}}\right)_{i\leqslant k-1}} Q_{n,\underline{\ell}^*_{j_1}}(h_1)\dots Q_{N_{k-2},\underline{\ell}^*_{j_{k-1}}}(h_{k-1}) \sum_{j_k > 2^k} \sum_{\underline{\ell}^*_{j_k} \in \Lambda_{j_k}} p_\mu^{L_k} Q_{N_{k-1},\underline{\ell}^*_{j_k}}(h_k) F_{N_k}(t_k^{\mathrm{p}}). \tag{3.27}$$

Based on the limit expansion (3.13), one can write the same decomposition for the limit family $(G_n)_{n\geqslant 0}$:

$$G_n(t) = \sum_{(j_i \leqslant 2^i)_{1\leqslant i\leqslant K}} Q^{\lim}_{n,\underline{0}_{j_1}}(h_1)\dots Q^{\lim}_{N_{K-1},\underline{0}_{j_K}}(h_K) G_{N_K}(0) + R_n^{[K],\lim}(t). \tag{3.28}$$

Let us denote $\widehat{G}_n(t) \doteq G_n(t) - R_n^{[K],\lim}(t)$ the limit pruned expansion. Since the chosen condition of a sub-exponential number of collisions is very restrictive at first, and then gradually relaxed, the adaptive cutting times are chosen small at first and then progressively bigger (see Fig. 2). The bound on the pruned-out remainder is given in the following proposition.

Proposition 3.6.1 (Estimate of the pruned-out term)**.** *With the previous notation, for any $\alpha \in (0, 3/4)$ and K large enough satisfying $t \lesssim K^{\frac{3}{4}-\alpha}$, a good choice of time cutting $\underline{h} = (h_1,\dots,h_K)$ provides the following estimate*

$$\left\| R_n^{[K]}(t) \right\|_{\mathbb{L}^\infty(\mathcal{D}^d)} + \left\| R_n^{[K],\lim}(t) \right\|_{\mathbb{L}^\infty(\mathcal{D}^d)} \leq C^{C_0\lambda} n^{cn} e^{-2^{K-K^\alpha}}. \tag{3.29}$$

Note that this estimate imposes a technical condition on the mixed scaling of λ and K for the error to be small; we will tune this scaling in Section 3.8. The factor n^{cn} stems from the fact that we generalize the result to all the correlation functions, and not only the first one. Indeed, one will see that this factor is due to the usual bound C^n stacking at each iteration of the cutting.

Proof. The proof is very similar to the one found in [17]. We give it for the hard sphere version, since the limit version is identical. Using the a priori estimates (3.21) on the densities and the continuity estimate on successive-collision operators given in Proposition 3.5.1, at given $(\underline{\ell}^*_{j_i})_{i\leq k-1}$ one has for every $k \in [\![1, K]\!]$

$$\begin{aligned} \left\| \sum_{j_k>2^k} \sum_{\underline{\ell}^*_{j_k}\in\Lambda_{j_k}} p_\mu^{L_k} Q_{N_{k-1},\underline{\ell}^*_{j_k}}(h_k) F^\varepsilon_{N_k}(t_k^{\mathrm{p}}) \right\|_{N_{k-1},\beta/2} &\leq e^{N_{k-1}} \sum_{j_k>2^k} \sum_{\underline{\ell}^*_{j_k}\in\Lambda_{j_k}} p_\mu^{L_k} \left(\frac{\sqrt{2}C_d h_k}{\beta^{(d+1)/2}} \right)^{j_k} \|F^\varepsilon_{N_k}(t_k^{\mathrm{p}})\|_{N_k,\beta} \\ &\leq e^{N_{k-1}} \sum_{j_k>2^k} \sum_{\underline{\ell}^*_{j_k}\in\Lambda_{j_k}} p_\mu^{L_k} \left(\frac{\sqrt{2}C_d h_k}{\beta^{(d+1)/2}} \right)^{j_k} C^{N_k} C^{C_0\lambda}. \end{aligned}$$

For μ large enough in the scaling $(S_{\varepsilon,\mu,\lambda})$, $p_\mu \leq 1$, so that the sum over $\underline{\ell}^*_{j_k} \in \Lambda_{j_k}$ only gives a factor 2^{N_k} that can be resorbed in the term C^{N_k}. We then iterate Proposition 3.5.1, downgrading the parameter $\beta/2$ by $\beta/(4k)$ at each step, so that it remains greater than $\beta/4$. Hence we can write, grouping $C^{N_{k-1}}$ and all the appearing terms of the form e^{N_i} together as a power of a constant $\hat{C}$, that

$$\left\| Q_{n,\underline{\ell}^*_{j_1}}(h_1)\dots Q_{N_{k-2},\underline{\ell}^*_{j_{k-1}}}(h_{k-1}) \sum_{j_k>2^k} \sum_{\underline{\ell}^*_{j_k}\in\Lambda_{j_k}} p_\mu^{L_k} Q_{N_{k-1},\underline{\ell}^*_{j_k}}(h_k) F_{N_k}(t_k^{\mathrm{p}}) \right\|_{n,\beta/4} \tag{3.30}$$
$$\leq C^{C_0\lambda} \hat{C}^{\sum_{i=0}^{k-1} N_i} \left(\left(\frac{4}{\beta}\right)^{\frac{d+1}{2}} \sqrt{4k} C_d h_1 \right)^{j_1} \dots \left(\left(\frac{4}{\beta}\right)^{\frac{d+1}{2}} \sqrt{4k} C_d h_{k-1} \right)^{j_{k-1}} \sum_{j_k>2^k} \left(\frac{\sqrt{2} C_d h_k}{\sqrt{\beta}} \right)^{j_k}.$$

We now observe, on the one hand, that recalling notation (3.25) for N_i and since for $i \leq k-1$, $j_i \leq 2^i$, we have $\sum_{i=0}^{k-1} N_i \leq nk + 2^{k+1}$. On the other hand, we can also put aside from the sum the following factors

$$\left(\left(\frac{4}{\beta}\right)^{\frac{d+1}{2}} 2k^{\frac{1}{4}-\alpha} C_d \right)^{\sum_{i=0}^{k-1} j_i} \leq (\hat{C}_\beta k^{\frac{1}{4}-\alpha})^{2^k}.$$

This computation is what allows us to gain a power $\frac{1}{4}$ on the time scaling that we imposed in Theorem 3.1, compared to [17]. Indeed, in the following, the computation above harnesses the full power decay of the last time interval h_k, whereas some of it was lost in [17]. In the end, for a possibly larger constant C depending on d and β, we get

$$\left\| R_n^{[K]}(t) \right\|_{\mathbb{L}^\infty} \leq C^{C_0\lambda} \sum_{k=1}^{K} C^{nk} (Ck^{\frac{1}{4}-\alpha})^{2^k} \sum_{j_1=0}^{2} \dots \sum_{j_{k-1}=0}^{2^{k-1}} (k^{\frac{1}{4}+\alpha} h_1)^{j_1} \dots (k^{\frac{1}{4}+\alpha} h_{k-1})^{j_{k-1}} \sum_{j_k>2^k} (Ch_k)^{j_k}.$$

Eventually, we consider similarly as in [17], for all $1 \leq i \leq K$,

$$\tilde{h}_i \doteq \frac{e^{-2^{(K-K^{1-\alpha}-i)}}}{2K^{\frac{1}{4}+\alpha}} \leq \frac{1}{2K^{\frac{1}{4}+\alpha}}, \tag{3.31}$$

and renormalize them such that

$$h_i \doteq \frac{t}{\sum_{j=1}^K \tilde{h}_j} \tilde{h}_i \leq \tilde{h}_i.$$

Indeed, as soon as $t \lesssim K^{\frac{3}{4}-2\alpha}$, we can write

$$\frac{1}{t}\sum_{i=1}^{K} \tilde{h}_i \geq \frac{1}{t} \sum_{j=0}^{\lfloor K^{1-\alpha} \rfloor} \tilde{h}_{K-j}$$
$$\geq \frac{1}{t} \sum_{j=0}^{\lfloor K^{1-\alpha} \rfloor} \frac{e^{-1}}{2K^{\frac{1}{4}+\alpha}} \geq 1.$$

Now, the time interval length $(h_i)_{1\leqslant i\leqslant K}$ cover t as imposed by (3.24), and their choice provides, summing the geometric series over $(j_i)_{1\leqslant i\leqslant K}$,

$$\left\| R_n^{[K]}(t) \right\|_{\mathbb{L}^\infty} \leq C^{C_0\lambda} \sum_{k=1}^{K} C^{nk} (Ck^{\frac{1}{4}-\alpha})^{2^k} \times 2^k \sum_{j_k>2^k} \left(\frac{e^{-2^{K-K^{1-\alpha}-k}}}{2K^{\frac{1}{4}+\alpha}} \right)^{j_k}$$
$$\leq C^{C_0\lambda} \sum_{k=1}^{K} C^{nk} \left(\frac{CK^{\frac{1}{4}-\alpha}}{K^{\frac{1}{4}+\alpha}} \right)^{2^k} \left(\frac{e^{-2^{K-K^{1-\alpha}-k}}}{2} \right)^{2^k}.$$

Observe now that there exists a constant c such that $(k \geq c \log n) \Rightarrow (nk \leq 2^k)$, so that in this case the factor C^{nk} is absorbed by C^{2^k}, and for $k \leq c \log n$, then $C^{nk} \leq C^{cn \log n} = n^{(c \log C)n}$. Now, the denominator $K^{\frac{1}{4}+\alpha}$ crushes the term $CK^{\frac{1}{4}-\alpha}$ for K large enough and we end up with

$$\left\| R_n^{[K]}(t) \right\|_{\mathbb{L}^\infty} \leq C^{C_0 \lambda} \tilde{C}^{n \log n} e^{-2^{K-K^{1-\alpha}}}.$$

□

3.7 Discarding trajectories implying several labelled particles

Since the limit pseudo-trajectories defined from the limit hierarchy (3.15) only imply collisions with particles at equilibrium (which are in wide majority), we must get rid of the ones including collisions with perturbed particles. We hence write our pruned expansion as

$$\begin{aligned} &\sum_{\left(\substack{j_i \leqslant 2^i \\ \underline{\ell}^*_{j_i} \in \Lambda_{j_i}}\right)_{1 \leqslant i \leqslant K}} p_\mu^{L_K} Q_{n, \underline{\ell}^*_{j_1}}(h_1) \dots Q_{N_{K-1}, \underline{\ell}^*_{j_K}}(h_K) F^\varepsilon_{N_K}(0) \\ &\quad = \sum_{(j_i \leqslant 2^i)_{1 \leqslant i \leqslant K}} Q_{n, \underline{0}_{j_1}}(h_1) \dots Q_{N_{K-1}, \underline{0}_{j_K}}(h_K) F^\varepsilon_{N_K}(0) \qquad (3.32) \\ &\quad + \sum_{(j_i \leqslant 2^i)} \sum_{k=1}^{K} \sum_{\substack{(\underline{\ell}^*_{j_i})_{i \leqslant K} \\ \underline{\ell}^*_{j_k} \neq \underline{0}_{j_k}}} p_\mu^{L_K} Q_{n, \underline{\ell}^*_{j_1}}(h_1) \dots Q_{N_{K-1}, \underline{\ell}^*_{j_K}}(h_K) F^\varepsilon_{N_K}(0), \qquad (3.33) \end{aligned}$$

and we denote $\widehat{F}^\varepsilon_n(t)$ the main term (3.32) containing only collisions with equilibrium, and $F^{\varepsilon, \text{u.e.}}_n(t)$ the one implying unwanted encounters (3.33). We bound the latter in the following proposition.

Proposition 3.7.1 (Encountering tagged particles is rare). *As long as*

$$p_\mu \leq \frac{1}{2K},$$

in the same time setting as in Proposition 3.6.1, the unwanted pseudotrajectories implying tagged particles are bounded by

$$\| F^{\varepsilon, \text{u.e.}}_n(t) \|_{\mathbb{L}^\infty(\mathcal{D}^d)} \leq C^{nK + A^K} \; p_\mu. \qquad (3.34)$$

The reader may think of the factor C^{A^K} as a small negative power of ε in the final scaling we will compute in the following Section 3.8, so that the scaling proportion p_μ will have to compensate it.

Proof of the proposition. A computation directly adapted from the initial proximity bound (3.18) leads to the estimate

$$\left\| F^\varepsilon_{N_K}(0, \underline{\tilde{\ell}}_{N_K}) \right\|_{N_K, \beta} \leq (C_0)^{N_K},$$

so that, using the binomial identity

$$\sum_{\underline{\ell}^*_{j_i} \in \Lambda_{j_i}} p_\mu^{|\underline{\ell}^*_{j_i}|} = (1 + p_\mu)^{j_i}$$

and the same continuity estimates on the successive-collision operators as in the proof of Proposition 3.6.1, one has

$$\| F^{\varepsilon, \text{u.e.}}_n(t) \|_{\mathbb{L}^\infty(\mathcal{D}^d)} \leq C^{nK} (C K^{\frac{1}{4}-\alpha})^{2^K} \sum_{(j_i \leqslant 2^i)_{i \leqslant K}} \sum_{k=1}^{K} \left((1 + p_\mu)^{j_k} - 1 \right) \prod_{i=1}^{K} \left(\frac{1}{2} \right)^{j_i}.$$

Notice on the one hand that

$$\begin{aligned}(CK^{\frac{1}{4}-\alpha})^{2^K} &= \exp\left[2^K \log(CK^{\frac{1}{4}-\alpha})\right] \\ &\leq \exp\left(A^K\right)\end{aligned}$$

for any $A > 2$ as long as K is large enough (depending on A). On the other hand, using $j_k \leq 2^k$ and taking $p_\mu \leq 2^{-K} \leq 2^{-k}$, we have by convexity on $[0, 2^{-k}]$ that

$$\begin{aligned}(1+p_\mu)^{j_k} - 1 &\leq (1+p_\mu)^{2^k} - 1 \\ &\leq (e-1)2^k p_\mu,\end{aligned}$$

so that eventually

$$\begin{aligned}\|F_n^{\varepsilon,\text{u.e.}}(t)\|_{\mathbb{L}^\infty(\mathcal{D}^d)} &\leq (e-1)p_\mu \times C^{nK+A^K} \sum_{k=1}^{K} 2^k \sum_{(j_i \leqslant 2^i)_{i \leqslant K}} \prod_{i=1}^{K} \left(\frac{1}{2}\right)^{j_i} \\ &\leq p_\mu \, \widehat{C}^{nK+A^K}\end{aligned}$$

for another constant $\widehat{C}$ absorbing the factor $(e-1)2^{K+1} \times 2^K$, concluding the proof. □

3.8 Proof of the convergence

Now that the pruned-out terms have been controlled, we want to compare the pruned terms $\widehat{F}_n^\varepsilon(t)$ and $\widehat{G}_n(t)$, which is the last step before our choice of scaling for K and the conclusion of Theorem 3.1. As explained in Section 3.2, our strategy relies on considering pseudo-trajectories without recollisions. The method is an adaptation of [7, Section 5] which is now classical; it follows from several approximations: an energy truncation and a time separation of the collisions are operated, so as to be able to construct a small set of bad collision parameters, outside of which there will be no recollisions.

This time separation method does not yield the best quantitative estimates, but in our study in long time the worst error is made with the pruned-out terms, so that here we use this method anyway. This allows concision on the one hand, and on the other hand one can thus compare it with the optimized method stated in Appendix D.

Like in [17], we still refine the quantitative bounds using the same continuity estimates on $Q_{n,\underline{\ell}^*_{j_1}}(h_1) \ldots Q_{N_{K-1},\underline{\ell}^*_{j_K}}(h_K)$ as in Propositions 3.6.1 and 3.7.1, which induces our factor ${C_\beta}^{nK+A^K}$, instead of the original crude bound with $|Q|_{1,J_k}(t)$, which gave a factor $(Ct)^{2^K}$ (see [7]). From a physical perspective, we decompose the time interval into small pieces whose lengths are adapted to the maximum number of particles that may appear in them, so that the dynamics behaves similarly during each one of them. Hence, as long as time does not get too big with respect to the number of pieces, none of the estimates depends on the total time length.

Pseudo-trajectory formulation First of all, let us notice that the pruned expansion (3.26) has a pseudo-trajectory formulation similar to the original one (3.11), summing over the successive numbers of collisions $(j_i \leq 2^i)_{i \leq K}$, with the following additional condition on the collision times, located between the time steps (3.24):

$$\underline{t}_{J_K} \in T_{\underline{j}_K}(t) \doteq \left\{(t_1, \ldots, t_{J_K}) \in T_{J_K}(t)\ ,\ \{t_{J_k+1}, \ldots, t_{J_{k+1}}\} \subset [t^{\mathrm{p}}_{k+1}, t^{\mathrm{p}}_k]\right\},$$

where $J_K \doteq N_K - n = j_1 + \cdots + j_K$ denotes the step number of collisions. As announced above, we start by computing a few approximations, initiated by an energy truncation, so as to work with bounded velocities.

Energy truncation We hence consider the following functionals with truncated energy

$$\widehat{F}_n^{\varepsilon,[\mathbf{V}]}(t) \doteq \sum_{\underline{j}_K} \sum_{\substack{\chi_{J_K} \\ L_K=0}} \int_{T_{\underline{j}_K}(t)} \mathrm{d}\underline{t}_{J_K} \int \mathrm{d}\underline{\omega}_{J_K} \mathrm{d}v_{n+1} \dots \mathrm{d}v_{N_K} \prod_{i=1}^{J_K} s_i \langle \omega_i, v_{n+i} - v_{m_i}^{[t_i^+]} \rangle_+ \times F_{N_K}^{\varepsilon}\big(0, \underline{z}_{N_K}^{[0]}\big) \mathbb{1}_{\left\| \underline{v}_{N_K}^{[0]} \right\|^2 \leqslant \mathbf{V}^2}, \tag{3.35}$$

and similarly the truncated limit functions $\widehat{G}_n^{[\mathbf{V}]}$. The error made by truncating the velocities is

$$\widehat{F}_n^{\varepsilon}(t) - \widehat{F}_n^{\varepsilon,[\mathbf{V}]}(t) = \sum_{(j_i \leqslant 2^i)} Q_{n,\underline{0}_{j_1}}(h_1) \dots Q_{N_{K-1},\underline{0}_{j_K}}(h_K) \left[F_{N_K}^{\varepsilon}(0) \mathbb{1}_{\left\| \underline{v}_{N_K}^{[0]} \right\|^2 \geqslant \mathbf{V}^2} \right],$$

with the following initial estimate

$$\begin{aligned} \left\| F_{N_K}^{\varepsilon}\big(0, \underline{z}_{N_K}^{[0]}, \underline{\tilde{\ell}}_{N_K}\big) \mathbb{1}_{\left\| \underline{v}_{N_K}^{[0]} \right\|^2 \geqslant \mathbf{V}^2} \right\|_{N_K,\beta/2} &\leq \sup_{\underline{u}_{N_K} \in \mathbb{R}^{dN_K}} \left| F_{N_K}^{\varepsilon}(0) e^{\beta \left\| \underline{u}_{N_K} \right\|^2} e^{-\frac{\beta}{2} \left\| \underline{u}_{N_K} \right\|^2} \mathbb{1}_{\left\| \underline{u}_{N_K} \right\|^2 \geqslant \mathbf{V}^2} \right| \\ &\leq \left\| F_{N_K}^{\varepsilon}(0) \right\|_{N_K,\beta} e^{-\frac{\beta}{2} \mathbf{V}^2}. \end{aligned}$$

Hence, applying the same bounds as in Propositions 3.6.1 and 3.7.1 in the same setting, we end up with the following lemma.

Lemma 3.8.1 (Energy truncation error)**.** *The error due to the energy truncation is bounded by*

$$\| \widehat{F}_n^{\varepsilon}(t) - \widehat{F}_n^{\varepsilon,[\mathbf{V}]}(t) \|_{\mathbb{L}^\infty} \leq C^{nK+A^K} \exp\left(-\frac{\beta}{2} \mathbf{V}^2 \right). \tag{3.36}$$

The same holds for its limit version $\widehat{G}_n(t) - \widehat{G}_n^{[\mathbf{V}]}(t)$.

Time separation We now need the successive collisions to be separated enough in time, to avoid pathological geometric recollisions. Like in the previous section, let us define

$$\widehat{F}_n^{\varepsilon,[\mathbf{V},\delta]}(t) \doteq \sum_{\underline{j}_K} \sum_{\substack{\chi_{J_K} \\ L_K=0}} \int_{T_{\underline{j}_K}^{[\delta]}(t)} \mathrm{d}\underline{t}_{J_K} \int \mathrm{d}\underline{\omega}_{J_K} \mathrm{d}v_{n+1} \dots \mathrm{d}v_{N_K} \prod_{i=1}^{J_K} s_i \langle \omega_i, v_{n+i} - v_{m_i}^{[t_i^+]} \rangle_+ \times F_{N_K}^{\varepsilon}\big(0, \underline{z}_{N_K}^{[0]}\big) \mathbb{1}_{\left\| \underline{v}_{N_K}^{[0]} \right\|^2 \leqslant \mathbf{V}^2}, \tag{3.37}$$

with the separation condition encoded in the time set

$$T_{\underline{j}_K}^{[\delta]}(t) = \Big\{ \underline{t} \in T_{\underline{j}_K}(t) \ , \ t_{i-1} - t_i > \delta \Big\}. \tag{3.38}$$

The limit version $\widehat{G}_n^{[\mathbf{V},\delta]}$ is defined by the same time restriction. The error of time separation is

$$\widehat{F}_n^{\varepsilon,[\mathbf{V}]} - \widehat{F}_n^{\varepsilon,[\mathbf{V},\delta]} \doteq \sum_{\underline{j}_K} \sum_{\substack{\chi_{J_K} \\ L_K=0}} \int_{\left(T_{\underline{j}_K}^{[\delta]}(t)\right)^c} \mathrm{d}\underline{t}_{J_K} \int \mathrm{d}\underline{\omega}_{J_K} \mathrm{d}v_{n+1} \dots \mathrm{d}v_{N_K} \times \prod_{i=1}^{J_K} s_i \langle \omega_i, v_{m_i}^{[t_i^+]} - v_{n+i} \rangle_+ \times F_{N_K}^{\varepsilon}\big(0, \underline{z}_{N_K}^{[0]}\big) \mathbb{1}_{\left\| \underline{v}_{N_K}^{[0]} \right\|^2 \leqslant \mathbf{V}^2},$$

and one can write

$$\left(T^{[\delta]}_{\underline{j}_K}(t)\right)^c = \bigcup_{i=1}^{J_K-1} \left\{\underline{t} \in T_{\underline{j}_K}(t)\ ,\ t_i - t_{i+1} \leqslant \delta\right\}. \tag{3.39}$$

Now, the integral in time over one of these sets, using the same method as before, changes the estimate of the corresponding successive-collision operator from $\dfrac{h_k^{j_k}}{j_k!}$ to $\dfrac{\delta h_k^{j_k-1}}{(j_k-1)!}$. Since the loss of $\frac{h_k}{j_k}$ and the factor $(J_K - 1)$ coming from the union (3.39) easily resorb into the bigger factor C^{A^K}, we get in the end the following lemma.

Lemma 3.8.2 (Time separation error)**.** *For this error, we have the following estimate*

$$\|\widehat{F}_n^{\varepsilon,[\mathbf{V}]} - \widehat{F}_n^{\varepsilon,[\mathbf{V},\delta]}\|_{\mathbb{L}^\infty} \leq C^{nK+A^K}\delta. \tag{3.40}$$

Once again, the same holds for its limit version $\widehat{G}_n^{[\mathbf{V}]} - \widehat{G}_n^{[\mathbf{V},\delta]}$ for similar reasons.

Restriction to non-pathological collision parameters Finally, at fixed ε, we have to restrict the collision parameters to non-pathological configurations, leading to no recollision during the free transport flow. To compute the convergence of the n-th marginal, we have to consider final configurations $\underline{z}_n = \underline{z}_n^{[t]}$ that do not directly lead to recollisions, i.e. that belong to the following set of past-excluding configurations, with some extra room $\varepsilon_d \doteq \varepsilon^{\frac{d}{d+1}}$:

$$\mathcal{E}_n^t(\varepsilon_d) \doteq \left\{\underline{z}_n \in \mathcal{D}_n^\varepsilon \;\middle|\; \forall \tau \in [0,t],\ (\underline{x}_n - \tau\underline{v}_n) \in \mathcal{D}_n^{\varepsilon_d}\right\}. \tag{3.41}$$

This set is the whole domain for $n = 1$, like in [7, 17]. Then, the restriction on the collision parameters is done based on the following geometric result, proved in [18, 7] and formalizing the arguments of Lanford [25], based on billiards theory.

Similarly to the notation $\underline{z}_n^{[\tau]} = (\underline{x}_n^{[\tau]}, \underline{v}_n^{[\tau]})$ for the hard sphere pseudo-trajectories, let us denote $\underline{\zeta}_n^{[\tau]} = (\underline{y}_n^{[\tau]}, \underline{u}_n^{[\tau]})$ the limit trajectories, and $N[\tau]$ the number of particles in the trajectory at time $\tau \in [0,t]$, from $N[t] = n$ to $N[0] = N_K$. Hence, this proposition asserts that once the previous truncations computed, choosing collision parameters away from a set of small measure, the hard sphere and limit pseudo-trajectories are easy to compare.

Lemma 3.8.3. *Considering a history $(\underline{j}_K, \chi_{J_K})$ with collision times $\underline{t}_{J_K} \in T^{[\delta]}_{\underline{j}_K}(t)$ (δ-separated) and a final configuration $\underline{z}_n^{[t]}$, given a maximum energy $\mathbf{V}^2 > 0$, there exists a set of pathological collision parameters*

$$\Pi(\underline{z}_n^{[t]}, \underline{j}_K, \chi_{J_K}) \subset (\mathbb{S}^{d-1} \times \mathbb{R}^d)^{J_K}$$

with small volume

$$|\Pi(\underline{z}_n^{[t]}, \underline{j}_K, \chi_{J_K})| \leq C J_K N_K \left(\varepsilon^{\frac{d}{d+2}} + \mathbf{V}^d \times \varepsilon^{\frac{d-1}{3(d+1)}} + \mathbf{V}^{\frac{d+1}{2}} \left(\frac{\varepsilon^{\frac{d}{d+1}}}{\delta}\right)^{\frac{d-1}{2}}\right), \tag{3.42}$$

and such that, assuming

i. the collision parameters are non-pathological: $(\underline{\omega}_{J_K}, v_{n+1}, \ldots, v_{N_K}) \notin \Pi(\underline{z}_n^{[t]}, \underline{j}_K, \chi_{J_K})$

ii. the energy of the corresponding pseudo-trajectory remains bounded: $\|\underline{v}_{N_K}^{[0]}\|^2 \leq \mathbf{V}^2$

iii. the final configuration is past-excluding for the free-flow: $\underline{z}_n^{[t]} \in \mathcal{E}_n^t(\varepsilon_d)$, *then*

1. *the hard sphere pseudo-positions remains sufficiently far away:* $\forall \tau \in [0,t],\ \underline{x}^{[\tau]}_{N[\tau]} \in \mathcal{D}^{\varepsilon_d/2}_{N[\tau]}$
2. *the* velocities *of the hard sphere and limit trajectories coincide:* $\forall \tau \in [0,t],\ \underline{v}^{[\tau]}_{N[\tau]} = \underline{v}^{[\tau]}_{N[\tau]}$
3. *the* positions *of both trajectories remain close:* $\forall \tau \in [0,t], \forall i \leq N[\tau], d(x^{[\tau]}_i, y^{[\tau]}_i) \leq J_K\varepsilon$.

Let us observe that *3.* is a consequence of *2.* since, when the velocities coincide, the only difference between the positions is the shift of $\varepsilon\omega_i$ that happens at each particle adjunction. Furthermore, *2.* is a consequence of *1.* since when the collision parameters are the same, and if the particles do not collide between the particle adjunctions, the velocities of both pseudo-trajectories are identically determined. This result has been proved [18, 7] for a one-species gas, but the dynamics is strictly identical for our mixture gas.

Hence, we consider the functional restricted to non-pathological collision parameters

$$\widetilde{F}^{\varepsilon,[\mathbf{V},\delta]}_n(t) \doteq \sum_{\underline{j}_K} \sum_{\substack{\chi_{J_K} \\ L_K=0}} \int_{T^{[\delta]}_{\underline{j}_K}(t)} \mathrm{d}\underline{t}_{J_K} \int_{\Pi(\underline{z}^{[t]}_n, \underline{j}_K, \chi_{J_K})^c} \mathrm{d}\underline{\omega}_{J_K} \mathrm{d}v_{n+1} \dots \mathrm{d}v_{N_K} \\ \times \prod_{i=1}^{J_K} s_i \langle \omega_i, v_{n+i} - v^{[t_i^+]}_{m_i} \rangle_+ \times F^{\varepsilon}_{N_K}\big(0, \underline{z}^{[0]}_{N_K}\big) \mathbb{1}_{\left\|\underline{v}^{[0]}_{N_K}\right\|^2 \leqslant \mathbf{V}^2}, \tag{3.43}$$

and its limit version $\widetilde{G}^{[\mathbf{V},\delta]}_n$. Now, the error $\widetilde{F}^{\varepsilon,[\mathbf{V},\delta]}_n - \widehat{F}^{\varepsilon,[\mathbf{V},\delta]}_n$ is supported on $\Pi(\underline{z}^{[t]}_n, \underline{j}_K, \chi_{J_K})$, so that we will use the control on its volume (3.42) to control the successive collision operators, concluding the bounds with the usual computation. More precisely, in the proof [17] of Proposition 3.5.1, we bound the collision operators (2.12) in the following way:

$$\begin{aligned} e^{\beta \|\underline{v}_j\|^2} \left| \mathcal{C}^{\ell}_j f_{j+1} \right| &\leq \sum_{i=1}^{j} \int \mathrm{d}\omega_j \mathrm{d}v_{j+1} \left| v_{j+1} - v_i \right| \cdot \|f_{j+1}\|_{j+1,\beta'} \times e^{-(\beta'-\beta)\|\underline{v}_j\|^2 - \beta'|v_{j+1}|^2} \\ &\leq \|f_{j+1}\|_{j+1,\beta'} \int \mathrm{d}\omega_j \mathrm{d}v_{j+1} \left(j\left|v_{j+1}\right| + \sum_{i=1}^{j} |v_i| \right) e^{-(\beta'-\beta)\|\underline{v}_j\|^2 - \beta'|v_{j+1}|^2}. \end{aligned}$$

By the Cauchy-Schwarz inequality, we used to bound [17]

$$\begin{aligned} e^{\beta \|\underline{v}_j\|^2} \left| \mathcal{C}^{\ell}_j f_{j+1} \right| &\leq \|f_{j+1}\|_{j+1,\beta'} \int \mathrm{d}\omega_j \mathrm{d}v_{j+1} \left(j\left|v_{j+1}\right| + \sqrt{\frac{j}{2e(\beta'-\beta)}} \right) e^{-\beta'|v_{j+1}|^2} \\ &\leq \|f_{j+1}\|_{j+1,\beta'} \left(j \frac{c_d}{\sqrt{\beta'^{d+1}}} + \sqrt{\frac{j}{2e(\beta'-\beta)}} \frac{\tilde{c}_d}{\sqrt{\beta'^d}} \right), \end{aligned}$$

for some constants $c_d, \tilde{c}_d$ depending only on the dimension. Nonetheless, for the present bound, one can write instead

$$\int \mathrm{d}\omega_j \mathrm{d}v_{j+1} \left(j|v_{j+1}| + \sqrt{\frac{j}{2e(\beta'-\beta)}} \right) e^{-\beta'|v_{j+1}|^2} \leq \int \mathrm{d}\omega_j \mathrm{d}v_j \left(\frac{1}{\sqrt{2e\beta}} + \sqrt{\frac{e^{-1}}{\beta - \beta'}} \right),$$

making appear the volume of collision parameters bounded in (3.42), leading to the following estimate (the factors J_K and N_K resorb in the bigger factor C^{nK+A^K}, for a slighlty different constant C depending on β).

Lemma 3.8.4 (Error of restriction to non-pathological collision parameters)**.** *Considering only collision parameters chosen so as to avoid recollisions lead to an error of order*

$$\|\widetilde{F}_n^{\varepsilon,[\mathbf{V},\delta]} - \widehat{F}_n^{[\mathbf{V},\delta]}\|_{\mathbb{L}^\infty} \leq C^{nK+A^K}\left(\varepsilon^{\frac{d}{d+2}} + \mathbf{V}^d \times \varepsilon^{\frac{d-1}{3(d+1)}} + \mathbf{V}^{\frac{d+1}{2}}\left(\frac{\varepsilon^{\frac{d}{d+1}}}{\delta}\right)^{\frac{d-1}{2}}\right), \tag{3.44}$$

and similarly for the limit $\widehat{G}_n^{[\mathbf{V}]} - \widehat{G}_n^{[\mathbf{V},\delta]}$ from the same computation.

Harnessing initial proximity Now that we have constructed approximations of our distributions that avoid recollisions, we can at last compare both the BBGKY and limiting distributions thanks to the coupled pseudo-trajectories. Indeed, we can write the coupling

$$\begin{aligned}\widetilde{F}_n^{\varepsilon,[\mathbf{V},\delta]} - \widetilde{G}_n^{[\mathbf{V},\delta]} = &\sum_{\underline{j}_K}\sum_{\chi_{J_K}} p_\mu^{|\underline{\ell}^*_{J_K}|}\int_{T^{[\delta]}_{\underline{j}_K}(t)} \mathrm{d}\underline{t}_{J_K}\int_{\Pi(\underline{z}_n^{[t]},\underline{j}_K,\chi_{J_K})^c} \mathrm{d}\underline{\omega}_{J_K}\mathrm{d}v_{n+1}\dots\mathrm{d}v_{N_K}\\ &\times\prod_{i=1}^{J_K} s_i\langle\omega_i, v_{n+i} - v_{m_i}^{[t_i^+]}\rangle_+\left[F^\varepsilon_{N_K}(0,\underline{z}^{[0]}_{N_K}) - G_{N_K}(0,\underline{\zeta}^{[0]}_{N_K})\right]\mathbb{1}_{\left\|\underline{v}^{[0]}_{N_K}\right\|^2\leqslant\mathbf{V}^2},\end{aligned}$$

with the same collision parameters for both pseudo-trajectories, the hard-sphere one and the limit one. Since by construction $\underline{x}_n^{[t]} = \underline{y}_n^{[t]}$, and by Lemma 3.8.3 for $\underline{z}_n^{[t]} \in \mathcal{E}_n^t(\varepsilon_d)$, the n first particles also have identical velocities on $[0,t]$, then for all times $\tau\in[0,t]$, we have $\underline{z}_n^{[\tau]} = \underline{\zeta}_n^{[\tau]}$. Henceforth, since by the work done in Section 3.7 all the added particles are particles at equilibrium tagged 0, one has (as $\underline{\ell}_n \subset [\![1,n]\!]$),

$$\begin{aligned}G_{N_K}(0,\underline{\zeta}^{[0]}_{N_K}) &= M_\beta^{\otimes N_K}(\underline{u}^{[0]}_{N_K})\rho^{\otimes\underline{\ell}_n}(\underline{\zeta}^{[0]}_{\underline{\ell}_n})\\ &= M_\beta^{\otimes N_K}(\underline{v}^{[0]}_{N_K})\rho^{\otimes\underline{\ell}_n}(\underline{z}^{[0]}_{\underline{\ell}_n}),\end{aligned}\tag{3.45}$$

and so

$$\begin{aligned}\left|F^\varepsilon_{N_K}(0,\underline{z}^{[0]}_{N_K}) - G_{N_K}(0,\underline{\zeta}^{[0]}_{N_K})\right| &= \left|F^\varepsilon_{N_K}(0,\underline{z}^{[0]}_{N_K}) - G_{N_K}(0,\underline{z}^{[0]}_{N_K})\right|\\ &\leq (C_0)^{N_K}e^{-\beta\|\underline{v}^{[0]}_{N_K}\|^2}\cdot\varepsilon\end{aligned}$$

by the initial proximity result (3.18). Hence, the previous estimates on the successive-collision operators eventually yield the following result.

Lemma 3.8.5 (Initial value error)**.** *Conditionally to $\underline{z}_n \in \mathcal{E}_n^t(\varepsilon_d)$, the initial error is bounded by*

$$\left\|\widetilde{F}_n^{\varepsilon,[\mathbf{V},\delta]} - \widetilde{G}_n^{[\mathbf{V},\delta]}\right\|_{\mathbb{L}^\infty(\mathcal{E}_n^t(\varepsilon_d))} \leq C^{nK+A^K}\varepsilon. \tag{3.46}$$

Coherent choice of truncation parameters $(K,\mathbf{V},\delta)$ Eventually, the last step is to tune the truncation parameters according to ε, so as to obtain the convergence we want. Actually, there is room to choose the scaling, since one can set all the errors as powers of ε, except for the pruning error (3.29) which is significantly bigger. Explicitly, stacking all the errors [pruning (3.29), removing additional tagged particle (3.34), energy truncation (3.36), time separation (3.40) and removing pathological trajectories (3.44)] then using the coupling result (3.46), one has

$$\begin{aligned}\|F_n^\varepsilon(t) - G_n(t)\|_{\mathbb{L}^\infty(\mathcal{E}_n^t(\varepsilon_d))} \leq\ & C^{C_0\lambda}n^{cn}e^{-2^{K-K^\alpha}}\\ &+ C^{nK+A^K}\left(p_\mu + e^{-\frac{\beta}{2}\mathbf{V}^2} + \delta + \left[\varepsilon^{\frac{d}{d+2}} + \mathbf{V}^d\times\varepsilon^{\frac{d-1}{3(d+1)}} + \frac{\mathbf{V}^{\frac{d+1}{2}}\varepsilon^{\frac{d(d-1)}{2(d+1)}}}{\delta^{\frac{d-1}{2}}}\right] + \varepsilon\right).\end{aligned}$$

Hence, choosing the scaling

$$\delta = \varepsilon^{\frac{d-1}{d+1}}, \quad \text{and } \mathbf{V}^2 = \frac{2}{\beta}|\log \varepsilon|,$$

one gets

$$e^{-\frac{\beta}{2}\mathbf{V}^2} + \delta + \mathbf{V}^d \times \varepsilon^{\frac{d-1}{4(d+1)}} + \frac{\mathbf{V}^{\frac{d+1}{2}} \varepsilon^{\frac{d(d-1)}{2(d+1)}}}{\delta^{\frac{d-1}{2}}} \leq \varepsilon + \varepsilon^{\frac{d-1}{d+1}} + \left|\frac{2}{\beta}\log\varepsilon\right|^{\frac{d}{2}} \varepsilon^{\frac{d-1}{3(d+1)}} + \left|\frac{2}{\beta}\log\varepsilon\right|^{\frac{d+1}{4}} \varepsilon^{\frac{d-1}{2(d+1)}}$$
$$\leq \varepsilon^{\frac{d-1}{4(d+1)}}$$

for ε small enough. This way, if we pick

$$K = \left\lfloor \frac{1}{\log A} \log\left(\frac{(d-1)|\log\varepsilon|}{8(d+1)\log C}\right)\right\rfloor,$$

then $C^{A^K} \leq \varepsilon^{-\frac{d-1}{8(d+1)}}$, and we can deal with the term C^{nK} like in the proof of Proposition 3.6.1, yielding the same factor n^{cn}. Hence, for K large enough, denoting $c_\beta \doteq (d-1)/8(d+1)\log C$,

$$e^{-2^{K-K^\alpha}} \leq e^{-2^{(1-\alpha)K}} \leq \exp\left(-c_\beta|\log\varepsilon|^{(1-\alpha)\frac{\log 2}{\log A}}\right).$$

As $A > 2$ is arbitrary, let us choose A such that $(1-\alpha)\dfrac{\log 2}{\log A} \geq 1-2\alpha$. This gives us the final condition on

$$\lambda \leq \frac{c_\beta}{2C_0\log C}|\log\varepsilon|^{1-2\alpha}.$$

Note that we take this scaling so as to have the biggest λ possible in the hypothesis, pushing the associated error at the same level as the truncation error above. It yields

$$C^{C_0\lambda}e^{-2^{K-K^\alpha}} \leq \exp\left(-\frac{c_\beta}{2}|\log\varepsilon|^{1-2\alpha}\right).$$

The final verification is the condition $p_\mu \leq 2^{-K}$ of Proposition 3.7.1, which is satisfied with room to spare considering the choices above. Piling all of this, the pruning error is bigger than all the other ones, and we end up with the very last inequality for ε small enough

$$\|F_n^\varepsilon(t) - G_n(t)\|_{\mathbb{L}^\infty(\mathcal{E}_n^t(\varepsilon_d))} \leq n^{cn}\exp\left(-\frac{c_\beta}{2}|\log\varepsilon|^{1-2\alpha}\right).$$

Eventually, the indicator $\mathbb{1}_{\mathcal{E}_n^t(\varepsilon_d)}$ of the set of past-excluding configurations (3.41) pointwise converges as ε goes to 0 to the indicator of the following set of full measure

$$\left\{\underline{z}_n \in \mathcal{D}^n\ ,\ \forall 1 \leq i < j \leq n,\ x_i \neq x_j \text{ and } v_i - v_j \notin \mathrm{Vect}(x_i - x_j)\right\},$$

which concludes the proof of Theorem 3.1. □

4 Cumulants and statistical refinements

To study further than the first order convergence of the empirical measure (Corollary 3.1.1), we introduce in this section its fluctuation field around its expected value, and present its limit. Eventually, we state a large deviation principle for the empirical measure, that we prove in the following sections.

This study is performed thanks to finer objects than the correlation functions, called *cumulants*, that capture finer scales of the dynamics and allow to rescale its rare events to characterize them. In the end, we show the convergence in short times of these cumulants to limit objects, with a full convergence rate in ε thanks to a new precise computation of the geometric estimates presented in Appendix D.

4.1 Statistical refinements: fluctuations and large deviations

The empirical measure of the tagged particles, defined in (3.3) for an observable H, can be seen as the observation of a measure $\tilde{\pi}_t^\varepsilon \in \mathcal{M}(\mathcal{D})$ on the domain $\mathcal{D}$, writing

$$\tilde{\pi}_t^\varepsilon[H] = \int H(z) \mathrm{d}\tilde{\pi}_t^\varepsilon(z). \tag{4.1}$$

The family $(\tilde{\pi}_s^\varepsilon)_{0\leqslant s\leqslant t}$ defines a measure on the trajectories of $\mathcal{D}^{[0,t]}$. The set $\mathrm{Traj}([0,t],\mathcal{M}(\mathcal{D}))$ of such measures is endowed with the Skorokhod topology. On the other hand, one may define the *fluctuation field*

$$\zeta_t^\varepsilon = \sqrt{\lambda}\big(\tilde{\pi}_t^\varepsilon - \mathbb{E}\,[\tilde{\pi}_t^\varepsilon]\big), \tag{4.2}$$

to capture the next small order after the law of large numbers (3.5).

Theorem 4.1 (Convergence of the fluctuation field)**.** *In the mixed scaling* ($S_{\varepsilon,\mu,\lambda}$)*, the fluctuation field defined above converges in law to a Gaussian process ζ_t, whose covariance is given by*

$$\mathbb{E}\Big[\zeta_t[g]\zeta_t[h]\Big] = \int_{\mathcal{D}} M(v)\varphi(t,z)g(z)h(z)\mathrm{d}z, \tag{4.3}$$

where φ denotes the solution to the linear Rayleigh–Boltzmann equation (2.7)*.*

Indeed, unlike in [9] where the limit fluctuation field satisfies a linear stochastic equation of the form

$$\mathrm{d}\tilde{\zeta}_t = \mathcal{L}\tilde{\zeta}_t \,\mathrm{d}t + \mathrm{d}\tilde{\eta}_t,$$

here the limit fluctuation field is trivial and do not depend on the second cumulant (see next Section 4.2); there is no interference between the tagged particles, on the mere condition that $\lambda \ll \varepsilon^{1-d}$, without any phase transition between this scaling and the nonlinear scaling $\lambda \sim \alpha\varepsilon^{1-d}$. The proof of this theorem is given in Section 8.2, using the convergence of cumulants, which is the point of the following Sections 4.2, 5, 6 and 7.

To study the *large deviations* of the dynamics, we will harness a weaker topology than the Skorokhod one. For a measure $\mathfrak{m} = (\mathfrak{m}_s)_{s\in[0,t]} \in \mathrm{Traj}([0,t],\mathcal{M}(\mathcal{D}))$, and an observable $h \in \mathcal{C}_c^\infty([0,t]\times\mathcal{D})$, we define the *filtered mean*

$$\{h,\mathfrak{m}\}_t = \int_{\mathcal{D}} h(t,z)\mathrm{d}\mathfrak{m}_t(z) - \int_0^t \int_{\mathcal{D}} (\partial_s + v\cdot\nabla_x)h(s,z)\mathrm{d}\mathfrak{m}_s(z), \tag{4.4}$$

that filters the transported part of the considered observables. The first quantity that will be relevant for the large deviation principle will be the limit object $\mathcal{I}(t,h)$, that is defined in (7.3) as the limit of an expectancy, and which can also be defined as the mild solution (Proposition 7.5.2) to the Hamilton–Jacobi system

$$\begin{cases} (\partial_s - v\cdot\nabla_x)q^{[t]} = \dfrac{\partial\mathcal{H}}{\partial p}(q^{[t]},p^{[t]}) & ,\quad q^{[t]}(0) = M\varphi_0, \\ (\partial_s - v\cdot\nabla_x)(p^{[t]} - h) = -\dfrac{\partial\mathcal{H}}{\partial q}(q^{[t]},p^{[t]}) & ,\quad p^{[t]}(t) = h(t), \end{cases} \tag{4.5}$$

with the Hamiltonian

$$\mathcal{H}\,(q,p) \doteq \int \mathrm{d}v_2\mathrm{d}\omega\langle v_1 - v_2,\omega\rangle_+ M_\beta(v_2)q(z_1)(e^{p(x_1,v_1)-p(x_1,v_1')} - 1),$$

in the sense of

$$\mathcal{I}(t,h) = \mathcal{I}(t,0) + \int_0^t \mathrm{d}s\int_{\mathcal{D}} q^{[t]}(s)(\partial_s - v\cdot\nabla_x)(p^{[t]}(s) - h(s)) + \int_0^t \mathcal{H}(q^{[t]}(s),p^{[t]}(s))\mathrm{d}s. \tag{4.6}$$

More precisely, we are interested in its Legendre transform

$$\boldsymbol{\Lambda}(t,\mathfrak{v}) \doteq \sup_{h\in\mathbb{B}_{t,\beta}} \Big[\{h,\mathfrak{v}\} - \mathcal{I}(t,h) - 1\Big], \tag{4.7}$$

where the supremum is taken over observables in $\mathbb{B}_{t,\beta}$, defined in (7.4) as the set of observables with bounded transport, and such that e^h is uniformly dominated by the β–Gaussian.

Eventually, for the lower bound of the large deviation principle, we need to consider the set $\mathbf{S}_t$ of strong solutions, on $[0,t]$, to a biased linear Boltzmann equation of the form

$$(\partial_s - v\cdot\nabla_x)\mathfrak{v} = \int \mathrm{d}v_2\mathrm{d}\omega\langle v_1 - v_2,\omega\rangle_+ \Big(M_\beta(v_2')\mathfrak{v}(v_1')e^{p(z_1)-p(z_1')} - M_\beta(v_2)\mathfrak{v}(v_1)e^{p(z_1')-p(z_1)}\Big), \tag{4.8}$$

for some $p\in\mathbb{B}_{t,\beta}$. The large deviation principle might then be formulated as follows.

Theorem 4.2 (Large deviations of the empirical measure)**.** *Considering the tagged empirical measure (4.1), there exists a time $T>0$ such that, for any $t\in(0,T]$, we have the following large deviation upper bound*

$$\limsup_{\varepsilon\to 0}\frac{1}{\lambda}\log\mathbb{P}\big(\tilde{\pi}_t^\varepsilon \in \mathbf{F}\big) \leq -\inf_{\mathfrak{v}\in\mathbf{F}}\boldsymbol{\Lambda}(t,\mathfrak{v}), \tag{4.9}$$

when $\mathbf{F}$ is a closed set in the Skorokhod topology. Additionally, when $\mathbf{O}$ is an open set in this topology, one has the large deviation lower bound

$$\liminf_{\varepsilon\to 0}\frac{1}{\lambda}\log\mathbb{P}\big(\tilde{\pi}_t^\varepsilon \in \mathbf{O}\big) \geq -\inf_{\mathfrak{v}\in\mathbf{O}\cap\mathbf{S}_t}\boldsymbol{\Lambda}(t,\mathfrak{v}). \tag{4.10}$$

Note that the most useful result is the upper bound on the probability of deviation, which has no restriction on its infimum. Nevetheless, the lower bound, which precises that the upper bound is optimal, is here restricted (like in [9]) to solutions not that far from the Boltzmann linear equation, since they must be solutions to the biased equation (4.8). The proof of this theorem is the subject of Section 8.3, once again based on the convergence of the cumulants.

4.2 Cumulant generating function

The cumulant generating function is the functional $\log\mathbb{E}\left[\exp\left(\mu\pi_t^\varepsilon[H]\right)\right]$, containing all the information on the moments of the empirical measure. Using the identity (2.10) between the correlation functions and observables, the formal expansion of this functional leads naturally [9] to objects called *cumulants*, which we define below. We then prove rigourously the said formal expansion of the cumulant generating function (4.16).

4.2.1 Cumulants

The definition of the cumulants is based on a decomposition into partitions, so that for $\sigma\in\mathcal{P}_n$ a partition of $[\![1,n]\!]$, we denote $|\sigma|$ the number of subsets $(\sigma_i)_{1\leqslant i\leqslant|\sigma|}$ that compose this partition, which is not to be confused with the cardinal $|\sigma_i|$ of one of these subsets. Eventually, we denote $\mathcal{P}_n^k\subset\mathcal{P}_n$ the set of partitions $\sigma\in\mathcal{P}_n$ that contain exactly $k=|\sigma|$ subsets.

Definition 4.1 (Cumulants)**.** The cumulants associated to a family $(G_n)_{n\geqslant 1}$ are defined as

$$g_n(\underline{z}_n,\underline{\ell}_n) \doteq \sum_{\sigma\in\mathcal{P}_n}(-1)^{|\sigma|-1}(|\sigma|-1)!\,G_{[\sigma]}(\underline{z}_n,\underline{\ell}_n), \tag{4.11}$$

where for any partition $\sigma \in \mathcal{P}_n$, we denote

$$G_{[\sigma]}(\underline{z}_n, \underline{\ell}_n) = \prod_{i=1}^{|\sigma|} G_{|\sigma_i|}(\underline{z}_{\sigma_i}, \underline{\ell}_{\sigma_i}).$$

Note that the n-th cumulant g_n is constructed from all the correlation functions (G_i) for $i \leq n$. Indeed, it decomposes the interaction of n particles into products of interactions within the subsets of every possible partition, to measure the defects of independence. One may easily check that for a tensorized family ($G_n = G_1^{\otimes n}$), all the cumulants g_n will vanish for $n \geq 2$.

We denote (f_n^ε) and (f_n) the cumulants respectively associated to the hierarchies (F_n^ε) and (F_n). Note that the second cumulant is $f_2^\varepsilon = F_2^\varepsilon - F_1^{\varepsilon\otimes 2}$, encoding the defect of independence between pairs of particles. One will see in Section 5.2 that when expanding the pseudo-trajectory formula, this corresponds to the rare dynamics in which a distinguished couple of particles interact together. Appendix B.2 is dedicated to the study of the cumulants of the exclusion indicators $(\mathbb{1}_{\mathcal{X}_n^\varepsilon})$, denoted ϕ_n.

Proposition 4.2.1 (Inversion formula)**.** *With the definition above, the injectivity of cumulants is a consequence of the following inversion formula*

$$G_n = \sum_{\sigma \in \mathcal{P}_n} g_{[\sigma]}. \tag{4.12}$$

Proof. We start this proof with the following combinatorial identity, satisfied for any $r \geq 2$,

$$\sum_{k=1}^{r} \sum_{\sigma \in \mathcal{P}_r^k} (-1)^k \prod_{i=1}^{k} (|\sigma_i| - 1)! = 0, \tag{4.13}$$

which is shown in [9, Lemma 2.5.1] using the Taylor series of $x \mapsto \exp(\log(1+x))$. Note that it can also be proved using combinatorics arguments (see Appendix B.2, Lemma B.1.1). We can hence introduce artificially the sum of the $G_{[\rho]}$ appearing in the cumulants' formula, making each of them vanish thanks to the latter identity (4.13), summing over partitions σ *of the subsets* (ρ_i) that compose the partition ρ (since in the notation below $r = |\rho|$):

$$\begin{aligned} G_n &= G_n + \sum_{r=2}^{n} \sum_{\rho \in \mathcal{P}_n^r} (-1)^r G_{[\rho]} \sum_{k=1}^{r} \sum_{\sigma \in \mathcal{P}_r^k} (-1)^k \prod_{i=1}^{k} (|\sigma_i| - 1)! \\ &= \sum_{r=1}^{n} \sum_{\rho \in \mathcal{P}_n^r} \sum_{k=1}^{r} \sum_{\sigma \in \mathcal{P}_r^k} \prod_{i=1}^{k} \left[(-1)^{|\sigma_i|-1} (|\sigma_i| - 1)! \prod_{j \in \sigma_i} G_{\rho_j} \right] \end{aligned}$$

using the definition of $G_{[\rho]}$ and the fact that $r = \sum |\sigma_i|$. The trick now is to invert the partitions: seeing $\sigma \in \mathcal{P}_{|\rho|}$ as a coarser partition than ρ, it is similar as taking a partition $\tilde{\sigma}$ of $[\![1, n]\!]$, and then a partition $\rho \in \mathcal{P}_{\tilde{\sigma}_i}$ of *each subset* $\tilde{\sigma}_i$ (cf. Fig. 3). Doing so, the cardinal of a subset $|\sigma_i|$ becomes the

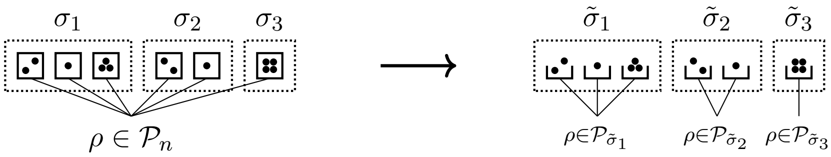

Figure 3: Partition indexing

number of subsets in the partition ρ of σ_i, so that inverting the partitions we get

$$G_n = \sum_{k=1}^{n} \sum_{\sigma\in\mathcal{P}_n^k} \prod_{i=1}^{k} \sum_{\rho\in\mathcal{P}_{\sigma_i}} (-1)^{|\rho|-1}(|\rho|-1)! G_{[\rho]},$$

which concludes the proof. □

4.2.2 Expanding the cumulant generating function

We go now to the proof of an expansion for the cumulant generating function. By linearity, and using the fact that before having been assigned a tag, the particles are exchangeable, one has

$$\begin{aligned}\mathbb{E}\left[\exp\left(\sum_{i=1}^{\mathcal{N}} H(Z_{\varepsilon,i}^{[t]}, L_i)\right)\right] &= 1 + \sum_{k\geqslant 1}\frac{1}{k!}\mathbb{E}\left[\left(\sum_{i=1}^{\mathcal{N}} H(Z_{\varepsilon,i}^{[t]}, L_i)\right)^k\right] \\ &= 1 + \sum_{k\geqslant 1}\frac{1}{k!}\mathbb{E}\left[\sum_{n=1}^{k}\frac{1}{n!}\sum_{\substack{k_1+\cdots+k_n=k\\(k_j\geqslant 1)_{j\leq n}}}\frac{k!}{k_1!\dots k_n!}\sum_{\substack{(i_j\leqslant\mathcal{N})_{j\leq n}\\ i_j\neq i_{j'}}} H_{i_1}^{k_1}\dots H_{i_n}^{k_n}\right],\end{aligned}$$

denoting $H_i \doteq H(Z_{\varepsilon,i}^{[t]}, L_i)$, and expanding the power k by partitioning the resulting sum according to the number of different particles implied in each product. Hence, using the relation (2.10) between observables and correlation functions, and permuting the sums to get rid of the sum over k, we get (once again extending the correlation functions by 0 outside of their domain)

$$\begin{aligned}\mathbb{E}\left[\exp\left(\sum_{i=1}^{\mathcal{N}} H(Z_i^t, L_i)\right)\right] &= 1 + \sum_{n\geqslant 1}\frac{1}{n!}\sum_{\underline{\ell}_n\in\Lambda_n}\lambda^{|\underline{\ell}_n|}\mu^{n-|\underline{\ell}_n|}\int F_n^\varepsilon(t,\underline{\ell}_n)\prod_{i=1}^{n}\left(\sum_{k_i\geqslant 1}\frac{1}{k_i!}H(z_i,\ell_i)^{k_i}\right) \\ &= 1 + \sum_{n\geqslant 1}\frac{1}{n!}\sum_{\underline{\ell}_n\in\Lambda_n}\lambda^{|\underline{\ell}_n|}\mu^{n-|\underline{\ell}_n|}\int F_n^\varepsilon(t,\underline{\ell}_n)(e^H-1)^{\otimes n}(\underline{\ell}_n). \qquad (4.14)\end{aligned}$$

Now that we dispose of an expansion for this expectancy in terms of correlation functions, let us see how the cumulants appear along with a logarithm. We start from the formula (4.14) above and use the inversion formula (4.12). Hence, using the exchangeability of particles to reduce the partitions to the number of elements in each subset, we get

$$\begin{aligned}&\mathbb{E}\left[\exp\left(\sum_{i=1}^{\mathcal{N}} H(Z_i^{[t]}, L_i)\right)\right] \\ &\quad= 1 + \sum_{n\geqslant 1}\frac{1}{n!}\sum_{\underline{\ell}_n\in\Lambda_n}\lambda^{|\underline{\ell}_n|}\mu^{n-|\underline{\ell}_n|}\sum_{\sigma\in\mathcal{P}_n}\prod_{i=1}^{|\sigma|}\int_{\mathcal{D}^{|\sigma_i|}} f_{\sigma_i}^\varepsilon(t,\underline{\ell}_{\sigma_i})(e^H-1)^{\otimes\sigma_i}(\underline{\ell}_{\sigma_i}) \\ &\quad= 1 + \sum_{n\geqslant 1}\frac{1}{n!}\sum_{s=1}^{n}\frac{1}{s!}\sum_{\substack{p_1+\ldots+p_s=n\\(p_i\geqslant 1)_{i\leqslant s}}}\frac{n!}{p_1!\dots p_s!}\prod_{i=1}^{s}\sum_{\underline{\ell}^{(i)}\in\Lambda_{p_i}}\lambda^{|\underline{\ell}^{(i)}|}\mu^{p_i-|\underline{\ell}^{(i)}|}\int_{\mathcal{D}^{p_i}} f_{p_i}^\varepsilon(t,\underline{\ell}^{(i)})(e^H-1)^{\otimes p_i}(\underline{\ell}^{(i)}),\end{aligned}$$

where the denominator $s!$ stems from the arbitrary order that we impose on the partitions' subsets. We have also split the labels $\underline{\ell}_n\in\Lambda_n$ into the labels $\underline{\ell}^{(i)}\in\Lambda_{p_i}$ on each subset. Eventually, we sum over n to relax the condition on the subset cardinals p_i, and factorize everything as

$$\mathbb{E}\left[\exp\left(\sum_{i=1}^{\mathcal{N}} H(Z_i^{[t]}, L_i)\right)\right] = 1 + \sum_{s\geqslant 1}\frac{1}{s!}\left(\sum_{p\geqslant 1}\frac{1}{p!}\sum_{\underline{\ell}_p\in\Lambda_p}\lambda^{|\underline{\ell}_p|}\mu^{p-|\underline{\ell}_p|}\int_{\mathcal{D}^p} f_p^\varepsilon(t,\underline{\ell}_p)(e^H-1)^{\otimes p}(\underline{\ell}_p)\right)^s,$$

which makes appear the exponential of the quantity defined below.

Definition 4.2 (Cumulant generating function)**.** Thanks to the computation above, the two following definitions are equivalent, defining the *cumulant generating function*:

$$\mathfrak{G}_\varepsilon^{[t]}[H] \doteq \log \mathbb{E}\left[\exp\left(\sum_{i=1}^{\mathcal{N}} H(Z_i^{[t]}, L_i)\right)\right] \tag{4.15}$$

$$\doteq \sum_{p\geqslant 1}\frac{1}{p!}\sum_{\underline{\ell}_p\in\Lambda_p}\lambda^{|\underline{\ell}_p|}\mu^{p-|\underline{\ell}_p|}\int_{\mathcal{D}^p} f_p^\varepsilon(t,\underline{\ell}_p)(e^H-1)^{\otimes p}(\underline{\ell}_p). \tag{4.16}$$

Note that it is not directly a generating function as one can be used to, since it is not an expansion in powers of the observable H, but in powers of its exponential, which makes appear combinations of the cumulants when deriving along H, not directly cumulants.

Let us finally observe that if the observable is of the form $\hat{H}(z,\ell) = h(z)\mathbb{1}_{\ell=1}$, (i.e. counting only the tagged particles) then the cumulant generating function writes

$$\mathfrak{G}_\varepsilon^{[t]}[\hat{H}] = \sum_{p\geqslant 1}\frac{\lambda^p}{p!}\int_{\mathcal{D}^p} f_p^\varepsilon(t,\underline{1}_p)(e^{\hat{H}}-1)^{\otimes p}. \tag{4.17}$$

We do not renormalize yet the cumulant generating function, since according to whether the observable H weights all the particles or only the tagged ones, the suitable scale will be μ or λ.

5 Reparametrization of the trajectories using dynamics trees

The first step to understand the cumulants is to find an equation on them implying only other cumulants, which we do by computing an expansion of the dynamics, based on the interactions between the particles (Lemma 5.2.1).

This expansion may be found in [9, Chapter 3, *Tree expansions of the hard-sphere dynamics*], for the general indistinguishable case. The purpose is to start with the pseudo-trajectory equation on the correlation functions, and to identify the cumulants in an expansion similar to the one characterizing them (see the inversion formula in Proposition 4.2.1).

5.1 Pseudo-trajectory measure

We start from the pseudo-trajectory formulation (3.11); recall that a pseudo-trajectory is fully determined by its final configuration $\underline{z}_n = \underline{z}_n^{[t]}$, and the following parameters: number of collisions, a collision history, collision times, collision angles and collision velocities, summed up in the parametrizing vector

$$\Psi_n \doteq (k,\chi_k,\underline{t}_k,\underline{\omega}_k,\underline{v}_k^*) \in \coprod_{k\geqslant 0}\{k\}\times\mathcal{H}_{n,k}\times T_k(t)\times\left(\mathbb{S}^{d-1}\times\mathbb{R}^d\right)^k,$$

referring to the definitions of the collision times (3.8) and collision history (3.9)

$$\chi_k \doteq (\underline{m}_k,\underline{\ell}_k^*,\underline{s}_k)\in\mathcal{H}_{n,k}\doteq\mathcal{M}_{n,k}\times\Lambda_k\times\{\pm 1\}^k.$$

Hence, denoting $v_{n+i}=v_i^*$ the velocities of added particles, and introducing the measure

$$\mathrm{d}\nu_{[t]}(\Psi_n) \doteq p_\mu^{|\underline{\ell}_k^*|}\mathrm{d}\underline{t}_k\mathrm{d}\underline{\omega}_k\mathrm{d}\underline{v}_k^*\prod_{i=1}^k s_i\langle\omega_i, v_{n+i}-v_{m_i}^{[t_i^+]}\rangle_+, \tag{5.1}$$

the pseudo-trajectory formulation rewrites as

$$F_n^\varepsilon(t) = \int F^\varepsilon\left(0, \underline{z}_{\Psi_n}^{[0]}\right) \, \mathrm{d}\nu_{[t]}(\Psi_n), \tag{5.2}$$

where $\underline{z}_{\Psi_n}^{[0]}$ denotes the initial configuration deduced from the pseudo-trajectory, according to the construction given in Section 3.2 (the initial configuration $\underline{z}_{\Psi_n}^{[0]}$ including also the *tags* of the corresponding particles). Note finally that the sums over k and χ_k in the pseudo-trajectory formulation result from the domain of integration of the measure $\nu_{[t]}$, and so remains implicit.

The formula giving an expression for the expectation of the empirical measure according to the correlation functions (2.10), may be generalized [9, Proposition 3.3.1] to observables $H_n(\underline{z}_n^{[0,t]})$ depending on the whole trajectory on $[0,t]$, as

$$\mathbb{E}\left[\sum_{1\leq i_k\neq i_j\leq\mathcal{N}} H_n(Z_{i_1}^{[0,t]}, L_{i_1}, \dots, Z_{i_n}^{[0,t]}, L_{i_n})\right] = \mu^n \sum_{\underline{\ell}_n\in\Lambda_n} p_\mu^{|\underline{\ell}_n|} \int_{\mathcal{D}_n^\varepsilon} \mathrm{d}\underline{z}_n \int F^\varepsilon(0, \underline{z}_{\Psi_n}^{[0]}) H_n(\underline{z}_{\Psi_n}^{[0,t]}) \mathrm{d}\nu_{[t]}(\Psi_n),$$

where $H_n(\underline{z}_{\Psi_n}^{[0,t]})$ is projecting the whole trajectory on the trajectories of the n first particles. The object that we will study, for tensorized observables, is hence the *H-weighted correlation function*

$$F_n^\varepsilon[H](t, \underline{z}_n, \underline{\ell}_n) \doteq \int F^\varepsilon(0, \underline{z}_{\Psi_n}^{[0]}) H^{\otimes n}(\underline{z}_{\Psi_n}^{[0,t]}) \, \mathrm{d}\nu_{[t]}(\Psi_n). \tag{5.3}$$

Note that for $H \equiv 1$, we retrieve the correlation functions. Moreover, the cumulant generating function (4.15) also generalizes to obervables depending on the whole trajectory $H_n(\underline{z}_n^{[0,t]})$, as follows

$$\mathfrak{G}_\varepsilon^{[0,t]}[H] \doteq \log \mathbb{E}\left[\exp\left(\sum_{i=1}^{\mathcal{N}} H(Z_i^{[0,t]}, L_i)\right)\right] = \sum_{p\geqslant 1} \frac{1}{p!} \sum_{\underline{\ell}_p\in\Lambda_p} \lambda^{|\underline{\ell}_p|} \mu^{p-|\underline{\ell}_p|} \int_{\mathcal{D}^p} f_p^\varepsilon \left[e^H - 1\right](t, \underline{\ell}_p). \tag{5.4}$$

Indeed, the computation of Section 4.2.2 are identical for the expansion of this generalized version of the cumulant generating function. The following section gives a formula for the H-weighted cumulants $f_p^\varepsilon[H]$.

5.2 Expansion and equation on the cumulants

Starting from the integral formulation (5.2) for the pseudo-trajectory formula, we will perform an expansion based on the interactions between the particles, to make appear an expansion in cumulants (4.11), eventually identifying the cumulants by injectivity, due to the inversion formula (Proposition 4.2.1). This expansion is a first illustration of the link between cumulants and dynamical interactions.

We start by splitting the pseudo-trajectories into *non-interacting aggregates*, to factorize the pseudo-trajectory measure. At a fixed pseudo-trajectory $\Psi_n = (k, \chi_k, \underline{t}_k, \underline{\omega}_k, \underline{v}_k)$, for a set of particles $A \subset [\![1, n]\!]$, we will denote

$$P(A) \subset A \cup [\![n+1, n+k]\!] \tag{5.5}$$

the set of all particles contained in the pseudo-trajectory stemming from A (depending on the history χ_k). Then, if $\mathbb{1}^{[0,t]}_{p\not\sim q}$ is the indicator that particles p and q never collided on $[0,t]$ during the pseudo-trajectory $(\underline{z}_n, \Psi_n)$, we denote for A, A' two subsets of $[\![1,n]\!]$,

$$\mathbb{1}_{A\not\sim A'} \doteq \prod_{p\in P(A)} \prod_{q\in P(A')} \mathbb{1}^{[0,t]}_{p\not\sim q}$$

the indicator that no particle stemming from A encountered any particle stemming from A', during the whole pseudo-dynamics on $[0,t]$. Under the condition $A \not\sim A'$, the pseudo-trajectories are independent and the measure and observable factorize as

$$\mathrm{d}\nu_{[t]}(\Psi_{A\cup A'}) H^{\otimes A\cup A'}(\underline{z}^{[0,t]}_{\Psi_{A\cup A'}}) = \mathrm{d}\nu_{[t]}(\Psi_A) H^{\otimes A}(\underline{z}^{[0,t]}_{\Psi_A}) \times \mathrm{d}\nu_{[t]}(\Psi_{A'}) H^{\otimes A'}(\underline{z}^{[0,t]}_{\Psi_{A'}}).$$

We keep this factorized writing for the moment, to remember that the pseudo-trajectories are defined and constructed independently on each aggregate. In practice, they behave as if the particles from two different aggregates could not interact: they overlap one another when they meet.

On the other hand, we denote the aggregating condition $\mathrm{agg}(A)$ the indicator that all the particles in the aggregate A are connected through collisions in their pseudo-trajectories, i.e. there exists a path $(i_1, \dots, i_{|A|}) \in [\![1, |A|]\!]^{|A|}$ such that

$$\{i_1, \dots, i_{|A|}\} = [\![1, |A|]\!] \quad \text{and} \quad \forall j \in [\![1,\ |A|-1]\!],\ \mathbb{1}^{[0,t]}_{A_{i_j} \sim A_{i_{j+1}}} = 1. \tag{5.6}$$

This eventually leads to the following conditioning according to the partition $\kappa \in \mathcal{P}_n$ into aggregates:

$$F^{\varepsilon}_n[H](t) = \sum_{\kappa\in\mathcal{P}_n} \int \left[\prod_{i=1}^{|\kappa|} \mathrm{agg}(\kappa_i)\ \mathrm{d}\nu_{[t]}(\Psi_{\kappa_i}) H^{\otimes\kappa_i}(\underline{z}^{[0,t]}_{\Psi_{\kappa_i}})\right] \left(\prod_{1\leqslant i<j\leqslant|\kappa|} \mathbb{1}_{\kappa_i\not\sim\kappa_j}\right) F^{\varepsilon}(0, \underline{z}^{[0]}_{\Psi_n}).$$

Henceforth, we denote the measure weighted by the observable

$$\mathrm{d}\nu^{[H]}_{[0,t]}(\Psi_n) \doteq H^{\otimes n}(\underline{z}^{[0,t]}_{\Psi_n}) \mathrm{d}\nu_{[t]}(\Psi_n). \tag{5.7}$$

Now, still in the idea of factorizing, we want to decorrelate the condition of exclusion between the aggregates $\mathbb{1}_{\kappa_i\not\sim\kappa_j}$, using the cumulants of the exclusion (Definition 4.11, studied specifically in Appendix B.2) thanks to which we write

$$\prod_{1\leqslant i<j\leqslant|\kappa|} \mathbb{1}_{\kappa_i\not\sim\kappa_j} = \sum_{\rho\in\mathcal{P}_{|\kappa|}} \phi_{[\rho]}(\kappa_1, \dots, \kappa_{|\kappa|})$$

recalling the notation

$$\phi_{[\rho]}(\kappa_1, \dots, \kappa_{|\kappa|}) = \prod_{i=1}^{|\rho|} \phi_{|\rho_i|}(\underline{\kappa}_{\rho_i}).$$

Finally, we still need to expand the initial distribution $F^{\varepsilon}(0, \underline{z}^{[0]}_{\Psi_n})$ to completely factorize the formula, to make appear a cumulant expansion. We need to expand it in a coarser way than the partition ρ, to be compatible with the product $\phi_{[\rho]}$ above, so that we consider the *cluster cumulants*, defined by the expansion

$$F^{\varepsilon}(0, \underline{z}^{[0]}_{\Psi_{\rho_1}}, \dots, \underline{z}^{[0]}_{\Psi_{\rho_{|\rho|}}}) = \sum_{\sigma\in\mathcal{P}_{|\rho|}} f^{\varepsilon,\rho}_{[\sigma]}(0, \underline{z}^{[0]}_{\Psi_{\rho_1}}, \dots, \underline{z}^{[0]}_{\Psi_{\rho_{|\rho|}}}) \tag{5.8}$$

due to the inversion formula (once again Proposition 4.2.1). This expansion is made by clusters, and so depends on the partition of the aggregates $\rho \in \mathcal{P}_{|\kappa|}$. The notation $\sigma \lhd \rho \lhd \kappa$ means that the partition σ is coarser than ρ, which is coarser then κ, and in the end we write

$$F_n^\varepsilon[H](t) = \sum_{\kappa \in \mathcal{P}_n} \int \left[\prod_{i=1}^{|\kappa|} \mathrm{agg}(\kappa_i) \, \mathrm{d}\nu_{[0,t]}^{[H]}(\Psi_{\kappa_i}) \right] \sum_{\sigma \lhd \rho \lhd \kappa} \phi_{[\rho]} f_{[\sigma]}^{\varepsilon,\rho}(0).$$

To identify with a cumulant expansion, the last step is to use the same trick as in the proof of the inversion formula to invert the partitions (Proposition 4.2.1), starting by considering a partition σ of $[\![1,n]\!]$, then considering a partition ρ *on each subset* σ_j. This is made possible by to the compatibility condition

$$f_{\sigma_j}^{\varepsilon,\rho} = f_{\sigma_j}^{\varepsilon,\rho|_{\sigma_j}},$$

and thanks to the factorizing identity

$$\phi_{[\rho]} f_{[\sigma]}^{\varepsilon,\rho} = \prod_{i=1}^{|\rho|} \phi_{\rho_i} \prod_{j=1}^{|\sigma|} f_{\sigma_j}^{\varepsilon,\rho} = \prod_{j=1}^{|\sigma|} \left[f_{\sigma_j}^{\varepsilon,\rho} \prod_{i \in \sigma_j} \phi_{\rho_i} \right].$$

Since the initial cluster cumulants $f_n^{\varepsilon,\rho}$ do not depend on the subdivision κ, we can also invert κ and ρ to finally get the cumulant expansion

$$F_n^\varepsilon[H](t) = \sum_{\sigma \in \mathcal{P}_n} \prod_{j=1}^{|\sigma|} \int \sum_{\rho \in \mathcal{P}_{\sigma_j}} f_{\sigma_j}^{\varepsilon,\rho}(0) \prod_{j=1}^{|\rho|} \left[\sum_{\kappa \in \mathcal{P}_{\rho_j}} \phi_{\rho_j} \prod_{i=1}^{|\kappa|} \mathrm{agg}(\kappa_i) \, \mathrm{d}\nu_{[0,t]}^{[H]}(\Psi_{\kappa_i}) \right].$$

Lemma 5.2.1. *The cumulants satisfy the following integral equation, in terms of aggregates and clusters of interaction*

$$f_n^\varepsilon[H](t, \underline{z}_n, \underline{\ell}_n) = \sum_{\rho \in \mathcal{P}_n} \int f_n^{\varepsilon,\rho}(0, \underline{z}_{\Psi_{\rho_1}}^{[0]}, \dots, \underline{z}_{\Psi_{\rho_{|\rho|}}}^{[0]}) \prod_{j=1}^{|\rho|} \left[\sum_{\kappa \in \mathcal{P}_{\rho_j}} \phi_{\rho_j} \prod_{i=1}^{|\kappa|} \mathrm{agg}(\kappa_i) \, \mathrm{d}\nu_{[0,t]}^{[H]}(\Psi_{\kappa_i}) \right]. \tag{5.9}$$

Note that the indicators $\mathrm{agg}(\kappa_i)$ and the cumulants $\phi_{[\rho]}$ depend on the whole pseudo-trajectory, and hence depend on the time t.

5.3 Initial cluster cumulants

This section is dedicated to prove Lemma 5.3.1 below, yielding an explicit formulation of the initial cluster cumulants implicitly defined in (5.8), and useful to find bounds in Section 6.2. We denote

$$\aleph_{|\rho|} \doteq \{\underline{z}_{\Psi_{\rho_i}}^{[0]} \ , \ 1 \leq i \leq |\rho|\} \quad \text{and} \quad \Gamma_p \doteq \{z_i^* \ , \ 1 \leq i \leq p\}$$

the set of clusters and the set of integrated particles, $N \doteq |\Psi_{\rho_1}| + \cdots + |\Psi_{\rho_{|\rho|}}|$ the total number of particles contained in the clusters, and

$$\left[\mathbb{1}_{\mathcal{X}_{(\cdot)}}\right]^{\otimes \aleph_{|\rho|}} \doteq \prod_{\underline{z} \in \aleph_{|\rho|}} \mathbb{1}_{\mathcal{X}_{|\underline{z}|}}(\underline{z})$$

the indicator that *within* each cluster, the particles exclude themselves. Finally, we denote as before $\mathbb{1}_{x \sim y} = 1 - \mathbb{1}_{x \not\sim y}$ the exclusion condition between two subsets of particles x and y (potentially clusters), and for $S \subset \aleph_{|\rho|}$, $k \leq p$ and $l \leq q$, we introduce the *cumulants of the cluster exclusion*

$$\phi_{S,k+l} \doteq \sum_{G \in \mathcal{C}_{S \cup [\![k+l]\!]}} \prod_{\{x,y\} \in E_G} (-\mathbb{1}_{x \sim y}),$$

where $\mathcal{C}_A$ stands for the connected graphs defined on a set A.

Lemma 5.3.1 (Explicit formula for the initial cluster cumulants)**.** *For any $\lambda, \mu > 0$, with the notation above, the initial cluster cumulants can be written as*

$$f_n^{\varepsilon,\rho}(0, \underline{z}_{\Psi_{\rho_1}}, \dots, \underline{z}_{\Psi_{\rho_{|\rho|}}}) = \Big[\mathbb{1}_{\mathcal{X}_{(\cdot)}}\Big]^{\otimes \aleph_{|\rho|}} M_\beta^{\otimes N} \varphi_0^{\otimes \ell_N} \sum_{p,q\geqslant 0} \frac{\lambda^p \mu^q}{p!q!} \int \varphi_0^{\otimes p} M_\beta^{\otimes p+q} \phi_{\aleph_{|\rho|},p+q}.$$

Proof. To obtain an explicit formulation for the initial cluster cumulants, we will identify them using the following characterization

$$F^\varepsilon(0, \underline{z}^{[0]}_{\Psi_{\rho_1}}, \dots, \underline{z}^{[0]}_{\Psi_{\rho_{|\rho|}}}) = \sum_{\sigma\in\mathcal{P}_{|\rho|}} f^{\varepsilon,\rho}_{[\sigma]}(\underline{z}^{[0]}_{\Psi_{\rho_1}}, \dots, \underline{z}^{[0]}_{\Psi_{\rho_{|\rho|}}}).$$

Using the exchangeability of particles, we have

$$\begin{aligned} F^\varepsilon(0, \underline{z}^{[0]}_{\Psi_{\rho_1}}, \dots, \underline{z}^{[0]}_{\Psi_{\rho_{|\rho|}}}) &= \frac{1}{\mathcal{Z}_\mu} \sum_{p\geqslant 0} \sum_{\underline{\ell}^*_p\in\Lambda_p} \frac{\lambda^{|\underline{\ell}^*_p|}\mu^{p-|\underline{\ell}^*_p|}}{p!} \int \varphi_0^{\otimes \tilde{\underline{\ell}}_{N+p}} M_\beta^{\otimes N+p} \mathbb{1}_{\mathcal{X}^\varepsilon_{N+p}} \mathrm{d}\underline{z}^*_p \\ &= \frac{1}{\mathcal{Z}_\mu} \sum_{p\geqslant 0} \sum_{q=1}^{p} \frac{\lambda^q \mu^{p-q}}{q!(p-q)!} \int \varphi_0^{\otimes \underline{\ell}_N} \varphi_0^{\otimes q} M_\beta^{\otimes N+p} \mathbb{1}_{\mathcal{X}^\varepsilon_{N+p}} \mathrm{d}\underline{z}^*_p \end{aligned}$$

To retrieve a cumulant formulation like (5.3) above, we will expand into cumulants the exclusion condition—which is the only obstruction to independence. To get rid of the partition function $\mathcal{Z}_\mu$ in the process, we will not expand the condition according to the integrating variables $\underline{z}^*_p$, to make appear the partition function at the numerator. More precisely, decomposing the exclusion condition within each cluster, we write

$$\mathbb{1}_{\mathcal{X}_{N+p}}(\underline{z}^{[0]}_{\Psi_{\rho_1}}, \dots, \underline{z}^{[0]}_{\Psi_{\rho_{|\rho|}}}, \underline{z}^*_p) = \Big[\mathbb{1}_{\mathcal{X}_{(\cdot)}}\Big]^{\otimes \aleph_{|\rho|}} \times \mathbb{1}_{\hat{\mathcal{X}}_{|\rho|+p}}(\underline{z}^{[0]}_{\Psi_{\rho_1}}, \dots, \underline{z}^{[0]}_{\Psi_{\rho_{|\rho|}}}, \underline{z}^*_p),$$

with

$$\begin{aligned} \mathbb{1}_{\hat{\mathcal{X}}_{|\rho|+p}}(\underline{z}^{[0]}_{\Psi_{\rho_1}}, \dots, \underline{z}^{[0]}_{\Psi_{\rho_{|\rho|}}}, \underline{z}^*_p) &= \prod_{x,y\in\aleph_{|\rho|}} \mathbb{1}_{x\not\sim y} \prod_{x\in\aleph_{|\rho|}, y\in\Gamma_p} \mathbb{1}_{x\not\sim y} \prod_{x,y\in\Gamma_p} \mathbb{1}_{x\not\sim y} \\ &= \sum_{G\in\mathcal{G}_{(\aleph_{|\rho|}\cup\Gamma_p)}} \prod_{\{x,y\}\in E_G} (-\mathbb{1}_{x\sim y}). \end{aligned}$$

We want now to split the sum above according to the partition induced by the connected components of the graph G, as for the standard cumulants of the exclusion (B.2). Nevertheless, since we want to make appear the partition function, we will compute a coarser splitting. Indeed, we will merely isolate the part of the graph which is not connected to any cluster of $\aleph_{|\rho|}$, i.e. the subset $B \subset \Gamma_p$ of integrated particles that are only connected between themselves. We still decompose the rest of the graph into its connected components, yielding a partition of $\aleph_{|\rho|} \cup B^c$ in which each subset has to contain a cluster (otherwise it would have been kept in B), which we denote $\sigma \in \tilde{\mathcal{P}}[\aleph_{|\rho|}, B^c]$. Hence, one can write

$$\begin{aligned} \mathbb{1}_{\hat{\mathcal{X}}_{|\rho|+p}}(\underline{z}^{[0]}_{\Psi_{\rho_1}}, \dots, \underline{z}^{[0]}_{\Psi_{\rho_{|\rho|}}}, \underline{z}^*_p) &= \sum_{B\subset\Gamma_p} \sum_{G\in\mathcal{G}_B} \prod_{\{x,y\}\in E_G} (-\mathbb{1}_{x\sim y}) \sum_{\sigma\in\tilde{\mathcal{P}}[\aleph_{|\rho|},B^c]} \prod_{i=1}^{|\sigma|} \sum_{G\in\mathcal{C}_{\sigma_i}} \prod_{\{x,y\}\in E_G} (-\mathbb{1}_{x\sim y}) \\ &= \sum_{B\subset\Gamma_p} \mathbb{1}_{\mathcal{X}_{|B|}}(\underline{z}^*_B) \sum_{\sigma\in\tilde{\mathcal{P}}[\aleph_{|\rho|},B^c]} \prod_{i=1}^{|\sigma|} \phi_{\sigma_i}, \end{aligned}$$

gathering the first sum over $\mathcal{G}_B$ by inverting the usual expansion, and recognizing cumulants of the exclusion in their form (B.2) (see Appendix B.2). In the end, a partition $\sigma \in \tilde{\mathcal{P}}[\aleph_{|\rho|}, B^c]$ is a partition of $\aleph_{|\rho|}$ on which some particles from B^c are added, either tagged by $\underline{\ell}_p^*$ or not. We will use the symmetry of particles (like in the study of the partition function, Appendix C) to reduce these added particles to their numbers of 1-tags $k_1, \dots, k_{|\sigma|} \geq 0$ and of 0-tags $l_1, \dots, l_{|\sigma|} \geq 0$, and similarly for the particles in B, in numbers $k_0, l_0 \geq 0$. Omitting the indicator $\left[\mathbb{1}_{\mathcal{X}_{(\cdot)}}\right]^{\otimes \aleph_{|\rho|}}$ to gain lisibility (it factorizes and distributes very easily), we thus write

$$
\begin{aligned}
&F^\varepsilon\left[0, \underline{z}^{[0]}_{\Psi_{\rho_1}}, \dots, \underline{z}^{[0]}_{\Psi_{\rho_{|\rho|}}}\right] \\
&= \frac{\varphi_0^{\otimes \underline{\ell}_N}}{\mathcal{Z}_\mu} \sum_{p \geqslant 0} \sum_{q=1}^{p} \frac{\lambda^q \mu^{p-q}}{q!(p-q)!} \sum_{\sigma \in \mathcal{P}_{\aleph_{|\rho|}}} \sum_{\substack{k_0+\dots+k_{|\sigma|}=q \\ l_0+\dots+l_{|\sigma|}=p-q}} \frac{q!}{k_0!\dots k_{|\sigma|}!} \frac{(p-q)!}{l_0!\dots l_{|\sigma|}!} \int \varphi_0^{\otimes q} M_\beta^{\otimes N+p} \mathbb{1}_{\mathcal{X}^\varepsilon_{k_0+l_0}} \prod_{i=1}^{|\sigma|} \phi_{\sigma_i, k_i+l_i} \\
&= \frac{\varphi_0^{\otimes \underline{\ell}_N} M_\beta^{\otimes N}}{\mathcal{Z}_\mu} \sum_{k_0, l_0 \geqslant 0} \frac{\lambda^{k_0} \mu^{l_0}}{k_0! l_0!} \int \varphi_0^{\otimes k_0} M_\beta^{\otimes l_0+k_0} \mathbb{1}_{\mathcal{X}^\varepsilon_{k_0+l_0}} \times \sum_{\sigma \in \mathcal{P}_{\aleph_{|\rho|}}} \prod_{i=1}^{|\sigma|} \sum_{k_i, l_i \geqslant 0} \frac{\lambda^{k_i} \mu^{l_i}}{k_i! l_i!} \int \varphi_0^{\otimes k_i} M_\beta^{\otimes l_i+k_i} \phi_{\sigma_i, k_i+l_i} \\
&= \sum_{\sigma \in \mathcal{P}_{\aleph_{|\rho|}}} \prod_{i=1}^{|\sigma|} M_\beta^{\otimes \sigma_i} \varphi_0^{\otimes \underline{\ell}_{\sigma_i}} \sum_{k_i, l_i \geqslant 0} \frac{\lambda^{k_i} \mu^{l_i}}{k_i! l_i!} \int \mathrm{d}\underline{z}^*_{l_i+k_i} \varphi_0^{\otimes k_i} M_\beta^{\otimes l_i+k_i} \phi_{\sigma_i, k_i+l_i},
\end{aligned}
$$

recognizing the partition function $\mathcal{Z}$ at the numerator. The lemma is thus proved by identification once added back the indicator $\mathbb{1}_{\mathcal{X}_{(\cdot)}}^{\otimes \aleph_{|\rho|}}$. □

5.4 Dynamics trees

Starting from the formulation (5.9) of the cumulants in terms of pseudo-trajectories, we will rewrite this integral to make appear the *correlations*' history between the particles. Indeed, the cumulants encode the small defects of particles' independence, which correspond to fortuitous encounters happening between the pseudo-trajectories of the n considered particles. In fact, we prove that asymptotically, the n-th cumulant f_n is supported over trajectories implying exactly $n-1$ of these fortuitous encounters, thus connecting all the pseudo-trajectories. This way, each of these interactions has to be clustering, to eventually form a minimally connected graph.

We will control these fortuitous interactions between pseudo-trajectories in the integrated form

$$
\sum_{\rho \in \mathcal{P}_n} \int \mathrm{d}\underline{z}_n \int f_n^{\varepsilon,\rho}(0, \underline{z}^{[0]}_{\Psi_{\rho_1}}, \dots, \underline{z}^{[0]}_{\Psi_{\rho_{|\rho|}}}) \prod_{j=1}^{|\rho|} \left[\sum_{\kappa \in \mathcal{P}_{\rho_j}} \phi_{\rho_j} \prod_{i=1}^{|\kappa|} \mathrm{agg}(\kappa_i)\, \mathrm{d}\nu^{[H]}_{[0,t]}(\Psi_{\kappa_i})\right], \tag{5.10}
$$

which is the integral over $\underline{z}_n$ of $f_n^\varepsilon[H](t, \underline{z}_n, \underline{\ell}_n)$ in its form (5.9). The integral in positions allow to act on the arrival positions $\underline{x}_n$ of the trajectories to make the fortuitous interactions appear or disappear. In the end, we will perform a change of these variables $\underline{x}_n$ that harnesses these interactions.

To that end, these fortuitous encounters will be recorded in *dynamics trees*, allowing us to express the conditions of their existence. They stem from two different sources: first, *recollisions* appear within the aggregates (κ_i) to make them connected; and on the other hand, on each cluster ρ_j of aggregates, the extended cumulants of their mutual exclusion ϕ_{ρ_j} make appear *overlapping* conditions between the aggregates, recalling that when two different aggregates meet, they overlap one another without interacting. This formulation in dynamics trees will be the first step to prove bounds on the cumulants, and to compute their limit (Sections 6.2 and 7.3).

Clustering recollisions As explained above, within an aggregate $\kappa_i \subset [\![1,n]\!]$, denoting $A \doteq \kappa_i$ and $a \doteq |A|$, each pseudo-trajectory Ψ_A contains recollisions (5.6) that connect the aggregate altogether, according to the condition $\mathrm{agg}(A)$.

Definition 5.1 (Clustering recollisions)**.** In a given pseudo-trajectory, looked at backwards in time starting from time t, we call *clustering recollision* the first recollision connecting the pseudo-trajectories stemming from two different particles. In an aggregate A, we order these clustering recollisions backwards in time from 1 to $a-1$, which define a *recollision tree* $T^{\mathrm{c}}(\underline{z}^{[0,t]}_{\Psi_A}) \in \mathcal{T}^{\prec}_A$ whose ordered edges record these recollisions: two particles $m, m' \in A \subset [\![1,n]\!]$ are connected in this tree if and only if a clustering recollision happens between the pseudo-trajectories stemming from m and m'. Eventually, we define

$$(\tau^{\mathrm{c}}_i, \omega^{\mathrm{c}}_i)_{1\leqslant i\leqslant a-1} \in ([0,t]\times \mathbb{S}^{d-1})^{a-1}$$

the ordered collision times and angles at which the recollision appear.

For simplicity, we denote $T^{\mathrm{c}}_A \doteq T^{\mathrm{c}}(\underline{z}^{[0,t]}_{\Psi_A})$. We now partition according to the associated tree

$$\mathrm{agg}(A) = \sum_{\Upsilon\in\mathcal{T}^{\prec}_A} \mathbb{1}_{T^{\mathrm{c}}_A=\Upsilon}.$$

Under the condition $T^{\mathrm{c}}_A = \Upsilon$, we will perform a change of variables that harnesses each of the clustering recollision conditions. To do so, we need to know *which particles*, from the pseudo-trajectories stemming from $m, m' \in A$, collided to connect them. Recall that $P(m)$ denotes the set of all particles added in the pseudo-trajectory stemming from m. Then, for every edge $e = (m, m') \in E_\Upsilon$, we define $\mathrm{col}_e(A)$ the pair of particles $(p_e, q_e) \in P(m) \times P(m')$ that collided on $(0,t)$ to connect the pseudo-trajectories of m and m', when creating the edge e. Eventually, we also partition according to which particle collided with which for each clustering recollision:

$$\mathrm{agg}(A) = \sum_{\Upsilon\in\mathcal{T}^{\prec}_A} \mathbb{1}_{T^{\mathrm{c}}_A=\Upsilon} \sum_{\substack{p_e\in P(m), q_e\in P(m')\\ e=(m,m')\in\mathbb{E}_\Upsilon}} \prod_{e\in E_\Upsilon} \mathbb{1}_{(p_e,q_e)=\mathrm{col}_e(A)}.$$

We introduce the following notation for the indicator that the clustering particles are those we want (the Fraktur $\mathfrak{S}$ is chosen after the historical German word for collisions *Stoß*, at the origin of the concept of molecular chaos *Stoßzahlansatz* [15])

$$\mathfrak{S}_{(p_e,q_e)_\Upsilon}[A] \doteq \prod_{e\in E_\Upsilon} \mathbb{1}_{(p_e,q_e)=\mathrm{col}_e(A)}.$$

We can now harness the successive clustering conditions, which correspond to the ordered edges of the tree T^{c}_A, enumerated $(e_1, \dots, e_{a-1}) = \big((m_1, m'_1), \dots, (m_{a-1}, m'_{a-1})\big)$ backwards in time. The relevant variables that we want to make appear are the times and angles of the collisions, when the current integration variables are the arrival positions $\underline{x}_A$ at time t.

As we are interested in the relative positions of the particles implied in the clustering recollisions, we start with the following change of variable of Jacobian 1, denoting $\overline{x}_A$ the barycenter of the positions $\underline{x}_A$,

$$\underline{x}_A \mapsto \Big(\overline{x}_A, (\vec{x}_e \doteq x_{p_e} - x_{q_e})_{e\in E_\Upsilon}\Big),$$

where we take the convention $p_e > q_e$, to be consistent in the following with the measure ν. Thus, once recursively determined the clustering conditions associated to all the edges preceding an edge e such that $\mathrm{col}_e(A) = (p_e, q_e)$, the condition associated with this edge states that at time τ^{c}_e, the relative position $x_{p_e}(\tau^{\mathrm{c}}_e) - x_{q_e}(\tau^{\mathrm{c}}_e)$ must belong to the sphere of radius ε, defining the recollision angle $\omega_e^{\rm c} \in \mathbb{S}^{d-1}$. Hence, denoting $\tau_{\rm d}$ the time of the closest deflection that underwent p_e or q_e between t and $\tau_e^{\rm c}$, and $\Delta x_{\rm d}$ the relative distance travelled between t and $\tau_{\rm d}$, the relative positions at time t might be retrieved from

$$x_{p_e} - x_{q_e} = \varepsilon\omega_e^{\rm c} - (\tau_e^{\rm c} - \tau_{\rm d})(v_{p_e}(\tau_e^{\rm c+}) - v_{q_e}(\tau_e^{\rm c+})) - \Delta x_{\rm d}. \tag{5.11}$$

The parameters $(\tau_e^{\rm c}, \omega_e^{\rm c})$ are unique because we consider the very first time (backwards) at which p_e and q_e meet, so that the mapping (5.11) above is invertible: we consider the local change of variable

$$x_{p_e} - x_{q_e} = \vec{x}_e \mapsto (\tau_e^{\rm c}, \omega_e^{\rm c}) \in [0, \tau_{e'}^{\rm c}] \times \mathbb{S}^{d-1},$$

where e' is the edge ordered right before e. Since $\Delta x_{\rm d}$ is a piecewise affine function recording the part of the dynamics that does not depend on $\tau_e^{\rm c}$ (but depends on the previous recursively determined clustering conditions, the dependency in $\tau_e^{\rm c}$ being explicit in (5.11)), the Jacobian of this change of variables is

$$\varepsilon^{d-1}\big\langle v_{p_e}(\tau_e^{\rm c+}) - v_{q_e}(\tau_e^{\rm c+}), \omega_e^{\rm c}\big\rangle_+ = \mu^{-1}\big\langle v_{p_e}(\tau_e^{\rm c+}) - v_{q_e}(\tau_e^{\rm c+}), \omega_e^{\rm c}\big\rangle_+. \tag{5.12}$$

Iterating this computation, we change the variables $(\vec{x}_e)_{e\in E_\Upsilon}$ to the ordered times $\underline{\tau}_{a-1}^{\rm c}$ and angles $\underline{\omega}_{a-1}^{\rm c}$ of clustering recollisions (also indexed by the ordered edges of E_Υ), finally leading to

$$\begin{aligned}\int & {\rm d}\underline{x}_A \Big[\mathbb{1}_{T_A^{\rm c}=\Upsilon}\mathfrak{S}_{(p_e,q_e)_\Upsilon}[A]\Big]{\rm d}\nu_{[0,t]}^{[H]}(\Psi_A) \\ &= \int \frac{{\rm d}\overline{x}_A}{\mu^{a-1}}{\rm d}\underline{\omega}_{a-1}^{\rm c}{\rm d}\underline{\tau}_{a-1}^{\rm c}\Big[\mathbb{1}_{T_A^{\rm c}=\Upsilon}\mathfrak{S}_{(p_e,q_e)_\Upsilon}[A]\Big]{\rm d}\nu_{[0,t]}^{[H]}(\Psi_A)\prod_{e\in E_\Upsilon}\langle v_{p_e}(\tau_e^{\rm c+}) - v_{q_e}(\tau_e^{\rm c+}), \omega_e^{\rm c}\rangle_+.\end{aligned}$$

Clustering overlaps Now for a general cluster $R = \rho_j$, of size $r = |R|$, we expand the cumulant

$$\phi_R = \sum_{G\in\mathcal{C}_{|\kappa|}}\prod_{\{i,j\}\in E_G}(-\mathbb{1}_{\kappa_i\sim\kappa_j})$$

making appear *overlap conditions* $\mathbb{1}_{\kappa_i\sim\kappa_j}$. At fixed pseudo-trajectory parameters Ψ_R, changing the barycenter $\overline{x}_{\kappa_i}$ of an aggregate moves rigidly the whole aggregate pseudo-trajectory, since by construction the trajectories are defined independently on each aggregate. That way, we can act on the barycenters to make them match the overlaps given by a given graph $G \in \mathcal{C}_{|\kappa|}$. In the same fashion as for the recollision trees, we define the following objects.

Definition 5.2 (Clustering overlaps)**.** In a given pseudo-trajectory, we call *clustering overlap* the first overlap (backwards in time) appearing between independently defined aggregates, connecting them according to the overlap condition discussed above. As for recollisions, clustering overlaps define an *overlap tree* $T^{\rm ov}(\underline{z}_{\Psi_R^{[0,t]}}) \in \mathcal{T}_{|\kappa|}^{\prec}$, and for each edge $e = \{m, m'\}$ associated to an overlap, we define

$$\tau_e^{\rm ov} = \sup\Big\{0 \le s \le t\ ,\ \mathbb{1}_{\kappa_m\sim\kappa_{m'}}^{[s,t]} = 1\Big\}$$

its overlap time (arbitrarily taking the latest possible, to make them consistent with recollisions when constructing pseudo-trajectories backwards in time), along with the associated overlap angle $\omega_e^{\rm ov}$.

As for the recollision trees, we denote $T_R^{\rm ov} \doteq T^{\rm ov}(\underline{z}_{\Psi_R^{[0,t]}})$ the overlap tree. Now, we partition the expansion of the cumulant according to the associated tree, treating separately when the graph appearing in the sum is the overlap tree, and when it presents some additional cycles. Denoting $\mathcal{C}_{|\kappa|}(\Upsilon)$ the connected graphs containing Υ, we have

$$\begin{aligned}\phi_R = & \sum_{\Upsilon\in\mathcal{T}_{|\kappa|}^{\prec}}\mathbb{1}_{T_R^{\rm ov}=\Upsilon}\prod_{\{i,j\}\in E_\Upsilon}(-\mathbb{1}_{\kappa_i\sim\kappa_j}) + \sum_{\Upsilon\in\mathcal{T}_{|\kappa|}^{\prec}}\mathbb{1}_{T_R^{\rm ov}=\Upsilon}\sum_{G\in\mathcal{C}_{|\kappa|}(\Upsilon)\setminus\{\Upsilon\}}\prod_{\{i,j\}\in E_G}(-\mathbb{1}_{\kappa_i\sim\kappa_j}) \\ \doteq & \qquad\qquad \phi_R^{[\rm tree]} \qquad\qquad + \qquad\qquad \phi_R^{[\rm cycle]}.\end{aligned} \tag{5.13}$$

Focusing first on the tree part, one can write

$$\sum_{\kappa \in \mathcal{P}_R} \phi_R^{[\text{tree}]} \prod_{i=1}^{|\kappa|} \operatorname{agg}(\kappa_i) \mathrm{d}\nu_{[0,t]}^{[H]}(\Psi_{\kappa_i}) = (-1)^{|\kappa|-1} \sum_{\Upsilon \in \mathcal{T}_{|\kappa|}^{\prec}} \mathbb{1}_{T_R^{\text{ov}} = \Upsilon} \prod_{i=1}^{|\kappa|} \operatorname{agg}(\kappa_i) \mathrm{d}\nu_{[0,t]}^{[H]}(\Psi_{\kappa_i}),$$

the overlap conditions being included in the overlap tree matching condition. At fixed overlap tree $\Upsilon = T_R^{\text{ov}}$, we act now on the aggregate barycenters that appeared during the change of variable harnessing the recollisions. Recall that moving these barycenters moves rigidly the aggregate pseudo-trajectories. Like for the clustering recollisions, we hence focus on the relative positions of these barycenters through the change of variable, of Jacobian 1,

$$(\overline{x}_{\kappa_i})_{i \leqslant |\kappa|} \mapsto \Big(\overline{x}_R, (\vec{x}_e \doteq x_{p_e} - x_{q_e})_{e \in E_\Upsilon} \Big),$$

conditionning on the overlapping particles (p_e, q_e) associated to the edge e, as it was done for recollisions.

Clustering trees At this point, we observe that the knowledge of a recollision tree on each aggregate, and of an overlap tree between these aggregates, is equivalent to the knowledge of global decorated *clustering tree* $T_R^\star \in \mathcal{T}_r^{\prec,\star}$, recording all the clustering interactions between the r particles of a cluster R, and recording whether it is a recollision or an overlap thourgh a decoration on each edge. This leads to the mapping

$$\begin{array}{ccc} \mathcal{T}_{|\kappa|}^{\prec} \times \prod_{i=1}^{|\kappa|} \mathcal{T}_{|\kappa_i|}^{\prec} & \longrightarrow & \mathcal{T}_r^{\prec,\star} \\ T_R^{\text{ov}}, (T_{\kappa_i}^{\text{c}})_{i \leqslant |\kappa|} & \longmapsto & T_R^\star. \end{array}$$

The decorations are denoted $\underline{s}_{E_\Upsilon}$ by analogy with the scattering labels of the pseudo-trajectory histories (3.9): indeed, this time again, we choose that a sign $+1$ corresponds to a scattering (i.e. a recollision), and a sign (-1) to no scattering (i.e. an overlap). In the decorated clustering tree, the aggregates correspond to the decorated connected components retrieved when removing the overlap edges (we denote $\mathfrak{CC}^\star(\Upsilon)$ the set of these decorated connected components), so that the tree part of our cluster measure writes

$$\sum_{\kappa \in \mathcal{P}_R} \phi_R^{[\text{tree}]} \prod_{i=1}^{|\kappa|} \operatorname{agg}(\kappa_i) \mathrm{d}\nu_{[0,t]}^{[H]}(\Psi_{\kappa_i}) = \sum_{\Upsilon \in \mathcal{T}_R^{\prec,\star}} (-1)^{|\mathfrak{CC}^\star(\Upsilon)|-1} \int \mathbb{1}_{T_R^\star = \Upsilon} \prod_{A \in \mathfrak{CC}^\star(\Upsilon)} \mathrm{d}\nu_{[0,t]}^{[H]}(\Psi_A).$$

For each global tree $\Upsilon \in \mathcal{T}_R^{\prec,\star}$, we thus have $(r-1)$ encounter times and angles $(\tau_e^\star, \omega_e^\star)_{e \in E_\Upsilon}$, either associated to a recollision or to an overlap, according to the decorations $(s_e)_{e \in E_\Upsilon}$. Once computed the recollision changes of variable in each aggregate (i.e. decorated connected component of Υ), we compute the same on their barycenters to harness the overlaps, eventually leading to the variables $(\overline{x}_R, \underline{\tau}_{r-1}^\star, \underline{\omega}_{r-1}^\star)$. Since the overlap change of variable has the same Jacobian as the recollision one (5.12), we end up with

$$\int \mathrm{d}\underline{x}_R \sum_{\kappa \in \mathcal{P}_R} \phi_R^{[\text{tree}]} \prod_{i=1}^{|\kappa|} \operatorname{agg}(\kappa_i) \mathrm{d}\nu_{[0,t]}^{[H]}(\Psi_{\kappa_i}) = \sum_{\Upsilon \in \mathcal{T}_R^{\prec,\star}} \sum_{\substack{p_e \in P(m), q_e \in P(m') \\ e=(m,m') \in E_\Upsilon}} \int \frac{\mathrm{d}\overline{x}_R}{\mu^{r-1}} \mathrm{d}\underline{\omega}_{E_\Upsilon}^\star \mathrm{d}\underline{\tau}_{E_\Upsilon}^\star$$
$$\times \mathbb{1}_{T_R^\star = \Upsilon} \mathfrak{S}_{(p_e,q_e)_\Upsilon}[R] \prod_{e \in E_\Upsilon} s_e \langle v_{p_e}(\tau_e^{\star+}) - v_{q_e}(\tau_e^{\star+}), \omega_e^\star \rangle_+ \prod_{A \in \mathfrak{CC}^\star(\Upsilon)} \mathrm{d}\nu_{[0,t]}^{[H]}(\Psi_A).$$

Now, keeping in mind that the pseudo-trajectories do not interact between decorated connected components of the clustering tree, one may gather back the measures $\nu_{[0,t]}^{[H]}$ on the whole set R. This means that we consider a unique history

$$(k, \underline{m}_k, \underline{\ell}_k^*, \underline{s}_k, \underline{t}_k, \underline{\omega}_k, \underline{v}_k^*)$$

for all the dynamics, although the pseudo-trajectories are still independently defined on each aggregate. Moreover, we want to keep track of the time ordering between the particle adjunctions and the clustering encounters. To that end, we will add to each edge $e \in E_\Upsilon$ of the clustering tree the information of the time interval $[t_{j+1}, t_j]$ (between two particle adjunctions) in which the clustering encounter occur. This way, adding this information to the knowledge of which particles are involved in the clustering encounter, each edge $e = \{m, m'\}$ is associated with the triplet

$$(p_e, q_e, i_e) \in P(m) \times P(m') \times [\![0, k]\!],$$

such that $\tau_e \in [t_{i_e+1}, t_{i_e}]$. Keeping the notation $\mathfrak{S}_{(p_e,q_e,i_e)}$ for this triplet's compatibility, we write

$$\int d\underline{x}_R \sum_{\kappa\in\mathcal{P}_R} \phi_R^{[\text{tree}]} \prod_{i=1}^{|\kappa|} \text{agg}(\kappa_i) d\nu_{[0,t]}^{[H]}(\Psi_{\kappa_i}) = \sum_{\Upsilon\in\mathcal{T}_R^{\prec,\star}} \sum_{\substack{p_e,q_e,i_e \\ e\in E_\Upsilon}} \int \frac{d\overline{x}_R}{\mu^{r-1}} d\underline{\omega}_{E_\Upsilon}^\star d\underline{\tau}_{E_\Upsilon}^\star \mathbb{1}_{T_R^\star=\Upsilon} \mathfrak{S}_{(p_e,q_e,i_e)_\Upsilon}[R]$$
$$\times \prod_{e\in E_\Upsilon} s_e \langle v_{p_e}(\tau_e^{\star +}) - v_{q_e}(\tau_e^{\star +}), \omega_e^\star\rangle_+ \, d\nu_{[0,t]}^{[H]}(\Psi_R).$$

With all this in mind, we are now able to gather all the particles adjunctions and clustering encounters into a single tree encoding the whole dynamics. Indeed, at fixed number of added particles k, the choice of the history $(\underline{s}_k, \underline{m}_k)$ contained in the measure ν, along with the clustering information, may be encoded in a dynamics tree $T_R^{\text{d}} \in \mathcal{T}_{r+k}^{\prec,\star}$, in which two vertices are connected either if a clustering encounter happens between them, or if one of them was added to the other one, i.e. if they form a couple $(n + j, m_j)$, with the associated signed decoration. The edges are time ordered from the ordered clustering conditions in $\Upsilon = T_R^\star$, and the intervals $(i_e)_{e\in E_\Upsilon}$ defined above. This provides the following mapping

$$\begin{array}{ccccccccccc} \mathcal{T}_R^{\prec,\star} & \times & [\![1, r+k]\!]^{E_\Upsilon} & \times & [\![0, k]\!]^{E_\Upsilon} & \times & \{\pm 1\}^k & \times & \mathcal{M}_{R,k} & \to & \mathcal{T}_{R,k}^{\text{d}} \subset \mathcal{T}_{r+k}^{\prec,\star} \\ T_R^\star & , & (p_e, q_e)_e & , & (i_e)_e & , & \underline{s}_k & , & \underline{m}_k & \mapsto & T_R^{\text{d}} \end{array}$$

with the condition $p_e, q_e \in P(m) \times P(m')$ for every $e = \{m, m'\} \in E_\Upsilon$. This mapping is injective, and made bijective by restriction to its image $\mathcal{T}_{R,k}^{\text{d}} \subset \mathcal{T}_{r+k}^{\prec,\star}$, containing all the admissible dynamics trees. More precisely, an admissible dynamics tree is such that the added particles from $[\![r+1, r+k]\!]$ appear in increasing order, and in particular do not have encounters before appearing. For such a dynamics tree, the global dynamics might be recovered, starting from the highest ordered edge (the closest to time 0) and reconstructing the trajectories up to time t, as in the example given in Figure 4 (note that in this example the edge 0 is not admissible as it would connect the vertices 4 and 6 before they appear). From the dynamics, the parameters $(T_R^\star, (p_e, q_e, i_e)_e, \underline{s}_k, \underline{m}_k)$ can be retrieved, making the mapping bijective. Recalling the expression (5.1) of the measure, this lets us with encounter times $\underline{\tau}_{E_\Upsilon}$, angles $\underline{\omega}_{E_\Upsilon}$, stemming either from the adjunction of a particle in the pseudo-trajectory expansion, or from a clustering encounter (recollision or overlap). Generalizing the notation $\langle v_m(\tau_e{}^+) - v_{m'}(\tau_e{}^+), \omega_e\rangle_+$ to the adjunction of particles (even if in this case one of the particles does not really exist at time τ_e^+),

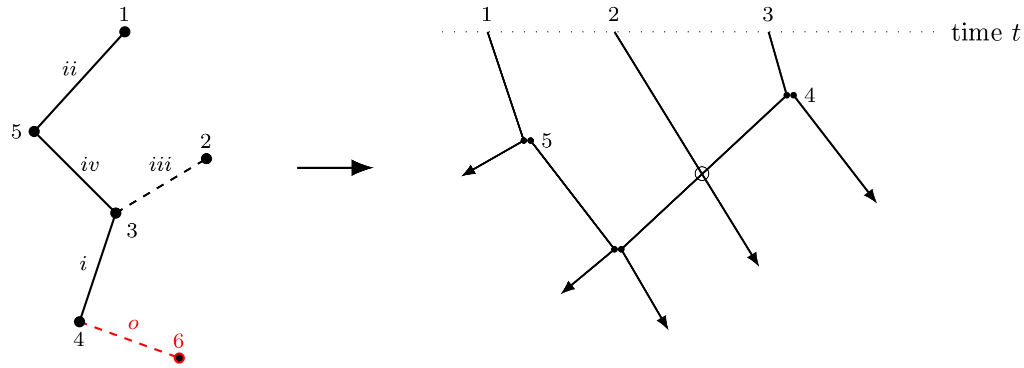

Figure 4: Retrieving the dynamics from the dynamics tree

one can write the condensed measure

$$\int \mathrm{d}\underline{x}_R \sum_{\kappa\in\mathcal{P}_R} \phi_R^{[\text{tree}]} \prod_{i=1}^{|\kappa|} \mathrm{agg}(\kappa_i)\mathrm{d}\nu_{[0,t]}^{[H]}(\Psi_{\kappa_i})$$

$$= \sum_{k\geqslant 0} \sum_{\underline{\ell}_k\in\Lambda_k} \sum_{\Upsilon\in\mathcal{T}^{\mathrm{d}}_{R,k}} \int \frac{p_\mu^{|\underline{\ell}_k|}}{\mu^{r-1}} \mathrm{d}\overline{x}_R \mathrm{d}\underline{v}_k^* H^{\otimes R} \mathbb{1}_{T_R^{\mathrm{d}}=\Upsilon} \prod_{e=\{m,m'\}\in E_\Upsilon} s_e\langle v_m({\tau_e}^+) - v_{m'}({\tau_e}^+), \omega_e\rangle_+ \mathrm{d}\omega_e \mathrm{d}\tau_e.$$

Although the order in the velocities' difference above do not change the calculus (as we integrate ω_e over the whole symmetric sphere), recall anyway that this order has been chosen such that $m > m'$. Summing everything up, we get the following proposition.

Proposition 5.4.1 (Dynamics tree formula for the integrated cumulants)**.** *The cumulants decompose into two terms*

$$f_n^\varepsilon[H](t) = f_n^\varepsilon[H]^{[\text{tree}]}(t) + f_n^\varepsilon[H]^{[\text{cycle}]}(t),$$

the first one containing only trees in its expansion, and the other including cycles. Indeed, the integral of the first tree term rewrites as the expansion

$$\int f_n^\varepsilon[H]^{[\text{tree}]}(\underline{z}_n)\mathrm{d}\underline{z}_n = \sum_{\rho\in\mathcal{P}_n} I_n^{[\rho]}[H](t,\underline{\ell}_n),$$

where

$$I_n^{[\rho]}[H](t,\underline{\ell}_n) \doteq \int f_n^{\varepsilon,\rho}(0, \underline{z}^{[0]}_{\Psi_{\rho_1}}, \dots, \underline{z}^{[0]}_{\Psi_{\rho_{|\rho|}}})\mathrm{d}\underline{v}_n \prod_{j=1}^{|\rho|} \frac{\mathrm{d}\overline{x}_{\rho_j}}{\mu^{|\rho_j|-1}} \sum_{k_j\geqslant 0} \sum_{\underline{\ell}_{k_j}} \sum_{\Upsilon\in\mathcal{T}^{\mathrm{d}}_{\rho_j,k_j}} p_\mu^{|\underline{\ell}_{k_j}|} H^{\otimes\rho_j} \mathbb{1}_{T^{\mathrm{d}}_{\rho_j}=\Upsilon}$$
$$\times\, \mathrm{d}\underline{v}_{k_j}^* \prod_{e\in E_\Upsilon} s_e\langle v_m({\tau_e}^+) - v_{m'}({\tau_e}^+), \omega_e\rangle_+ \mathrm{d}\omega_e \mathrm{d}\tau_e. \tag{5.14}$$

The term $f_n^\varepsilon[H]^{[\text{cycle}]}$ corresponds to a similar expansion, stemming from the part $\phi_{\rho_j}^{[\text{cycle}]}$ of the exclusion cumulants (5.13)*. Recall that the cluster cumulants f_n^ρ were defined in* (5.8)*.*

6 Integrability bounds

We present here integrability bounds for the cumulants, based on the writing above (Proposition 5.4.1). First we show how to reduce the study of the cycle part to the study of the tree part, making appear additional constraints on the way. Secondly, we bound this tree part in Proposition 6.2.1.

6.1 Discarding overlap cycles: a tree inequality

As announced, we start by controlling the part of the cumulant expansion containing cycles. A cycle in the overlap conditions imposes a strong geometric constraint, which will provide smallness (Appendix D). But first, we will compute a *tree inequality* to simplify the sum over all the connected graphs. To do this, we will harness the cancellations due to the signs, using the same trick as in the proof of Proposition B.2.1. Indeed, recall that the cycle part of the exlusion cumulants writes

$$\phi_R^{[\text{cycle}]} = \sum_{\Upsilon \in \mathcal{T}^{\prec}_{|\kappa|}} \mathbb{1}_{T^{\text{ov}}(R)=\Upsilon} \sum_{G \in \mathcal{C}_{|\kappa|}(\Upsilon)\setminus\{\Upsilon\}} \prod_{\{i,j\}\in E_G} (-\mathbb{1}_{\kappa_i \sim \kappa_j}).$$

Denoting $\mathbb{1}_R^{\between}$ the indicator that the global dynamics contains a cycle, the sum over the cycling graphs rewrites as

$$\begin{aligned}\sum_{G \in \mathcal{C}_{|\kappa|}(\Upsilon)\setminus\{\Upsilon\}} \prod_{\{i,j\}\in E_G} (-\mathbb{1}_{\kappa_i \sim \kappa_j}) &= \mathbb{1}_R^{\between} \prod_{\{i,j\}\in E_\Upsilon} (-\mathbb{1}_{\kappa_i \sim \kappa_j}) \sum_{E' \subset E_\Upsilon^c} \prod_{\{i,j\}\in E'} (-\mathbb{1}_{\kappa_i \sim \kappa_j}) \\ &= \mathbb{1}_R^{\between} \prod_{\{i,j\}\in E_\Upsilon} (-\mathbb{1}_{\kappa_i \sim \kappa_j}) \prod_{\{i,j\}\in E_\Upsilon^c} (1-\mathbb{1}_{\kappa_i \not\sim \kappa_j})\end{aligned}$$

computing the inverse expansion of the exclusion cumulants, the latter product being smaller than 1. We are hence brought back to the tree case, and dominate the cycle part by the tree part, with the additional strong cycle condition

$$\begin{aligned}\int f_n^\varepsilon[H]^{[\text{cycle}]} \leq \sum_{\rho \in \mathcal{P}_n} \int f_n^{\varepsilon,\rho}(0) \mathrm{d}\underline{v}_n \prod_{j=1}^{|\rho|} \frac{\mathrm{d}\overline{x}_{\rho_j}}{\mu^{|\rho_j|-1}} \sum_{k_j \geqslant 0} \sum_{\underline{\ell}_{k_j}} \sum_{\Upsilon \in \mathcal{T}^{\mathrm{d}}_{\rho_j,k_j}} p_\mu^{|\underline{\ell}_{k_j}|} H^{\otimes \rho_j} \mathbb{1}_{T^{\mathrm{d}}_{\rho_j}=\Upsilon} \\ \times \mathrm{d}\underline{v}^*_{k_j} \prod_{e \in E_\Upsilon} s_e \langle v_m(\tau_e{}^+) - v_{m'}(\tau_e{}^+), \omega_e \rangle_+ \mathrm{d}\omega_e \mathrm{d}\tau_e \times \mathbb{1}^{\between}_{\rho_j}.\end{aligned}$$

This term is negligible in front of the tree one, because of the strong geometric cycle condition, as stated in Section 7.2.

6.2 Bounding the cumulants on short times

We hence need to study the tree part of the integrated cumulants, written (Proposition 5.4.1)

$$\begin{aligned}\int f_n^\varepsilon[H]^{[\text{tree}]}(t) = \sum_{\rho \in \mathcal{P}_n} \int f_n^{\varepsilon,\rho}(0) \mathrm{d}\underline{v}_n \prod_{j=1}^{|\rho|} \frac{\mathrm{d}\overline{x}_{\rho_j}}{\mu^{|\rho_j|-1}} \sum_{k_j \geqslant 0} \sum_{\underline{\ell}_{k_j}} \sum_{\Upsilon \in \mathcal{T}^{\mathrm{d}}_{\rho_j,k_j}} \int \mathrm{d}\underline{v}^*_{k_j} \mathbb{1}_{T^{\mathrm{d}}_{\rho_j}=\Upsilon} \\ \times \prod_{e \in E_\Upsilon} s_e \langle v_m(\tau_e{}^+) - v_{m'}(\tau_e{}^+), \omega_e \rangle_+ \mathrm{d}\omega_e \mathrm{d}\tau_e.\end{aligned} \tag{6.1}$$

Proposition 6.2.1 (Cumulant bound). *For μ large enough in the mixed scaling ($S_{\varepsilon,\mu,\lambda}$), one has for any t small enough (depending on β) that for an absolute constant $C > 0$,*

$$\left| \int f_n^\varepsilon[H]^{[\text{tree}]}(t) \right| \leq \frac{C^n n!}{\mu^{n-1}} \|H\|_\infty^n.$$

More precisely,

$$\left| \int f_n^\varepsilon[H]^{[\text{tree}]}(t) - I_n^{[\![1,n]\!]}[H](t) \right| \leq \frac{C^n n!}{\mu^{n-1}} \|H\|_\infty^n \cdot \varepsilon. \tag{6.2}$$

The latter inequality specifies that the leading term in the integral formula for the cumulants is the one corresponding to the trivial partition $\rho = \{[\![1,n]\!]\}$.

Proof. **First step: initial cluster cumulants.** To get bounds on this cumulant, we will start integrating the *initial* cluster cumulants, of which we recall the following expression (Lemma 5.3.1), where $N \doteq |\Psi_{\rho_1}| + \cdots + |\Psi_{\rho_{|\rho|}}|$ denotes the total number of particles contained in the clusters,

$$f_n^{\varepsilon,\rho}(0, \underline{z}^{[0]}_{\Psi_{\rho_1}}, \ldots, \underline{z}^{[0]}_{\Psi_{\rho_{|\rho|}}}) = \left[\mathbb{1}_{\mathcal{X}_{(\cdot)}}\right]^{\otimes \aleph_{|\rho|}} M_\beta^{\otimes N} \varphi_0^{\otimes \ell_N} \sum_{p,q\geqslant 0} \frac{\lambda^p \mu^q}{p!q!} \int \mathrm{d}\underline{z}^*_{p+q} \varphi_0^{\otimes p} M_\beta^{\otimes p+q} \phi_{\aleph_{|\rho|},p+q}.$$

Using the definition (3.17) of C_0 to control M_β and φ_0, we have

$$\int \left| f_n^{\varepsilon,\rho}(0, \underline{z}^{[0]}_{\Psi_{\rho_1}}, \ldots, \underline{z}^{[0]}_{\Psi_{\rho_{|\rho|}}}) \right| \prod_{k=1}^{|\rho|} \mathrm{d}\overline{x}_{\rho_k} \leq C_0^N e^{-\beta \|\underline{v}_N\|^2} \sum_{p,q\geqslant 0} \frac{(C_0\lambda)^p \mu^q}{p!q!} \int M_\beta^{\otimes p+q} \left|\phi_{\aleph_{|\rho|},p+q}\right| \prod_{k=1}^{|\rho|} \mathrm{d}\overline{x}_{\rho_k} \mathrm{d}\underline{z}^*_{p+q},$$

and we will apply the tree inequality (Proposition B.2.1, *i.*) to the cumulant of the exclusion

$$\left|\phi_{\aleph_{|\rho|},p+q}\right| \leq \sum_{T\in\mathcal{T}_{\aleph_{|\rho|}\cup[\![p+q]\!]}} \prod_{\{x,y\}\in E_G} \mathbb{1}_{x\sim y}.$$

Like in the proof of the bound on these cumulants (Proposition B.2.1, *ii.*), we will use the integration variables $\underline{z}^*_{p+q}$ and $(\overline{x}_{\rho_k})_k$, the latter moving rigidly the clusters $(\underline{x}_{\Psi_k})_k$. Integrating over successive leaves of the tree, removing the edge $\{i,j\}$ leads to a factor $|\Psi_{\rho_i}|\cdot|\Psi_{\rho_j}|C_d\varepsilon^d$, depending on the number of particles in each cluster. For this reason, this time we need to discriminate according to the degrees $d_1, \ldots, d_{|\rho|+p+q}$ of the vertices $\Psi_{\rho_1}, \ldots, \Psi_{\rho_{|\rho|}}, x_1^*, \ldots, x^*_{p+q}$. Since the number of trees on $[\![1,m]\!]$ with prescribed degrees $d_1, \ldots, d_m$ is equal to

$$\frac{(m-2)!}{\prod_{i=1}^m (d_i-1)!},$$

we get (also integrating the density M_β over $\underline{v}^*_{p+q}$)

$$\begin{aligned}\int \left| f_n^{\varepsilon,\rho}(0, \underline{z}^{[0]}_{\Psi_{\rho_1}}, \ldots, \underline{z}^{[0]}_{\Psi_{\rho_{|\rho|}}}) \right| \prod_{k=1}^{|\rho|} \mathrm{d}\overline{x}_{\rho_k} \\ \leq C_0^N e^{-\beta\|\underline{v}_N\|^2} \sum_{p,q\geqslant 0} \frac{(C_0\lambda)^p\mu^q}{p!q!} \varepsilon^{d(|\rho|+p+q-1)} \sum_{d_1,\ldots,d_{|\rho|+p+q}} \frac{(|\rho|+p+q-2)!}{\prod_{i=1}^{|\rho|+p+q}(d_i-1)!} \prod_{i=1}^{|\rho|} |\Psi_{\rho_i}|^{d_i}.\end{aligned}$$

Now, we observe that

$$\frac{(|\rho|+p+q-2)!}{p!q!} = \binom{p+q}{p}\binom{p+q+|\rho|-2}{p+q}(|\rho|-2)! \leq \binom{p+q}{p} 2^{p+q+|\rho|-2}(|\rho|-2)!\ .$$

We write on the other hand

$$\prod_{i=1}^{|\rho|} \sum_{d_i\geqslant 1} \frac{|\Psi_{\rho_i}|^{d_i}}{(d_i-1)!} \leq \prod_{i=1}^{|\rho|} |\Psi_{\rho_i}| \exp(|\Psi_{\rho_i}|) \leq e^{2N},$$

and similarly

$$\prod_{i=1}^{p+q} \sum_{d_{|\rho|+i}\geqslant 1} \frac{1}{(d_{|\rho|+i}-1)!} \leq e^{p+q}.$$

We eventually get, since $|\rho| \leq N$ and $C_0\lambda \leq \mu$,

$$\int \left| f_n^{\varepsilon,\rho}(0, \underline{z}^{[0]}_{\Psi_{\rho_1}}, \dots, \underline{z}^{[0]}_{\Psi_{\rho_{|\rho|}}}) \right| \prod_{k=1}^{|\rho|} \mathrm{d}\overline{x}_{\rho_k} \leq (2e^2C_0)^N e^{-\beta\|\underline{v}_N\|^2} (|\rho|-2)! \varepsilon^{d(|\rho|-1)} \sum_{p,q \geqslant 0} \binom{p+q}{p} (2\mu\varepsilon^d)^{p+q}$$
$$\leq (2e^2C_0)^N e^{-\beta\|\underline{v}_N\|^2} (|\rho|-2)! \varepsilon^{d(|\rho|-1)} \sum_{r\geqslant 0} (4\varepsilon)^r,$$

eventually leading, for an absolute constant C, to

$$\int \left| f_n^{\varepsilon,\rho}(0, \underline{z}^{[0]}_{\Psi_{\rho_1}}, \dots, \underline{z}^{[0]}_{\Psi_{\rho_{|\rho|}}}) \right| \prod_{k=1}^{|\rho|} \mathrm{d}\overline{x}_{\rho_k} \leq (CC_0)^N e^{-\beta\|\underline{v}_N\|^2} (|\rho|-2)! \varepsilon^{d(|\rho|-1)}. \tag{6.3}$$

Second step: bounding the collision kernels. The observable H being bounded, we can dominate it roughly. Moreover, let us observe that the factors bounding the initial cumulant (6.3) decompose on the clusters as

$$(CC_0)^N e^{-\beta\|\underline{v}_N\|^2} = \prod_{j=1}^{r} (CC_0)^{|\Psi_{\rho_j}|} e^{-\beta\|\underline{v}_{\Psi_{\rho_j}}\|^2}, \tag{6.4}$$

so that for a generic cluster $R = \rho_j$, we harness the associated velocity decay to study

$$S_R \doteq \sum_{k\geqslant 0} \sum_{\underline{\ell}_k \in \Lambda_k} p_\mu^{|\underline{\ell}_k|} \sum_{\Upsilon \in \mathcal{T}^{\mathrm{d}}_{R,k}} \int \mathrm{d}\underline{v}_{r+k} \mathbb{1}_{T^{\mathrm{d}}_{\rho_j} = \Upsilon} \prod_{e \in E_\Upsilon} s_e \langle v_m(\tau_e{}^+) - v_{m'}(\tau_e{}^+), \omega_e \rangle_+ \mathrm{d}\omega_e \mathrm{d}\tau_e e^{-\|\underline{v}_{r+k}\|^2}.$$

With our notation, $r + k = |\Psi_R|$. Since we just bounded the dependency on the initial value, the trajectories do not depend anymore on the particles' tags, allowing us to bound roughly the sum over $\underline{\ell}_k$, using that $p_\mu \leq 1$. We will use the velocity decay to bound the velocities appearing in the collision kernels. We will bound each edge separately, starting from the furthest from time t, so that the other collision kernels will only contain velocities that do not depend on the considered edge.

For each edge corresponding to a clustering condition, using the Cauchy-Schwartz inequality, we bound all the possibilities for this edge by

$$2 \sum_{m,m'=1}^{r+k} (|v_m(\tau_e^{\star+})| + |v_{m'}(\tau_e^{\star+})|) \leq 4(r+k)\sqrt{(r+k)\|\underline{v}_{r+k}\|^2}, \tag{6.5}$$

where the factor 2 stems from the choice of the edge's decoration. For each edge associated to a particle's adjunction, since one of the vertices corresponds to which particle is added, we only have to sum over the choice of the second particle as

$$2 \sum_{m=1}^{r+k} (|v_m(\tau_e^{\star+})| + |v_{m'}(\tau_e^{\star+})|) \leq 2\sqrt{(r+k)\|\underline{v}_{r+k}\|^2}. \tag{6.6}$$

Determining whether the edge is a clustering condition or a particle's adjunction corresponds to an additional factor 2^{r+k}. In the end, harnessing the velocity decay (6.4) and integrating the ordered times, we get

$$S_R \leq \sum_{k\geqslant 0} 8^k \int \mathrm{d}\underline{v}_{r+k} \mathrm{d}\underline{\omega}_{E_\Upsilon} \mathrm{d}\underline{\tau}_{E_\Upsilon} (r+k)^{r-1} \left(\sqrt{(r+k)\|\underline{v}_{r+k}\|^2} \right)^{r+k-1} e^{-\|\underline{v}_{r+k}\|^2}$$
$$\leq \sum_{k\geqslant 0} 8^k \frac{t^{r+k-1}}{(r+k-1)!} (r+k)^{r-1} \sqrt{r+k}^{\,r+k-1} \left(C_d\sqrt{r+k} \right)^{r+k-1}$$
$$\leq \sum_{k\geqslant 0} (\hat{C}_d t)^{r+k-1} (r+k)^{r-1} e^{r+k}, \tag{6.7}$$

using in the end that $(r+k)^{r+k-1} \leq (r+k-1)! e^{r+k}$.

Last step: combinatorial manipulations. We now gather it all to bound the cumulant using its expression (6.1) from Proposition 5.4.1. Using the exchangeability of particles to reduce the partition $\rho \in \mathcal{P}_n$ to the cardinal of its subsets, and denoting $C_H \doteq \|H\|_\infty$ and $\hat{C} \doteq CC_0$, we write for a constant C_d depending only on the dimension that

$$\left| \int f_n^\varepsilon[H]^{[\text{tree}]}(t) \right| \leq \sum_{i=1}^n \sum_{r_1+\cdots+r_i=n} \frac{(i-2)!\varepsilon^{d(i-1)} n!}{i! r_1! \dots r_i!} \frac{C_H^n}{\mu^{n-i}} \prod_{j=1}^i \sum_{k_j \geqslant 0} (C_d t)^{r_j+k_j-1} (r_j+k_j)^{r_j-1}$$
$$\leq \frac{C_H^n n!}{\mu^{n-1}} \sum_{i=1}^n \varepsilon^{i-1} \prod_{j=1}^i \sum_{r_j \geqslant 0} \sum_{k_j \geqslant 0} (C_d t)^{r_j+k_j-1} e^{k_j+r_j}$$

in the scaling $\mu\varepsilon^d = \varepsilon$, and harnessing the denominators $r_j!$ to control the terms $(r_j+k_j)^{r_j-1}$. Now, for t small enough the series are convergent (and the term $r_j = k_j = 0$ actually does not appear, as it corresponds to the empty tree), so that we get

$$\left| \int f_n^\varepsilon[H]^{[\text{tree}]}(t) \right| \leq \frac{C_H^n n!}{\mu^{n-1}} \sum_{i=1}^n \varepsilon^{i-1} \cdot 4^i$$
$$\leq 4 \frac{C_H^n n!}{\mu^{n-1}} (1+8\varepsilon),$$

which concludes the proof. The second point (6.2) of the proposition is proved observing that the term 8ε corresponds to the sum for $i \geq 2$, i.e. to the sum over the non-trivial partitions. ☐

7 Limit cumulants

Since the cumulants f_n decrease very quickly as n increases (Proposition 6.2.1), we only need the convergence of the first cumulant to compute the large deviation principle, and the convergence of the fluctuation field. Indeed, in the rescaling $(\lambda^{n-1} f_n)$ that appears in the statistical study of the tagged particles, the cumulants vanish for $n \geqslant 2$, as soon as $\frac{\lambda}{\mu}$ goes to zero. Nevertheless, for completeness, we also compute their limit in the rescaling $(\mu^{n-1} f_n)$. The corresponding formula could be used to capture the fine scales of the dynamics in a further study.

7.1 Rare encounters of tagged particles

First of all, like in Section 3.7, we will show that the encounters between tagged particles are very rare. Indeed, when we have bounded roughly the sum over the possible tags $\underline{\ell}_k \in \Lambda_k$ by 2^k, one could have looked at the influence of the encounters with at least one tagged particle, which corresponds to the sum

$$\sum_{\substack{\underline{\ell}_k \in \Lambda_k \\ \underline{\ell}_k \neq \underline{0}_k}} p_\mu^{|\underline{\ell}_k|} = \sum_{i=1}^k \binom{k}{i} p_\mu^i$$
$$\leq p_\mu 2^k.$$

This leads to the exact same bound as the previous one, with an additional factor $p_\mu \ll 1$. In the end, we get the estimate presented in the proposition below.

Proposition 7.1.1 (Rare tagged encounters)**.** *Denoting*

$$\int f_n^\varepsilon[H]^{[\text{tree}],\underline{0}}(t)$$

the integrated cumulants where the domain of integration of $\mathrm{d}\nu$ *is restricted to* $\{\underline{\ell}_k^* = \underline{0}_k\}$*, one has in the usual mixed scaling* ($S_{\varepsilon,\mu,\lambda}$)*, for any* t *small enough and an absolute constant* $C > 0$*, that*

$$\left| \int f_n^\varepsilon[H]^{[\text{tree}]}(t) - \int f_n^\varepsilon[H]^{[\text{tree}],\underline{0}}(t) \right| \leq \frac{C^n n!}{\mu^{n-1}} \|H\|_\infty^n \cdot p_\mu.$$

7.2 Discarding cycles

In the expansion of the integrated cumulants in terms of dynamics trees (Section 5.4), we emphasized that non-clustering encounters may happen, either stemming from a non-clustering overlap, or from a non-clustering collision. We show here that the dynamics presenting such cycles are negligible in the said expansion, in particular proving the smallness of the cycle part of the expansion (Section 6.1). Using a more precise computation than the previous method [8], we achieve an optimal bounding factor in ε instead of $\varepsilon|\log\varepsilon|$.

Proposition 7.2.1 (Cycles are rare in the dynamics)**.** *We have the following estimate on the expansion proved in Proposition 5.4.1 for the integrated cumulants, under the constraints that a cycle happens in the dynamics:*

$$\Bigg| \sum_{\rho \in \mathcal{P}_n} \int f_n^{\varepsilon,\rho}(0) \mathrm{d}\underline{v}_n \prod_{j=1}^{|\rho|} \frac{\mathrm{d}\overline{x}_{\rho_j}}{\mu^{|\rho_j|-1}} \sum_{k_j \geqslant 0} \sum_{\Upsilon \in \mathcal{T}^{\mathrm{d}}_{\rho_j,k_j}} H^{\otimes \rho_j} \mathbb{1}_{T^{\mathrm{d}}_{\rho_j} = \Upsilon}$$
$$\times \, \mathrm{d}\underline{v}_{k_j}^* \mathrm{d}\underline{\omega}_{E_\Upsilon} \mathrm{d}\underline{\tau}_{E_\Upsilon} \prod_{e \in E_\Upsilon} s_e \langle v_m(\tau_e{}^+) - v_{m'}(\tau_e{}^+), \omega_e \rangle_+ \times \mathbb{1}^{\between}_{\rho_j} \Bigg| \leq \frac{C^n n!}{\mu^{n-1}} \|H\|_\infty^n \cdot \varepsilon.$$

The cycle condition imposes strong geometric constraints that result in a additional factor ε *in the usual bounds for the integrated cumulants. The dynamics implying cycles are thus negligible in the expansion above.*

The proof of this proposition is given in Appendix D, based on geometric estimates.

7.3 Convergence of the integrated cumulants

Now that we proved that the cycles and the other tagged particles asymptotically have a negligible impact on the dynamics of n fixed particles, we go back to the expression (5.14) given in Proposition 5.4.1 for the integral of cumulants, to determine their limit (Proposition 7.3.1 below). Indeed, by Propositions 7.1.1 and 7.2.1 above, up to an error of order $(\varepsilon + p_\mu)$, the pseudo-trajectories $\underline{z}_{n+k}^{[0,t]}$ only include clustering encounters, and none of the added particles is tagged. Hence, the velocities of these pseudo-trajectories only depend on the dynamics tree $T^{\mathrm{d}}_{[\![n]\!],k}$ and on the dynamics parameters in the integral, so that they are equal to the velocities of the limit pseudo-trajectories $\underline{\zeta}_{n+k}^{[0,t]}$. This way, similarly as in the discussion Lemma 3.8.3, and to the discussion around (3.45), the positions of both pseudo-trajectories are $(k\varepsilon)$-close, and in particular they are identical for the n final particles. Since the only tagged particles are among these n final particles, and since the equilibrium M_β only depends on the velocities, we get

$$[\varphi_0^{\otimes \ell_n} M_\beta^{\otimes n+k}](\underline{\zeta}_{n+k}^{[0]}) = [\varphi_0^{\otimes \ell_n} M_\beta^{\otimes n+k}](\underline{z}_{n+k}^{[0]}).$$

The observable $H^{\otimes n}$ also only depends on the n studied particles, so that one can bound

$$\begin{aligned}&\left|H^{\otimes n}(\underline{\zeta}_n^{[0,t]})[\varphi_0^{\otimes \underline{\ell}_n} M_\beta^{\otimes n+k}](\underline{\zeta}_{n+k}^{[0]}) - H^{\otimes n}(\underline{z}_n^{[0,t]}) F_{n+k}^\varepsilon(0,\underline{z}_{n+k}^{[0]})\right| \\ &\quad = \left|H^{\otimes n}(\underline{z}_n^{[0,t]})\right| \cdot \left|[\varphi_0^{\otimes \underline{\ell}_n} M_\beta^{\otimes n+k}](\underline{z}_{n+k}^{[0]}) - F_{n+k}^\varepsilon(0,\underline{z}_{n+k}^{[0]})\right| \\ &\quad \leq \left|H^{\otimes n}(\underline{z}_n^{[0,t]})\right| \cdot C_0^{n+k} e^{-\|\underline{v}_{n+k}\|^2}\varepsilon,\end{aligned}$$

by the initial error studied in Proposition 3.3.1. Eventually, recalling the result (6.2) of Proposition 6.2.1, the leading term in the cumulant expansion corresponds to the case of a single cluster $\rho_1 = [\![1,n]\!]$, once again with an error of order ε. We end up with the following limit formula for the integrated cumulants.

Proposition 7.3.1 (Convergence of the cumulants)**.** *For $t > 0$ small enough, the integrated cumulants $\int f_n^\varepsilon[H](t,\underline{\ell}_n)$ converge as ε goes to 0 in the scaling ($S_{\varepsilon,\mu,\lambda}$) towards the following limit formula*

$$\begin{aligned}&\int f_n[H](t,\underline{\ell}_n) \\ &\quad \doteq \sum_{k\geqslant 0} \sum_{\Upsilon \in \mathcal{T}^{\mathrm{d}}_{[\![n]\!],k}} \int \frac{\mathrm{d}\overline{x}\mathrm{d}\underline{v}_{n+k}}{\mu^{n-1}} \mathrm{d}\underline{\omega}_{E_\Upsilon} \mathrm{d}\underline{\tau}_{E_\Upsilon} \prod_{e\in E_\Upsilon} s_e \langle v_m^{[\tau_e^+]} - v_{m'}^{[\tau_e^+]}, \omega_e\rangle_+ H^{\otimes n}[\varphi_0^{\otimes \underline{\ell}_n} M_\beta^{\otimes n+k}](\underline{\zeta}_{n+k}^{[0]}),\end{aligned}$$

where none of the added particles in the pseudo-trajectories are tagged, and with the quantitative bound

$$\left|\int f_n[H](t,\underline{\ell}_n) - \int f_n^\varepsilon[H](t,\underline{\ell}_n)\right| \leq \frac{C^n n!}{\mu^{n-1}} \|H\|_\infty^n \cdot (\varepsilon + p_\mu)\,.$$

Let us observe finally that some of the trees in the formula above do not contribute to the sum. Indeed, imagine that in a tree two added particles (hence non-tagged ones), both meet as their first (forwards) encounter at a time τ_e. Then, changing the sign of the encounter leads to a weight

$$-s_e\langle v_m^{[\tau_e^+]} - v_{m'}^{[\tau_e^+]}, \omega_e\rangle_+ M_\beta\left(v_m^{[0]\prime}\right) M_\beta\left(v_{m'}^{[0]\prime}\right) = -s_e\langle v_m^{[\tau_e^+]} - v_{m'}^{[\tau_e^+]}, \omega_e\rangle_+ M_\beta\left(v_m^{[0]}\right) M_\beta\left(v_{m'}^{[0]}\right),$$

using the equilibrium structure. Note that since they are added, they do not contribute to the weight $\varphi_0^{\otimes \underline{\ell}_n} H^{\otimes n}$, so that the contributions of this tree and of its counterpart with a changed sign cancel out. In particular, for the first cumulant $f_1 = F_1$, as pictured in Figure 5, the only shape of tree that contributes is the linear one. Hence, it is enough to know the signs associated to each encounter, and one may write

$$\begin{aligned}&\int F_1[H](t,\ell) \\ &\quad = \sum_{k\geqslant 0} \sum_{\underline{s}_k} \int \mathrm{d}z_1 \mathrm{d}\underline{v}_k^* \mathrm{d}\underline{\omega}_k \mathrm{d}\underline{\tau}_k \prod_{i=1}^k s_i \langle v_1^{[\tau_i^+]} - v_{1+i}^{[\tau_i^+]}, \omega_i\rangle_+ H(\zeta_1^{[0,t]})[\varphi_0^{\otimes \ell} M_\beta^{\otimes 1+k}](\underline{\zeta}_{1+k}^{[0]}).\end{aligned} \tag{7.1}$$

Fortunately, this expansion coincides with the pseudo-trajectory formulation (3.15) of the solution to the linear Rayleigh–Boltzmann equation (2.7).

7.4 Convergence of the cumulant generating function

We study here the convergence of the cumulant generating function in the case (4.17) of an observable $H = \widehat{H}\mathbb{1}_{\ell=1}$ weighting only the tagged particles, generalized to observables depending on the

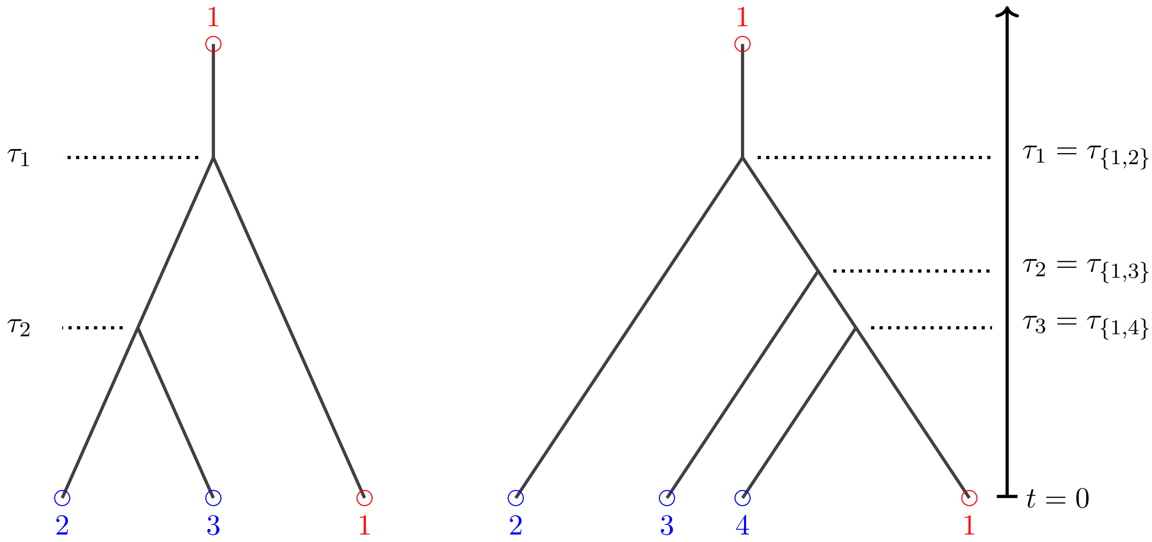

Figure 5: Left: non-contributing tree. Right: only contributing tree.

whole pseudo-trajectory $z^{[0,t]}$ as in (5.4). It writes

$$\mathfrak{G}_\varepsilon^{[0,t]}[H] = \sum_{p\geqslant 1} \frac{\lambda^p}{p!} \int f_p^\varepsilon \left[e^H - 1\right](t, \underline{1}_p).$$

For $p \geq 2$, using the bounds on the cumulants (Proposition 6.2.1), one has

$$\sum_{p\geqslant 2} \frac{\lambda^p}{p!} \left| \int f_p^\varepsilon \left[e^H - 1\right](t, \underline{1}_p) \right| \leq \lambda \sum_{p\geqslant 2} \left(\frac{\lambda}{\mu}\right)^{p-1} \left(C\|e^H - 1\|_\infty\right)^p$$
$$\leq \lambda \times 2\frac{\lambda}{\mu} \left(C\|e^H - 1\|_\infty\right)^2$$

as soon as $2\lambda C\|e^H - 1\|_\infty \leq \mu$ in the scaling ($S_{\varepsilon,\mu,\lambda}$). Hence, rescaling properly the generating function, we get the following proposition thanks to the convergence of the first cumulant (Proposition 7.3.1).

Proposition 7.4.1 (Convergence of the cumulant generating function)**.** *In the scaling* ($S_{\varepsilon,\mu,\lambda}$)*, the rescaled cumulant generating function converges as*

$$\frac{1}{\lambda} \mathfrak{G}_\varepsilon^{[0,t]}[H] \xrightarrow[\varepsilon\to 0]{} \int F_1[e^H - 1](t, 1) = \int F_1[e^H](t, 1) - 1. \tag{7.2}$$

All the cumulants of order greater than 2 vanish in this scaling, at the opposite of the non-linear version [9] in which they contribute to add non-linearity in the limiting generating function. Next section is dedicated to the equations driving this limit.

7.5 Hamilton–Jacobi equations for the limit cumulant generating function

For observables that behave according to the structure of the trajectories, i.e. of the form

$$H(z^{[0,t]}) = g(t, z^{[t]}) - \int_0^t [\partial_s - v \cdot \nabla_x] g(s, z(s)) \mathrm{d}s$$

as introduced in Section 4.1, we will study the limit cumulant generating function (see Section 7.4), denoted

$$\mathcal{I}(t, g) \doteq \int \tilde{F}_1 \left[\exp\left(g_t - \int_0^t (\partial_s - v \cdot \nabla_x) g_s\right)\right](t). \tag{7.3}$$

It is relevant to consider observables in the functional space

$$g \in \mathbb{B}_{T,\beta} = \left\{ h \;\Big|\; \exists C > 0 : \; \forall t \leqslant T, \; \left\| h(t,v) + \frac{\beta}{2}|v|^2 \right\|_{\mathbb{L}^\infty(\mathcal{D})} + \|(\partial_t - v\cdot\nabla_x)h(t)\|_{\mathbb{L}^\infty(\mathcal{D})} < C \right\}. \tag{7.4}$$

Indeed, the relevant variable that will appear as boundary condition of a Boltzmann equation will be $e^g(t)$ (see the proof of Proposition 7.5.2). Note that the time of validity of Theorem 4.2 depends on this norm C chosen for $g \in \mathbb{B}_{T,\beta}$. One may choose to extend this class of functions to observables growing as small inverse Gaussians, using part of the exponential decay of the cumulants to compensate this growth (see [9, Chapter 7]), yet that would mean losing control on the solutions to the Hamilton-Jacobi system below (Proposition 7.5.1).

We will use the formula (7.1) above to find the Hamilton–Jacobi equation it satisfies. Observe that the term $k = 0$ in this expansion corresponds to the initial value $\mathcal{I}(0,g)$. For any $k \geqslant 1$, we will split the dynamics tree in two, to isolate the influence of particle 2 at time τ_1. To this extent, let us define $\underline{\widetilde{s}}_{k-1} \doteq (s_2,\dots,s_k)$, and similarly $\underline{\widetilde{v}}_{k-1}, \underline{\widetilde{\omega}}_{k-1}, \underline{\widetilde{\tau}}_{k-1}$, and $\underline{\widetilde{\zeta}}_k \doteq (\zeta_1,\zeta_3,\dots,\zeta_{1+k})$. We split according to the sign s_1, which labels the first collision (between v_1 and $v_2 = v_1^*$), writing

$$\begin{aligned}
&\mathcal{I}(t,g) - \mathcal{I}(0,g) \qquad (7.5)\\
&= \int \mathrm{d}z_1 \mathrm{d}v_2 \mathrm{d}\omega_1 \mathrm{d}\tau_1 \langle v_1 - v_2, \omega_1\rangle_+ M_\beta(v_2') \sum_{k\geqslant 1} \sum_{\underline{\widetilde{s}}} \int \mathrm{d}\underline{\widetilde{v}} \mathrm{d}\underline{\widetilde{\omega}} \mathrm{d}\underline{\widetilde{\tau}} \prod_{i=2}^k s_i \langle v_1^{[\tau_i+]} - v_{1+i}^{[\tau_i+]}, \omega_i\rangle_+ \\
&\qquad \times \exp\left(g(t,\zeta_1^{[t]}) - \int_0^t (\partial_s - v\cdot\nabla_x) g(s,\zeta_1^{[s]}) \right) [\varphi_0 M_\beta^{\otimes k}](\underline{\widetilde{\zeta}}_k^{[0]}) \\
&- \int \mathrm{d}z_1 \mathrm{d}v_2 \mathrm{d}\omega_1 \mathrm{d}\tau_1 \langle v_1 - v_2, \omega_1\rangle_+ M_\beta(v_2) \sum_{k\geqslant 1} \sum_{\underline{\widetilde{s}}} \int \mathrm{d}\underline{\widetilde{v}} \mathrm{d}\underline{\widetilde{\omega}} \mathrm{d}\underline{\widetilde{\tau}} \prod_{i=2}^k s_i \langle v_1^{[\tau_i+]} - v_{1+i}^{[\tau_i+]}, \omega_i\rangle_+ \\
&\qquad \times \exp\left(g(t,\zeta_1^{[t]}) - \int_0^t (\partial_s - v\cdot\nabla_x) g(s,\zeta_1^{[s]}) \right) [\varphi_0 M_\beta^{\otimes k}](\underline{\widetilde{\zeta}}_k^{[0]}),
\end{aligned}$$

where the pseudo-trajectory $\underline{\widetilde{\zeta}}_k^{[0,t]}$ corresponds to the scattering case $s_1 = 1$ in the first term, and to the overlapping case $s_1 = -1$ in the second one. Now, by the fundamental theorem of calculus for the transport equation, since there is no collision in the pseudo-trajectory on $[\tau_1, t]$, one has

$$g(t,\zeta_1(t)) - \int_0^t (\partial_s - v\cdot\nabla_x) g(s,\zeta_1(s)) = g(\tau_1,\zeta_1(\tau_1^+)) - \int_0^{\tau_1} (\partial_s - v\cdot\nabla_x) g(s,\zeta_1(s)).$$

In the overlapping case, $g(\tau_1,\zeta_1(\tau_1^+)) = g(\tau_1, x_1^{[\tau_1]}, v_1) = g(\tau_1,\zeta_1(\tau_1^-))$, yet there is a discontinuity in the scattering case. In this case, writing

$$g(\tau_1,\zeta_1(\tau_1^+)) = g(\tau_1, x_1^{[\tau_1]}, v_1) - g(\tau_1, x_1^{[\tau_1]}, v_1') + g(\tau_1, x_1^{[\tau_1]}, v_1'),$$

and recalling that the mapping $(v_1', v_2') \mapsto (v_1, v_2)$ is of Jacobian 1, the first term in the expansion (7.5) above writes exactly as the second one, with an additional factor

$$\exp\left(g(\tau_1, x_1^{[\tau_1]}, v_1') - g(\tau_1, x_1^{[\tau_1]}, v_1) \right).$$

This way, the expansion (7.5) of the cumulant generating function corresponds to an integral over time of the same expansion at time τ_1, with an additional weight depending on $z_1(\tau_1)$: this is precisely the partial derivative of the functional $\mathcal{I}(t,g)$ with respect to $g(t)$ as an independent variable, in the direction of the said weight:

$$\mathcal{I}(t,g) - \mathcal{I}(0,g) = \int_0^t \mathrm{d}\tau \frac{\partial \mathcal{I}}{\partial g(t)}(\tau,g) \left[\int \mathrm{d}v_2 \mathrm{d}\omega \langle v_1 - v_2, \omega\rangle_+ M_\beta(v_2) (e^{g(\tau,z_1') - g(\tau,z_1)} - 1) \right].$$

One may identify the partial derivative with the function $z \mapsto \partial_{g(t)}\mathcal{I}(t,g,z)$ such that

$$\frac{\partial \mathcal{I}}{\partial g(t)}(t,g)\,[h] = \int_{\mathcal{D}} \partial_{g(t)}\mathcal{I}(t,g,z)h(z)\mathrm{d}z.$$

The function $z \mapsto \partial_{g(t)}\mathcal{I}(t,g,z)$ is shown [9] to be continuous in x, with values in the space of measures weighted by the inverse of the Maxwellian M_β^{-1}. Hence, we are left with

$$\mathcal{I}(t,g) = \mathcal{I}(0,g) + \int_0^t \mathrm{d}\tau \int \mathrm{d}v_2\mathrm{d}\omega\langle v_1 - v_2, \omega\rangle_+ M_\beta(v_2)\partial_{g(t)}\mathcal{I}(\tau,g,z_1)(e^{g(\tau,z_1')-g(\tau,z_1)} - 1), \tag{7.6}$$

yielding the following proposition.

Proposition 7.5.1 (Hamilton–Jacobi system for the limit cumulant generating function)**.** *Introducing the Hamiltonian*

$$\mathcal{H}\,(q,p) \doteq \int \mathrm{d}v_2\mathrm{d}\omega\langle v_1 - v_2, \omega\rangle_+ M_\beta(v_2)q(z_1)(e^{p(x_1,v_1)-p(x_1,v_1')} - 1),$$

the limit cumumant generating function is a mild solution of the equation

$$\partial_t\mathcal{I}(t,g) = \mathcal{H}\left(\partial_{g(t)}\mathcal{I}(t,g), g(t)\right),$$

in the sense of (7.6).

At fixed $t > 0$ and $g \in \mathbb{B}_{t,\beta}$ (defined above in (7.4)), this Hamiltonian equation incites to introduce the following Hamilton–Jacobi system

$$\begin{cases} (\partial_s - v\cdot\nabla_x)q^{[t]} = \dfrac{\partial \mathcal{H}}{\partial p}(q^{[t]},p^{[t]}) & , \quad q^{[t]}(0) = M\varphi_0\exp(p^{[t]}(0)), \\ (\partial_s - v\cdot\nabla_x)(p^{[t]} - g) = -\dfrac{\partial \mathcal{H}}{\partial q}(q^{[t]},p^{[t]}) & , \quad p^{[t]}(t) = g(t), \end{cases} \tag{7.7}$$

where the unknowns $(q^{[t]},p^{[t]})$ are meant to find the minimizing values of $\big(\partial_{g(t)}\mathcal{I}(t,g), g(t)\big)$ for this Hamiltonian. Next proposition is dedicated to prove that the mild Hamiltonian solution

$$\hat{\mathcal{I}}(t,g) \doteq \mathcal{I}(t,0) + \int_0^t \mathrm{d}s \int_{\mathcal{D}} q^{[t]}(s)(\partial_s - v\cdot\nabla_x)(p^{[t]}(s) - g(s)) + \int_0^t \mathcal{H}(q^{[t]}(s),p^{[t]}(s))\mathrm{d}s \tag{7.8}$$

is well-defined, and to identify it with the functional $\mathcal{I}(t,g)$.

Proposition 7.5.2 (Identification of the Hamiltonian solutions)**.** *For any time $t > 0$ and any observable $g \in \mathbb{B}_{t,\beta}$, the Hamilton–Jacobi system* (7.7) *admits a unique global solution $(q^{[t]},p^{[t]})$ such that $(q^{[t]}e^{-p^{[t]}}, e^{p^{[t]}}) \in \mathbb{L}^\infty\left([0,t],\mathcal{F}_{1,\beta/2}\right)^2$, and the functional $\hat{\mathcal{I}}(t,g)$ defined as* (7.8) *from this solution coincide with our functional*

$$\mathcal{I}(t,g) = \hat{\mathcal{I}}(t,g).$$

Proof. To prove the well-posedness of the equation, we compute the change of unknowns

$$\begin{cases} \gamma(s) = e^{g(s)} \\ \theta(s) = (\partial_s - v\cdot\nabla_x)g(s), \end{cases}$$

and we consider the functional $\mathcal{J}(t,\theta,\gamma) \doteq \mathcal{I}(t,g)$. For this functional, the formula for the partial derivative with respect with γ is even simpler: the term $\gamma = e^g$ disappear, so that we have to add it in the weight corresponding to the direction of the derivation. The associated Hamiltonian is hence

$$\hat{\mathcal{H}}\,(\chi,\eta) \doteq \int \mathrm{d}v_2\mathrm{d}\omega\langle v_1 - v_2, \omega\rangle_+ M_\beta(v_2)\chi(z_1)(\eta(z_1') - \eta(z_1)),$$

for the variables $\chi \doteq qe^{-p}$ and $\eta \doteq e^{p}$, and the Hamilton–Jacobi system writes as

$$\begin{cases} (\partial_s - v\cdot\nabla_x)\chi + \theta\chi = \dfrac{\partial\hat{\mathcal{H}}}{\partial\eta}(\chi,\eta) \quad , \quad \chi(0) = M_\beta\varphi_0, \\ (\partial_s - v\cdot\nabla_x)\eta - \theta\eta = -\dfrac{\partial\hat{\mathcal{H}}}{\partial\chi}(\chi,\eta) \quad , \quad \eta(t) = \gamma(t). \end{cases}$$

This system is equivalent to the following linear Boltzmann–Hamilton–Jacobi system

$$\begin{cases} (\partial_s - v\cdot\nabla_x)\chi = -\theta\chi + \displaystyle\int \mathrm{d}v_2\mathrm{d}\omega\langle v_1-v_2,\omega\rangle_+\Big(M_\beta(v_2')\chi(z_1') - M_\beta(v_2)\chi(z_1)\Big) \\ (\partial_s - v\cdot\nabla_x)\eta = +\theta\eta - \displaystyle\int \mathrm{d}v_2\mathrm{d}\omega\langle v_1-v_2,\omega\rangle_+\Big(M_\beta(v_2')\eta(z_1') - M_\beta(v_2)\eta(z_1)\Big). \end{cases} \tag{7.9}$$

Contrary to [9], these equations are decoupled, since the coupling was stemming from the non-linearity. These linear equations are very close to the Rayleigh–Boltzmann equation (2.7), and Appendix E is dedicated to show that in our functional setting, this system admits a unique global positive solution $(\chi,\eta) \in \mathbb{L}^\infty\left([0,t],\ \mathcal{F}_{1,\beta/2}\right)^2$, proving the first part of the proposition. The details of the identification of the functionals is similar as [9], showing that they share enough regularity in addition to be solutions to the same mild equation. □

8 Fluctuation and large deviation results

We eventually prove in this section Theorems 4.1 and 4.2. We start presenting tightness properties, so that we will be able to prove these theorems in a weak form, to extend them afterwards to stronger topologies.

8.1 Tightness

We expose here tightness results for the empirical measure and the fluctuation field, useful to extend weak results, in the sense of observables, to results in the strong Skorokhod topology on the set of trajectories $\mathrm{Traj}([0,t],\mathcal{M}(\mathcal{D}))$.

The results of this section are given without complete proofs, which can be found in the paper dealing with the fluctuations and large deviations of the general symmetric hard-sphere dynamics [9]. Indeed, these proofs rely on the bounds on the cumulants, that we proved in Section 6.2 in our mixed model, and are otherwise identical.

Proposition 8.1.1. *There exists a distance d, based on normalized bounded observables, and associated to the strong Skorokhod topology on $\mathrm{Traj}([0,t],\mathcal{M}(\mathcal{D}))$, such that*

$$\lim_{A\to+\infty}\lim_{\lambda\to+\infty}\frac{1}{\lambda}\log\mathbb{P}\left[\sup_{s\in[0,t]} d(0,\tilde{\pi}^\varepsilon_s) \geq A\right] = -\infty, \tag{8.1}$$

and for any $\eta > 0$,

$$\lim_{\delta\to+\infty}\lim_{\mu\to+\infty}\frac{1}{\lambda}\log\mathbb{P}\left[\sup_{|s-s'|<\delta} d(\tilde{\pi}^\varepsilon_s,\tilde{\pi}^\varepsilon_{s'}) > \eta\right] = -\infty. \tag{8.2}$$

Proof. For the example, we show the first result (8.1), which is the easiest one and for this reason not completely detailed in [9]. Let us define the random set containing the labels of the tagged particles

$$\mathcal{S}_\lambda \doteq \Big\{1 \leq i \leq \mathcal{N}\ ,\ L_i = 1\Big\}.$$

Noticing that the number $|\mathcal{S}_\lambda|$ of tagged particles does not change with time, and since whichever normalized observable $(h_i)_{i\in\mathbb{N}}$ appearing in the distance d is bounded, one has

$$\begin{aligned} d(0,\tilde{\pi}_s^\varepsilon) &= \sum_{i\in\mathbb{N}} \frac{1}{\lambda} \sum_{j\in\mathcal{S}_\lambda} h_i(Z_{\varepsilon,j}^{[0,s]}) \\ &\leq \frac{|\mathcal{S}_\lambda|}{\lambda} \sum_{i\in\mathbb{N}} \|h_i\|_\infty. \end{aligned}$$

The latter sum being normalized, the condition $d(0,\tilde{\pi}_t^\varepsilon) \geq A$ is reduced to asking, for a constant $C > 0$ depending on these observables, that

$$\frac{|\mathcal{S}_\lambda|}{\lambda} \geqslant CA.$$

Thus, we have

$$\mathbb{P}\left[\sup_{s\in[0,t]} d(0,\tilde{\pi}_t^\varepsilon) \geq A\right] \leq \mathbb{P}\big[|\mathcal{S}_\lambda| \geq CA\lambda\big] \leq \frac{\mathbb{E}\left[2^{|\mathcal{S}_\lambda|}\right]}{2^{CA\lambda}}$$

by the Markov inequality. We know very well the initial distribution discussed in Section 2.3, and in particular we have a formula (C.2) for the partition function, that harnesses symmetry to write $q = |\underline{\ell}_p|$ the number of tagged particles in the following computation

$$\begin{aligned} \mathbb{E}\left[2^{|\mathcal{S}_\lambda|}\right] &= \frac{1}{\mathcal{Z}} \sum_{p,q\geqslant 0} 2^q \cdot \frac{\lambda^q \mu^p}{p!q!} \int M_\beta^{\otimes p+q} \varphi_0^{\otimes p} \mathbb{1}_{\mathcal{X}_{p+q}^\varepsilon} \\ &= \frac{1}{\mathcal{Z}} \sum_{p,q\geqslant 0} \left[\sum_{l=0}^q \frac{q!}{l!(q-l)!}\right] \frac{\lambda^q \mu^p}{p!q!} \int M_\beta^{\otimes p+q} \varphi_0^{\otimes q} \mathbb{1}_{\mathcal{X}_{p+q}^\varepsilon}. \end{aligned}$$

Hence, we use that $\mathbb{1}_{\mathcal{X}_{p+q}^\varepsilon} \leq \mathbb{1}_{\mathcal{X}_{p+q-l}^\varepsilon}$ and invert the sums, eventually computing an index shift to get

$$\begin{aligned} \mathbb{E}\left[2^{|\mathcal{S}_\lambda|}\right] &\leq \frac{1}{\mathcal{Z}} \sum_{l\geqslant 0} \frac{(\lambda\|\varphi_0\|)^l}{l!} \sum_{\substack{p\geqslant 0\\ q\geqslant l}} \frac{\lambda^{q-l}\mu^p}{p!(q-l)!} \int M_\beta^{\otimes p+q-l} \varphi_0^{\otimes q-l} \mathbb{1}_{\mathcal{X}_{p+q-l}^\varepsilon} \\ &\leq \sum_{l=0}^q \frac{(\lambda\|\varphi_0\|)^l}{l!} \cdot \frac{1}{\mathcal{Z}} \sum_{p,q\geqslant 0} \frac{\lambda^q \mu^p}{p!q!} \int M_\beta^{\otimes p+q} \varphi_0^{\otimes q} \mathbb{1}_{\mathcal{X}_{p+q}^\varepsilon} \\ &= e^{\lambda\|\varphi_0\|}, \end{aligned}$$

recognizing the partition function $\mathcal{Z}$. In the end,

$$\mathbb{P}\left[\sup_{s\in[0,t]} d(0,\tilde{\pi}_t^\varepsilon) \geq A\right] \leq \exp\Big(\lambda\|\varphi_0\| - C\log 2A\lambda\Big),$$

which concludes the proof of the first statement in the mixed scaling ($S_{\varepsilon,\mu,\lambda}$).

The second statement (8.2) follows from the previous bound and the estimate on the cumulants given in Proposition 6.2.1, with the same proof as in [9, Proposition 7.3.2]. □

Proposition 8.1.2 (Tightness of the fluctuation field). *There exists a distance $\tilde{d}$, based on some normalized bounded observables, and associated to the strong Skorokhod topology on* $\mathrm{Traj}([0,t],\mathcal{M}(\mathcal{D}))$, *such that*

$$\lim_{A\to+\infty} \lim_{\mu\to+\infty} \mathbb{P}\left[\sup_{s\in[0,t]} \tilde{d}(0,\zeta_s^\varepsilon) \geq A\right] = 0$$

and for any $\eta > 0$,

$$\lim_{\delta\to 0} \lim_{\mu\to+\infty} \mathbb{P}\left[\sup_{|s-s'|<\delta} \tilde{d}(\zeta_s^\varepsilon, \zeta_{s'}^\varepsilon) > \eta\right] = 0. \tag{8.3}$$

The proof of this proposition is once again identical to the one in [9, Proposition 6.2.3], using the bounds proved in Proposition 6.2.1. Note that in the integrated form, dominating roughly the bounded observables, it is easy to remark that the bounding estimates that are true in the non-linear symmetric case remain true in our linear tagged model, since by symmetry one can rewrite the sum over the tags $\underline{\ell}_p$ as

$$\sum_{\underline{\ell}_p\in\Lambda_p} \lambda^{|\underline{\ell}_p|}\mu^{p-|\underline{\ell}_p|}\int M_\beta^{\otimes p}\varphi_0^{\otimes\underline{\ell}_p}\mathbb{1}_{\mathcal{X}_p^\varepsilon}\mathrm{d}\underline{z}_p = \mu^p\int M_\beta^{\otimes p}\left(\frac{\lambda}{\mu}\varphi_0+1\right)^{\otimes p}\mathbb{1}_{\mathcal{X}_p^\varepsilon}\mathrm{d}\underline{z}_p,$$

which formally corresponds to a bounded symmetric initial data $M_\beta(p_\mu\varphi_0+1)$.

8.2 Convergence of the fluctuation field

Since the fluctuation field (ζ_t^ε) defined in (4.2) is tight by Proposition 8.1.2, identifying its limit moments is enough to characterize its limit, as they decrease fast enough (see the method of moments in [6, Theorem 30.1]). Hence, to identify the limit fluctuation field with a Gaussian process, and to find its covariance, we consider the following sampled observable

$$H(z^{[0,t]},\ell) = \mathbb{1}_{\ell=1}\sum_{j=1}^{J}\psi_j(z^{[\theta_j]}).$$

For simplicity, we denote $\tilde{f}_n(t) \doteq f_n(t,\underline{\ell}_n=\underline{1}_n)$ and $\tilde{F}_1(t) \doteq F_1(t,\ell_1=1)$ the cumulants associated to the tagged particles only. Recall that the random set $\mathcal{S}_\lambda$ contains the labels of tagged particles. With the same notation as in Corollary 3.1.1, we have

$$\zeta_t^\varepsilon[H] = \frac{1}{\sqrt{\lambda}}\sum_{j=1}^{J}\left[\sum_{i\in\mathcal{S}_\lambda}\psi_j\left(Z_{\varepsilon,i}^{[\theta_j]}\right)-\lambda\int\tilde{F}_1^\varepsilon(\theta_j)\psi_j\right].$$

To characterize the law of this fluctuation field, we look at its Fourier transform, using the generalized cumulant generating function (5.4) to write

$$\mathbb{E}\left[e^{i\zeta_t^\varepsilon[H]}\right] = \exp\left(\mathfrak{G}_\varepsilon^{[0,t]}\left[\frac{iH}{\sqrt{\lambda}}\right]\right)\exp\left(-i\sqrt{\lambda}\sum_{j=1}^{J}\int\tilde{F}_1^\varepsilon(\theta_j)\psi_j\right). \tag{8.4}$$

Expanding the cumulant generating function (5.4) yields

$$\begin{aligned}\mathfrak{G}_\varepsilon^{[0,t]}\left[\frac{iH}{\sqrt{\lambda}}\right] &= \sum_{p\geqslant1}\frac{\lambda^p}{p!}\int\tilde{f}_p^\varepsilon\left[e^{\frac{i}{\sqrt{\lambda}}H}-1\right]\\ &= \sum_{p\geqslant1}\frac{\lambda^p}{p!}\int\tilde{f}_p^\varepsilon\left[\frac{i}{\sqrt{\lambda}}H-\frac{1}{2\lambda}H^2+O\left(\frac{\|H\|}{\sqrt{\lambda}}\right)^3\right].\end{aligned}$$

Using the estimates on the cumulants stated in Proposition 6.2.1, and the convexity of $x\mapsto(1+x)^p$, we bound for any $p\geqslant2$

$$\frac{\lambda^p}{p!}\left|\int\tilde{f}_p^\varepsilon\left[\frac{i}{\sqrt{\lambda}}H+O\left(\frac{\|H\|}{\sqrt{\lambda}}\right)^2\right]\right|\leq\frac{\lambda^pC^p}{\mu^{p-1}}\cdot\left[\frac{\|H\|^p}{\lambda^{\frac{p}{2}}}+2^pO\left(\frac{\|H\|}{\sqrt{\lambda}}\right)^{p+1}\right],$$

so that

$$\sum_{p\geqslant 2}\frac{\lambda^p}{p!}\left|\int \tilde f_p^\varepsilon\left[e^{\frac{i}{\sqrt{\lambda}}H}-1\right]\right|\leq 2(C\|H\|)^2\frac{\lambda}{\mu}.$$

Then the cumulant generating function writes

$$\mathfrak{G}_\varepsilon^{[0,t]}\left[\frac{iH}{\sqrt{\lambda}}\right]=\int \tilde F_1^\varepsilon\left[i\sqrt{\lambda}H-\frac{H^2}{2}+O\left(\frac{\|H\|}{\sqrt{\lambda}}\right)^3\right]+2(C\|H\|)^2\frac{\lambda}{\mu}$$

where the first term simplifies in the Fourier transform (8.4). Eventually, in the mixed scaling ($S_{\varepsilon,\mu,\lambda}$) this provides the following convergence

$$\mathbb{E}\left[e^{i\zeta_t^\varepsilon[H]}\right]\xrightarrow[\varepsilon\to 0]{}\exp\left(-\frac{1}{2}\sum_{i,j=1}^J\int \tilde F_1\left[\psi_i(z^{[\theta_i]})\psi_j(z^{[\theta_j]})\right]\right)$$

thanks to the convergence of the integrated cumulants shown in Proposition 7.3.1. By the method of moments mentioned above, and thanks to the tightness (8.3) of the fluctuation field, ζ_t^ε converges to a Gaussian process ζ_t, with time-observable covariance

$$\mathbb{E}\Big[\zeta_s[g]\zeta_t[h]\Big]=\int \tilde F_1\left[g(z^{[s]})h(z^{[t]})\right].$$

At each time, it is hence characterized by its covariance (4.3) with respect to observables. This concludes the proof of Theorem 4.1, recognizing the expansion (7.1) of the solution $M\varphi$ to the Rayleigh–Boltzmann equation. □

8.3 Large deviations of the empirical measure

We explain here how the convergence of the cumulant generative function (Sections 7.4 and 7.5) leads to the large deviation principle for the empirical measure (Theorem 4.2), following the method of [9].

Upper bound Let $\mathbf{F}$ be a closed set for the Skorokhod topology. In particular, $\mathbf{F}$ is also closed for the weaker topology given by the opens of the form

$$\mathbf{O}_{h,\delta}(\mathfrak{v})\doteq\Big\{\mathfrak{m}\in \mathrm{Traj}([0,t],\mathcal{M}(\mathcal{D}))\ ,\ |\{h,\mathfrak{m}-\mathfrak{v}\}|<\delta\Big\},$$

for observables $h\in\mathcal{C}_c^\infty$ and measures $\mathfrak{v}\in\mathrm{Traj}([0,t],\mathcal{M}(\mathcal{D}))$, where $\delta>0$ is fixed (we recall the definition (4.4) of the transport filtered mean). For any open set of this form, one can write

$$\mathbb{P}\big(\tilde\pi_t^\varepsilon\in\mathbf{O}_{h,\delta}(\mathfrak{v})\big)\leq\mathbb{E}\left[\exp\Big(\lambda\{h,\tilde\pi_t^\varepsilon\}-\lambda\{h,\mathfrak{v}\}+\lambda\delta\Big)\right].$$

Now, since by definition

$$\{h,\tilde\pi_t^\varepsilon\}=\frac{1}{\lambda}\sum_{i\in\mathcal{S}_\lambda}\left[h\left(t,Z_i^{[t]}\right)+\int_0^t(\partial_s+v\cdot\nabla_x)h\left(s,Z_i^{[s]}\right)\right],$$

and harnessing the definition (5.4) of the cumulant generating function, we get

$$\mathbb{P}\big(\tilde\pi_t^\varepsilon\in\mathbf{O}_{h,\delta}(\mathfrak{v})\big)\leq\exp\left(\mathfrak{G}_\varepsilon^{[0,t]}\left[h_t+\int(\partial_s+v\cdot\nabla_x)h_s\right]-\lambda\{h,\mathfrak{v}\}+\lambda\delta\right).$$

Taking the limit superior, thanks to the convergence (7.2) of the generating function, and using the notation $\mathcal{I}(t,g)$ for the limit cumulant generating function (7.3), one has

$$\limsup_{\varepsilon\to 0} \frac{1}{\lambda} \log \mathbb{P}\big(\tilde{\pi}_t^\varepsilon \in \mathbf{O}_{h,\delta}(\mathfrak{v})\big) \le \mathcal{I}(t,h) - 1 - \{h, \mathfrak{v}\} + \delta.$$

By definition of the Legendre transform (4.7), for our fixed $\delta > 0$ there exists an observable $g \in \mathbb{B}_{t,\beta}$ such that

$$\{g, \mathfrak{v}\} - \mathcal{I}(t,g) + 1 > \mathbf{\Lambda}(t, \mathfrak{v}) - \delta,$$

so that

$$\limsup_{\varepsilon\to 0} \frac{1}{\lambda} \log \mathbb{P}\big(\tilde{\pi}_t^\varepsilon \in \mathbf{O}_{g,\delta}(\mathfrak{v})\big) \le -\mathbf{\Lambda}(t,\mathfrak{v}) + 2\delta.$$

By the tightness result stated in Proposition 8.1.1, $\mathbf{F}$ is in fact compact, and one can extract a finite covering of open sets $\mathbf{F} \subset \cup_{i=1}^k \mathbf{O}_{g_i,\delta}(\mathfrak{v}_i)$, so that

$$\limsup_{\varepsilon\to 0} \frac{1}{\lambda} \log \mathbb{P}\big(\tilde{\pi}_t^\varepsilon \in \mathbf{F}\big) \le -\inf_{i\leqslant k} \mathbf{\Lambda}(t, g_i) + 2\delta \le -\inf_{\mathfrak{v}\in\mathbf{F}} \mathbf{\Lambda}(t,\mathfrak{v}) + 2\delta.$$

The result (4.9) follows, considering that $\delta > 0$ may be chosen arbitrarily small.

Lower bound The lower bound follows from more elaborate methods, that works only for measures $\mathfrak{v}$ such that the Legendre transform $\mathbf{\Lambda}(t,\mathfrak{v})$ is reached for an observable $h \in \mathbb{B}_{t,\beta}$, which is the case when $\mathfrak{v}$ is a strong solution of the Boltzmann–Hamilton–Jacobi equation (7.7)

$$(\partial_s - v\cdot\nabla_x)\mathfrak{v} = \int \mathrm{d}v_2 \mathrm{d}\omega \langle v_1 - v_2, \omega\rangle_+ \left(M_\beta(v_2')\mathfrak{v}(v_1') e^{p(z_1)-p(z_1')} - M_\beta(v_2)\mathfrak{v}(v_1) e^{p(z_1')-p(z_1)} \right),$$

for some $p \in \mathbb{B}_{t,\beta}$, whence the restriction on the infimum defining the lower bound. The algebraic details are exaclty the same as in [9, Chapter 7] and proceed of industrious topological considerations, so that we do not replicate them here. ☐

Appendix

A Scattering parametrization

This appendix is dedicated to the parametrizations of the scattering. Indeed, the scattering system (2.2), denoting $(v,w) \doteq (v_i, v_j)$, is equivalent to

$$\begin{cases} v' = \dfrac{v+w}{2} + \sigma\dfrac{|v-w|}{2} \\ w' = \dfrac{v+w}{2} - \sigma\dfrac{|v-w|}{2}, \end{cases} \tag{A.1}$$

with $\sigma = \dfrac{v_i{}' - v_j{}'}{|v_i{}' - v_j{}'|}$ denoting the deviation's direction with respect to the mean velocity. Since (v, w, v', w') are in the same plane, σ and ω also belong to the same plane (see Fig. 6). Thus, we might reduce their dependency to the 1-dimensional circle, where the following holds

$$\sigma = \pi - 2\omega,$$

σ going through the circle, and ω through the half-circle.

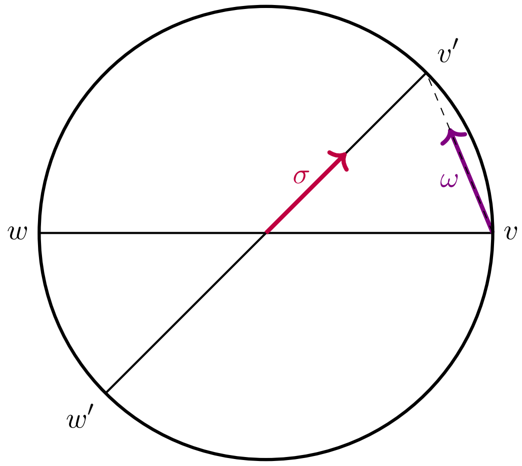

Figure 6: Deflection parameters

Change of variable $\omega \mapsto \sigma$. Let us denote $u = w - v$ the precollisional relative velocity. We want to change the variable $\omega \in \mathbb{S}^{d-1}_+$ into $\sigma \in \mathbb{S}^{d-1}$ in the following integral, written in hyperspherical coordinates

$$\int_{\mathbb{S}^{d-1}_+} f(\omega)\langle \omega, u\rangle \mathrm{d}\omega = \int f(\omega)|u|\cos(\theta_{d-2})\sin^{d-2}\theta_1 \sin^{d-3}\theta_2 \dots \sin^2\theta_{d-3}\sin\theta_{d-2}\, \mathrm{d}\underline{\theta}_{d-1},$$

where the $d-2$ first angles $(\theta_i)_{i\leqslant d-2}$ range $[0,\pi]$, and θ_{d-1} ranges $[0,2\pi]$. Thus, choosing the angle $\theta \doteq \widehat{(u,\omega)}$ as θ_{d-2}, the half-sphere that ω ranges corresponds to $\theta \in [0,\pi/2]$. In Fig. 6, one can see that $\widehat{(u,\sigma)} = \pi - 2\widehat{(u,\omega)}$, so that we will compute the natural change of variable $\theta = (\pi - \phi)/2$. Denoting $\check{\theta} = (\theta_1, \dots, \theta_{d-3}, \theta_{d-1})$, we get by trigonometry formulas that

$$\begin{aligned}
\int_{\mathbb{S}^{d-1}_+} f(\omega)\langle \omega, u\rangle \mathrm{d}\omega &= \int_0^\pi \int f(\omega)\frac{|u|}{2}\sin\left(\frac{\phi}{2}\right)\cos\left(\frac{\phi}{2}\right)\sin^{d-2}\theta_1 \sin^{d-3}\theta_2 \dots \sin^2\theta_{d-3}\, \mathrm{d}\check{\theta}\mathrm{d}\phi \\
&= \int_0^\pi \int f(\omega)\frac{|u|}{4}\sin(\phi)\sin^{d-2}\theta_1 \sin^{d-3}\theta_2 \dots \sin^2\theta_{d-3}\, \mathrm{d}\check{\theta}\mathrm{d}\phi \\
&= \frac{|u|}{4}\int_{\mathbb{S}^{d-1}} f(\omega)\mathrm{d}\sigma.
\end{aligned}$$

It is useful in the computations of this paper to be able to switch from a variable to another, with the following formula.

Lemma A.0.1 (Change of scattering angle)**.** *For any measurable function f, one has*

$$\int_{\mathbb{S}^{d-1}_+} f(\omega)\langle \omega, u\rangle \mathrm{d}\omega = \frac{|u|}{4}\int_{\mathbb{S}^{d-1}} f(\omega)\mathrm{d}\sigma. \tag{A.2}$$

B Cumulants

B.1 Combinatorics of the inversion formula

Before starting this appendix presenting the specificities of the cumulants of the exclusion indicator, we remark that the identity (4.13) at the origin of the inversion formula (4.12) for the cumulants can be seen through combinatorial methods. We recall this combinatorial formula in the following lemma.

Lemma B.1.1. *For any $r \geqslant 1$, one has*

$$\sum_{k=1}^{r+1}(-1)^k \sum_{\sigma\in\mathcal{P}_{r+1}^k}\prod_{i=1}^{k}(|\sigma_i|-1)! = 0. \tag{B.1}$$

Proof. We start introducing the following mapping

$$\begin{array}{ccc} \mathcal{P}_{r+1}^k & \longrightarrow & \mathcal{P}_r^{k-1} \amalg \mathcal{P}_r^k \\ \{\sigma_1,\ldots,\sigma_k\} & \mapsto & \{\breve{\sigma}_1,\ldots,\breve{\sigma}_k\} \end{array}$$

where $\breve{\sigma}_i = \sigma_i \setminus \{r+1\}$ denotes the subsets of σ from which we have removed $(r+1)$. Inverting this mapping, for a $\sigma \in \mathcal{P}_r^k$, the element $(r+1)$ must form a partition subset on its own; and otherwise for a $\sigma \in \mathcal{P}_r^{k-1}$, we have r choices for the subset $(r+1)$ belongs to. This way, one can write

$$\begin{aligned}\sum_{\sigma\in\mathcal{P}_{r+1}^k}\prod_{i=1}^{k}(|\sigma_i|-1)! &= \sum_{\sigma\in\mathcal{P}_r^k}\prod_{i=1}^{k}(|\sigma_i|-1)! + \sum_{j=1}^{k-1}\sum_{\sigma\in\mathcal{P}_r^{k-1}}|\sigma_j|\prod_{i=1}^{k-1}(|\sigma_i|-1)! \\ &= \sum_{\sigma\in\mathcal{P}_r^k}\prod_{i=1}^{k}(|\sigma_i|-1)! + r\sum_{\sigma\in\mathcal{P}_r^{k-1}}\prod_{i=1}^{k-1}(|\sigma_i|-1)!\ .\end{aligned}$$

Thus, decomposing as above the sum (B.1), we can perform an index shift on k to gather it back (the extreme terms, summing over $\mathcal{P}_r^{r+1}$ and $\mathcal{P}_r^0$, vanish). We get

$$\begin{aligned}\sum_{k=1}^{r+1}(-1)^k\sum_{\sigma\in\mathcal{P}_{r+1}^k}\prod_{i=1}^{k}(|\sigma_i|-1)! &= (1-r)\sum_{k=1}^{r}(-1)^k\sum_{\sigma\in\mathcal{P}_r^k}\prod_{i=1}^{k}(|\sigma_i|-1)! \\ &= (1-r)(2-r)\times\cdots\times 0\times\sum_{k=1}^{1}(-1)^1\times 0!\end{aligned}$$

when iterating the computation, which concludes the proof. □

B.2 Cumulants of the exclusion

This section is devoted to the study of the cumulants of the exclusion indicator associated with the hard sphere domain (2.1) (see Section 4.2.1 for the definition of the cumulants). The exclusion between particles may be generalized to the exclusion between aggregates and clusters (see Section 5.2), yet the following results and computations remain identical.

Denoting $\mathcal{G}_S$ the set of graphs on a set S and $\mathcal{G}_n \doteq \mathcal{G}_{[\![1,n]\!]}$ the set of graphs on $\{1,\ldots,n\}$, one may write

$$\begin{aligned}\mathbb{1}_{\mathcal{X}_n^\varepsilon}(x_1,\ldots,x_n) &= \prod_{1\leq i<j\leq n}(1-\mathbb{1}_{x_i\sim x_j}) \\ &= \sum_{G\in\mathcal{G}_n}\prod_{\{i,j\}\in E_G}(-\mathbb{1}_{x_i\sim x_j}).\end{aligned}$$

One can even further decompose the graphs into their connected components. We denote $\mathcal{C}_S \subset \mathcal{G}_S$ the set of *connected* graphs on S and $\mathcal{P}_n$ the set of partitions of $\{1,\ldots,n\}$, so that we can write

$$\mathbb{1}_{\mathcal{X}_n^\varepsilon}(x_1,\ldots,x_n) = \sum_{\sigma\in\mathcal{P}_n}\prod_{k=1}^{|\sigma|}\left(\sum_{G_k\in\mathcal{C}_{\sigma_k}}\prod_{\{i,j\}\in E_{G_k}}(-\mathbb{1}_{x_i\sim x_j})\right).$$

Hence, thanks to the uniqueness due to the inversion formula (Proposition 4.2.1), the cumulants of the exclusion are given by

$$\phi_k(x_1, \ldots, x_k) \doteq \sum_{G \in \mathcal{C}_k} \prod_{\{i,j\} \in E_G} (-\mathbb{1}_{x_i \sim x_j}). \tag{B.2}$$

The very specific structure of these cumulants yields a strong bound on them, called the tree inequality, exposed in the following Proposition. This bound on the integral of the cumulants allows a strong control of the particle correlations, used in Appendix C to gain estimates on the partition function, and in Section 6.2 to bound the initial cumulants.

Proposition B.2.1 (Tree inequality)**.**

(i) The modulus of the cumulants may be controlled restricting the sum defining them to the trees $\mathcal{T}_k \subset \mathcal{C}_k$ (i.e. to the minimally connected graphs) as such:

$$|\phi_k(x_1, \ldots, x_k)| \leq \sum_{T \in \mathcal{T}_k} \prod_{\{i,j\} \in E_T} \mathbb{1}_{x_i \sim x_j}.$$

(ii) As a consequence, we have the following control over their integral

$$\int |\phi_k(\underline{x}_k)| \mathrm{d}\underline{x}_k \leq k^{k-2} (|\mathcal{B}_d| \varepsilon^d)^{k-1}.$$

Proof. The proof of this proposition relies on a partition scheme due to Penrose [30], and may also be found in [9].

1. The key argument is to find a map $\pi : \mathcal{C}_k \to \mathcal{T}_k$ such that for any tree $T \in \mathcal{T}_k$, there is a connected graph $R(T) \in \mathcal{C}_k$ satisfying

   $$\pi^{-1}(T) = \{G \in \mathcal{C}_k \ , \ E_T \subset E_G \subset E_{R(T)}\}.$$

   This means that we can partition $\mathcal{C}_k$ into subsets corresponding each to a single tree, and containing all the graphs that are both compatible with this tree and smaller than an upper graph $R(T)$. We will prove later the existence of such a partition.

   Now, we decompose the sum defining the k-th cumulant according to this mapping, and using its structure we get

   $$\begin{aligned}
   \sum_{G \in \mathcal{C}_k} \prod_{\{i,j\} \in E_G} (-\mathbb{1}_{i \sim j}) &= \sum_{T \in \mathcal{T}_k} \sum_{G \in \pi^{-1}(T)} \prod_{\{i,j\} \in E_G} (-\mathbb{1}_{i \sim j}) \\
   &= \sum_{T \in \mathcal{T}_k} \left(\prod_{\{i,j\} \in E_T} (-\mathbb{1}_{i \sim j}) \right) \left(\sum_{E' \subset E_{R(T)} \setminus E_T} \prod_{\{i,j\} \in E'} (-\mathbb{1}_{i \sim j}) \right) \\
   &= \sum_{T \in \mathcal{T}_k} \left(\prod_{\{i,j\} \in E_T} (-\mathbb{1}_{i \sim j}) \right) \left(\prod_{\{i,j\} \in E_{R(T)} \setminus E_T} (1 - \mathbb{1}_{i \sim j}) \right)
   \end{aligned}$$

   reversing the usual computation to harness the cancellations, which yields the result since $1 - \mathbb{1}_{i \sim j} \in \{0, 1]\}$.

   Let us now expose the Penrose partition scheme. To construct the representative $T \in \mathcal{T}_k$ of a connected graph G, we will choose the vertex 1 as its root, and explore the graph from there, always favoring the lowest vertices when choosing a new edge. Formally, we proceed by generations of same distance to the root $d(1,\cdot)$, on the tree in construction. At each generation, starting from the currently connected vertices, we add the edges of E_G connecting these vertices to isolated ones, and whenever a conflict happens—which would create a cycle—the lowest currently connected vertex has priority to connect with the new one. This iterated procedure leads to a tree T such that $E_T \subset E_G$.

Then, to construct $R(T)$, we search an algorithm to retrieve some information on E_G from the knowledge of T. For each edge $\{i,j\} \in E_G \setminus E_T$, in the procedure to construct T, at the generation when the first among i and j was connected to the root 1 (let us say i), then either j gets connected at the same generation, and then $d(1,i) = d(1,j)$, either it gets connected at the next generation, since they are connected in G. Since we supposed $\{i,j\} \notin E_T$, it precisely means that some other vertex—within the generation of i—had priority to claim j. Hence, in that case, $d(1,j) = d(1,i) + 1$ and the parent $p(j)$ of j is lower than i.

A good candidate for $R(T)$ is thus T at which we add the vertices $\{i,j\}$ such that $d(1,i) = d(1,j)$, or such that $d(1,i) = d(1,j) + 1$ when $p(j) < i$. One may check that with this choice, every graph G satisfying $E_T \subset E_G \subset E_{R(T)}$ has T as its tree representative, which concludes the construction we needed.

2. By the first part of the proposition, the problem is reduced to the integration of the vertices of a tree, for which we know a very simple algorithm. We start integrating over a leaf i_ℓ of T, which appears in only one condition $\mathbb{1}_{i_\ell \sim j}$, and we then iterate for a new leaf until the tree is reduced to a single root, so that

$$\begin{aligned}\int |\phi_k| &\leq \int \sum_{T \in \mathcal{T}_k} \prod_{\{i,j\} \in E_T} \mathbb{1}_{i \sim j} \mathrm{d}\underline{x}_k \\ &\leq \sum_{T \in \mathcal{T}_k} |\mathcal{B}_d| \varepsilon^d \int \prod_{\{i,j\} \in E_{T \setminus \{i_\ell\}}} \mathbb{1}_{i \sim j} \mathrm{d}x_1 \dots \mathrm{d}\check{x}_{i_\ell} \dots \mathrm{d}x_k \\ &\leq \sum_{T \in \mathcal{T}_k} (|\mathcal{B}_d| \varepsilon^d)^{k-1} \\ &= k^{k-2} (|\mathcal{B}_d| \varepsilon^d)^{k-1},\end{aligned}$$

concluding with Cayley's formula, that gives the number of trees on $[\![1,k]\!]$. ☐

C Partition function estimates

This section is dedicated to the proof of Proposition 3.4.1, which yields the following estimate on the partition function, for μ large enough in our mixed low density scaling ($S_{\varepsilon,\mu,\lambda}$):

$$\frac{1}{\mathcal{Z}_\mu} \sum_{q,r \geqslant 0} \frac{(\lambda C_0)^q \mu^r}{q! r!} \int \mathbb{1}_{\mathcal{X}^\varepsilon_{q+r}} \leq C_d^{C_0 \lambda}.$$

First of all, let us use the symmetry of the particles (among each labelled group) to observe that the partition function can be rewritten:

$$\begin{aligned}\mathcal{Z}_\mu &= \sum_{p\geqslant 0}\sum_{\underline{\ell}_p\in\Lambda_p}\frac{\lambda^{|\underline{\ell}_p|}\mu^{p-|\underline{\ell}_p|}}{p!}\int (M_\beta\varphi_0)^{\otimes \underline{\ell}_p}(\underline{z}_{\underline{\ell}_p})\mathbb{1}_{\mathcal{X}_p^\varepsilon}(\underline{x}_p)\mathrm{d}\underline{x}_p\mathrm{d}\underline{v}_{\underline{\ell}_p}\\ &= \sum_{p\geqslant 0}\sum_{k=0}^{p}\frac{\lambda^k\mu^{p-k}}{k!(p-k)!}\int (M_\beta\varphi_0)^{\otimes k}(\underline{z}_k)\mathbb{1}_{\mathcal{X}_p^\varepsilon}(\underline{x}_p)\mathrm{d}\underline{x}_p\mathrm{d}\underline{v}_k \qquad \text{(C.1)}\\ &= \sum_{p,q\geqslant 0}\frac{\lambda^q\mu^p}{q!p!}\int (M_\beta\varphi_0)^{\otimes q}(\underline{x}_q)\mathbb{1}_{\mathcal{X}_{p+q}^\varepsilon}(\underline{x}_{p+q})\mathrm{d}\underline{x}_{p+q}\mathrm{d}\underline{v}_q. \qquad \text{(C.2)}\end{aligned}$$

We now start by proving the computational lemma below, giving an explicit and exact formulation of the partition function using the cumulants of the exclusion, defined in the Appendix B.2 above.

Lemma C.0.1. *For all $\lambda,\mu>0$, with (ϕ_k) the cumulants associated to the exclusion $(\mathbb{1}_{\mathcal{X}_k^\varepsilon})$, one has*

$$\mathcal{Z}_\mu = \exp\left(\sum_{(k,l)\neq(0,0)}\frac{\mu^k\lambda^l}{k!\ l!}\int (M_\beta\varphi_0)^{\otimes l}\phi_{k+l}\right). \qquad \text{(C.3)}$$

Proof. We start from the formulation (C.2) of the partition function, and we will write the following cumulant expansion, denoting $\mathcal{P}_{p+q}^s$ the partitions of $[\![1,p+q]\!]$ into s subsets,

$$\begin{aligned}\mathcal{Z}_\mu &= 1+\sum_{p,q\neq 0,0}\frac{\mu^p\lambda^q}{p!q!}\sum_{s=1}^{p+q}\sum_{\sigma\in\mathcal{P}_{p+q}^s}\int\prod_{k=1}^{s}\phi_{|\sigma_k|}(\underline{x}_{\sigma_k})(M_\beta\varphi_0)^{\otimes q}(\underline{x}_q)\mathrm{d}\underline{x}_{p+q}\\ &= 1+\sum_{p,q\neq 0,0}\frac{\mu^p\lambda^q}{p!q!}\sum_{s=1}^{p+q}\sum_{\substack{(k_i,l_i)\neq(0,0)\\ \sum k_j=p,\sum l_j=q}}\frac{1}{s!}\frac{p!}{k_1!\cdots k_s!}\frac{q!}{l_1!\cdots l_s!}\int\prod_{i=1}^{s}\phi_{k_i+l_i}(\underline{x}_{k_i+l_i})(M_\beta\varphi_0)^{\otimes l_i}(\underline{x}_{l_i})\mathrm{d}\underline{x}_{k_i+l_i}.\end{aligned}$$

To perform the last equality, denoting $P=[\![1,p]\!]$ and $Q=[\![p+1,p+q]\!]$, we consider the following surjection that counts the number of elements from each partition subset in both P and Q, defined up to an arbitrary choice of a partition order:

$$\Phi: \begin{array}{ccc}\mathcal{P}_{p+q}^s & \longrightarrow & \left\{(\underline{k}_s,\underline{\ell}_s)\ \middle|\ (k_i,l_i)\neq(0,0),\sum k_j=p,\sum l_j=q\right\}\\ \sigma & \mapsto & (|\sigma_1\cap P|,\ldots,|\sigma_s\cap P|,|\sigma_1\cap Q|,\ldots,|\sigma_s\cap Q|).\end{array}$$

Its defect of injectivity is indeed given by

$$\sharp\left\{\sigma\in\mathcal{P}_{p+q}^s\ ,\ \Phi(\sigma)=(\underline{k}_s,\underline{\ell}_s)\right\}=\frac{1}{s!}\cdot\frac{p!}{k_1!\cdots k_s!}\cdot\frac{q!}{l_1!\cdots l_s!},$$

providing the combinatorial factors. Now, permuting the sums so as to get rid of the condition on the value of the index sums, we get

$$\begin{aligned}\mathcal{Z}_\varepsilon &= 1+\sum_{s\geqslant 1}\frac{1}{s!}\prod_{i=1}^{s}\left(\sum_{(k_i,l_i)\neq(0,0)}\frac{\mu^{k_i}}{k_i!}\frac{\lambda^{l_i}}{l_i!}\int\phi_{k_i+l_i}(\underline{x}_{k_i+l_i})(M_\beta\varphi_0)^{\otimes l_i}(\underline{x}_{l_i})\mathrm{d}\underline{x}_{k_i+l_i}\right) \qquad \text{(C.4)}\\ &= \exp\left(\sum_{(k_i,l_i)\neq(0,0)}\frac{\mu^{k_i}\lambda^{l_i}}{k_i!l_i!}\int\phi_{k_i+l_i}\cdot(M_\beta\varphi_0)^{\otimes l_i}\right),\end{aligned}$$

concluding the proof. □

Proof of the proposition. Now, Lemma C.0.1 applied to both the numerator and the denominator – that have the same structure – gives

$$\frac{1}{\mathcal{Z}_\mu}\sum_{p,q\geqslant 0}\frac{\mu^p(\lambda C_0)^q}{p!q!}\int \mathbb{1}_{\mathcal{X}^\varepsilon_{p+q}} = \exp\left(\sum_{(k,l)\neq 0,0}\frac{\mu^k\lambda^l}{k!\,l!}\int \phi_{k+l}\cdot\left[C_0^l-(M_\beta\varphi_0)^{\otimes l}\right]\right).$$

The terms of the sum are vanishing for $l=0$, so that

$$\begin{aligned}\left|\sum_{(k,l)\neq 0,0}\frac{\mu^k\lambda^l}{k!\,l!}\int \phi_{k+l}\cdot\left[C_0^l-(M_\beta\varphi_0)^{\otimes l}\right]\right| &\leq \sum_{\substack{k\geqslant 0\\ l\geqslant 1}}\frac{\mu^k\lambda^l}{k!\,l!}\int|\phi_{k+l}|\cdot 2C_0^l\\ &\leq 2\sum_{r\geqslant 1}\sum_{l=1}^{r}\frac{\mu^{r-l}(\lambda C_0)^l}{(r-l)!\,l!}\int|\phi_r|\\ &\leq 2\sum_{r\geqslant 1}\sum_{l=1}^{r}\frac{\mu^{r-l}(\lambda C_0)^l}{(r-l)!\,l!}r^{r-2}(|\mathcal{B}_d|\varepsilon^d)^{r-1},\end{aligned}$$

by Proposition B.2.1 stemming from the tree inequality. Using Stirling's approximation, we have for any $l\leq r$

$$\frac{r^{r-2}}{(r-l)!\,l!}\leq\frac{e^r}{r^2}\cdot\frac{r!}{(r-l)!\,l!}\leq\frac{(2e)^r}{r^2},$$

so that

$$\left|\sum_{(k,l)\neq 0,0}\frac{\mu^k\lambda^l}{k!\,l!}\int \phi_{k+l}\cdot\left[C_0^l-(M_\beta\varphi_0)^{\otimes l}\right]\right|\leq 4eC_0\lambda\sum_{r\geqslant 1}\sum_{l=1}^{r}\frac{\mu^{r-l}(\lambda C_0)^{l-1}}{r^2}(2e|\mathcal{B}_d|\varepsilon^d)^{r-1}.$$

Eventually, thanks to the scaling $\mu\varepsilon^{d-1}=1$ and $1\ll\lambda\ll\mu$, we have

$$\begin{aligned}\sum_{r\geqslant 1}\sum_{l=1}^{r}\frac{\mu^{r-l}(\lambda C_0)^{l-1}}{r^2}(2e|\mathcal{B}_d|\varepsilon^d)^{r-1} &\leq \sum_{r\geqslant 1}\sum_{l=1}^{r}\frac{\mu^{r-1}}{r^2}(2e|\mathcal{B}_d|\varepsilon^d)^{r-1}\\ &\leq\sum_{r\geqslant 1}(2e|\mathcal{B}_d|\varepsilon)^{r-1}\\ &\leq 2,\end{aligned}$$

concluding the proof. □

D Geometric control of dynamics cycles

We prove here Proposition 7.2.1, adding the analysis of geometric cycle constraints to the bounds proved in Section 6.2. Thanks to a fine computation to handle the appearing singularities (Section D.4), we achieve a full convergence rate in ε, better than the previous existing one [8].

D.1 Parametrizing the cycle

First of all, it will be useful in the following to parametrize the encounters in terms of the angle σ instead of ω (see Appendix A for the associated change of variables). For a general cluster $R=\rho_j$, at fixed number of added particles k_j and fixed labels, we want to show that the cycle condition in the following integral

$$\sum_{\Upsilon \in \mathcal{T}^{\mathrm{d}}_{R,k}} \int \mathbb{1}_{T^{\mathrm{d}}_R = \Upsilon} \mathrm{d}\underline{v}_{r+k} \mathrm{d}\underline{\sigma}_{E_\Upsilon} \mathrm{d}\underline{\tau}_{E_\Upsilon} \prod_{e \in E_\Upsilon} s_e |v_m(\tau_e{}^+) - v_{m'}(\tau_e{}^+)| e^{-\beta \|\underline{v}_{r+k}\|^2} \times \mathbb{1}^{\between}_R \tag{D.1}$$

leads to similar bounds as without the cycle constraint (see (6.7)), with an additional small factor ε. This way, we will use the same computation as in Section 6.2, when bounding the integrated cumulants, gaining smallness from the cycle.

We have denoted $\underline{v}_{r+k} = (\underline{v}_R, \underline{v}^*_k)$, and we define $\mathbf{V} = \max(1, \|\underline{v}_{r+k}\|_2)$ to control the total energy. For simplicity, we will not precise in all the following, every time we use an adverb of time, that time is looked at backwards.

Let us hence suppose that two particles, say i and j, create the first cycle at a time $t_c \in [\tau_{\mathrm{c}+1}, \tau_{\mathrm{c}}]$, through a non-clustering collision or overlap. The cycling condition imposes strong geometric constraints, providing smallness when integrating over well-chosen parameters in the dynamics, restrained to small geometric volumes. Nevertheless, to integrate over a collision angle, we need to make sure that the velocities appearing in the product over the edges do not depend on this angle: we will hence start identifying the corresponding edges, and sum over them to make the associated velocities disappear like in (6.5) and (6.6).

The parametrization of the cycle will depend on whether one of both particles i or j has undergone a deflection between t and τ_c or not. We here define the relevant interactions to parametrize the cycle. Should it be the case, let us denote k the particle that deflected i at the closest time $\tau_{\mathrm{d}} \geq \tau_c$, and if such a deflection never happened let us denote k the particle that overlapped i at a closest time $\tau_{\mathrm{ov}} \geq \tau_c$. One can arbitrarily choose i, among i and j, to be the closest to have been deflected, or to have been overlapped if none of both has ever been deflected. Eventually, something that might happen is that the first connection between i and j (before their cycle) stems from i overlapping a particle (call it l), after having encountered k (see i, j, k, l in Fig. 7).

The choice of this configuration $(i, j, k, l) \in [\![1, r+k]\!]^4$ and of $(\tau_c, \tau_{\mathrm{d}}, \tau_{\mathrm{ov}}) \subset \{\tau_p\}_{p \leq r+k-1}$ corresponds to a combinatorial factor of order $(r+k)^7$.

Since the first cycle happens after τ_{c}, before that time every encounter is clustering, registered in the tree Υ, and every velocity in the dynamics thus only depends on the signs and angles associated to the edges of Υ. This way, we may sum over these edges like in Section 6.2, as soon as we do it in an order respecting the interdependence among the encounters.

The set $K \subset E_\Upsilon$ of edges possibly impacted by the collision between i and k at time τ_{d} is defined as follows: we start with all the edges between τ_{c+1} and 0, since the cycle happens before. Then, we can define the connected component of k when removing the edge τ_{d} between i and k, of which component we add to K all the edges after the deflection time τ_{d} (see the definition of the dynamics tree in Fig. 4). Eventually, we add to K all the potential overlaps undergone by i after τ_{d}, apart from the one with l if appropriate, since we need it to close the cycle. We hence condition on this set K, which corresponds to a factor 2^{c} once c is fixed, and we can sum beforehand over all these edges as in (6.5) and (6.6). To be finally able to integrate over the deflection angle σ_{d} associated with the time τ_{d} without impacting velocities in the product over the edges, it only remains to dominate roughly the one associated with the overlap time τ_{ov} in the deflection case, as follows:

$$|v_i(\tau_{\mathrm{ov}}{}^+) - v_l(\tau_{\mathrm{ov}}{}^+)| \leq 2\mathbf{V}.$$

This factor will be found back in the final estimation (D.10). We will now treat both cases separately: as said before, if one of both particles has been deflected before τ_{c}, we will use the angle σ_{d} of the corresponding collision to parametrize the cycle condition. Else, we will use the distance between the aggregates involved in the closest overlap. Both ways will raise some singularities in velocities, that we will control afterwards, in Section D.4.

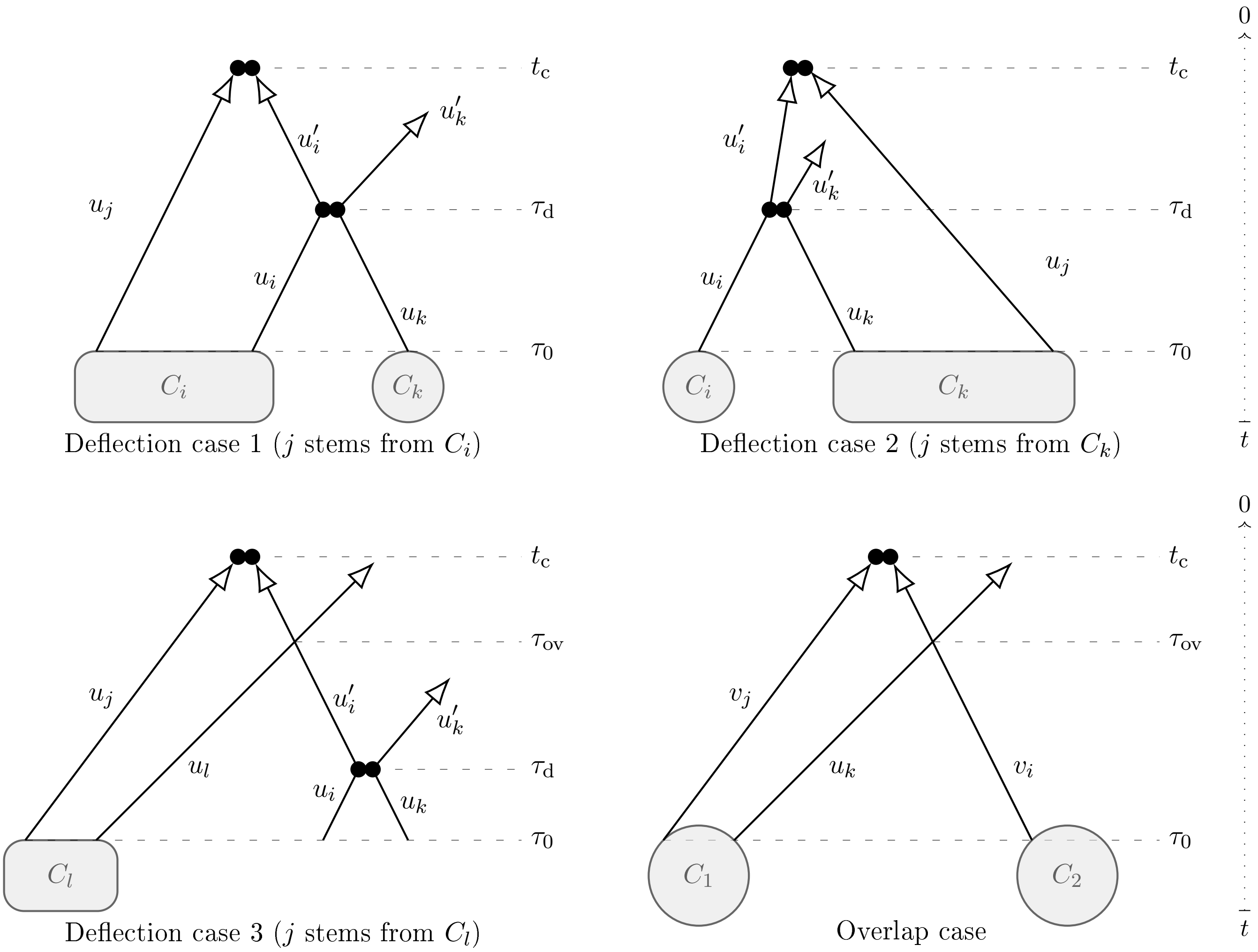

Figure 7: The three deflection cases, and the overlap case

D.2 Deflected case

If one of the particles has been deflected before τ_c, we have fixed the closest deflection, involving i and k at time τ_d. Taking the encounter time just before τ_d as a reference (call it τ_0), we shall denote u_i, u_j, u_k the velocities of particles i, j, k between τ_0 and τ_d, and u_i', u_j', u_k' these velocities between τ_d and t_c (cf. Fig. 7). We first suppose that $k \neq j$. Then, there are three possibilites for the origin of particle j, considering that at time t_c it creates a cycle with i: it can stem from the connected component C_i of i at time τ_0, from the connected component C_k of k at time τ_0, or from the connected component C_l of a particle l overlapping i at a time τ_{ov}, between τ_d and τ_c (Fig. 7). The dependency of $x_i - x_j$ along the setting at τ_0 is different in each case.

- ▷ If j is in the connected component of i (Fig. 7, top left), this is the simplest case and

$$[x_i - x_j](t_c) = [x_i - x_j](\tau_0) + (\tau_d - \tau_0)[u_i - u_j] + (t_c - \tau_d)[u_i' - u_j].$$

- ▷ If j is in the connected component of k (Fig. 7, top right), then

$$\begin{aligned}[x_i - x_j](t_c) &= [x_i - x_k + x_k - x_j](\tau_d) + (t_c - \tau_d)[u_i' - u_j] \\ &= \varepsilon\omega_d + [x_k - x_j](\tau_0) + (\tau_d - \tau_0)[u_k - u_j] + (t_c - \tau_d)[u_i' - u_j].\end{aligned}$$

▷ Eventually, if j is in the connected component of l (Fig. 7, bottom left), then

$$\begin{aligned}[x_i - x_j](t_c) &= [x_i - x_l + x_l - x_j](\tau_{ov}) + (t_c - \tau_{ov})[u_i' - u_j] \\ &= \varepsilon\omega_{ov} + [x_l - x_j](\tau_0) + (\tau_{ov} - \tau_0)[u_l - u_j] + (t_c - \tau_{ov})[u_i' - u_j].\end{aligned}$$

Actually, it may happen that u_l is deflected between τ_0 and τ_{ov}, yet it only requires to change the term $(\tau_{ov} - \tau_0)[u_l - u_j]$, splitting it according to each segment between two deflections.

In each case, using the cycle condition $[x_i - x_j](t_c) = \varepsilon\omega_c + \zeta$, for a $\zeta \in \mathbb{Z}^d$ due to the periodicity of the domain and satisfying $|\zeta| \leq t\mathbf{V}$, one can write

$$(t_c - \tau_*)[u_i' - u_j] = \Delta x(\tau_*) + \varepsilon\omega_*, \tag{D.2}$$

where $\tau_* \in \{\tau_d, \tau_{ov}\}$, $\Delta x(\tau_*)$ is independent of ω_d (see the explicit formula (D.5)), and $\omega_* \in \omega_c - \{0, \omega_d, \omega_{ov}\}$ satisfies $|\omega_*| \leq 2$. This relationship (D.2) defines a cone to which $u_i' - u_j$ must belong to achieve the cycle, whose height depends on the relative position of i and j at time τ_*: if $\Delta x(\tau_*)$ is large enough compared to ε, we will be able to control the height of this cone, and hence its angle, which constrains σ_d. Otherwise, it will provide a strong condition on the encounter times. First, if

$$|\Delta x(\tau_*)| \geq 4\varepsilon, \tag{D.3}$$

then

$$\Big|(t_c - \tau_*)[u_i' - u_j]\Big| \geq \frac{|\Delta x(\tau_*)|}{2},$$

so that

$$\frac{1}{|t_c - \tau_*|} \leq \frac{4\mathbf{V}}{|\Delta x(\tau_*)|}. \tag{D.4}$$

Now, the value of $\Delta x(\tau_*)$ is respectively given by

$$\Delta x(\tau_*) = \begin{cases} [x_j - x_i](\tau_0) + (\tau_d - \tau_0)[u_j - u_i] & \text{if } j \in C_i, \\ [x_k - x_j](\tau_0) + (\tau_d - \tau_0)[u_k - u_j] & \text{if } j \in C_k, \\ [x_l - x_j](\tau_0) + (\tau_{ov} - \tau_0)[u_l - u_j] & \text{if } j \in C_l, \end{cases} \tag{D.5}$$

which is affine in $\tau_* \in \{\tau_d, \tau_{ov}\}$, so that denoting $(\tau_* - \tau_0^*)\Delta u$ the projection of $\Delta x(\tau_*)$ on $\Delta u \in \{u_j - u_i, u_k - u_j, u_l - u_j\}$ according to the considered case, one has $|\Delta x(\tau_*)| \geq |(\tau_* - \tau_0^*)\Delta u|$. Thus, using (D.2) and (D.4), we know that

$$u_i' - u_j = \frac{\Delta x(\tau_*) + \varepsilon\omega_*}{|t_c - \tau_*|}$$

belongs to a cylinder of width at most

$$\frac{4\varepsilon\mathbf{V}}{|\Delta x(\tau_*)|} \leq \frac{4\varepsilon\mathbf{V}}{|(\tau_* - \tau_0^*)\Delta u|},$$

(which contains the previous cone of controlled height, cf Fig. 8, **a.**). Hence, u_i' also belongs to such a cylinder, since u_j remains unchanged. Now, the deflection condition states that $|u_i' - (u_i + u_k)/2| = |u_i - u_k|/2$, so that $u_i' - (u_i + u_k)/2$, whose direction is given by σ_d (see Fig. 6), also has to belong to a sphere of radius $|u_i - u_k|/2$. The intersection of this sphere with the previous cylinder is included in the union of two cones, of which solid angle is maximum when the cylinder is tangent to the sphere (cf. Fig. 8, **b.**), namely

$$C_d \min\left[1, \left(\frac{\varepsilon\mathbf{V}}{|(\tau_* - \tau_0^*)\Delta u| \cdot |u_i - u_k|}\right)^{d-1}\right].$$

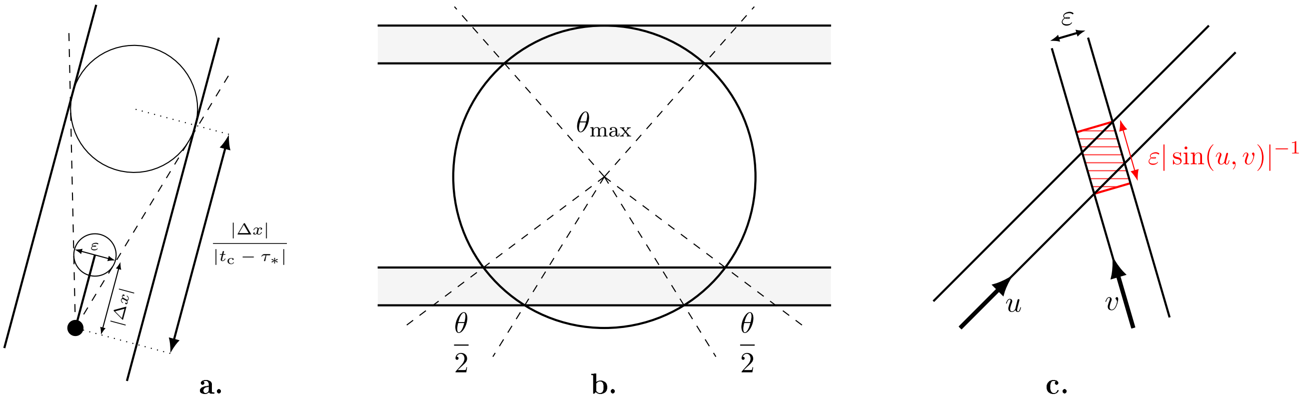

Figure 8: Geometrical estimates

Eventually, denoting $\mathbb{1}_{i\between j}$ the condition that i and j create the first cycle within the tree Υ, we can bound the following integral using that $d \geq 3$,

$$\int \mathbb{1}_{i\between j} \mathrm{d}\sigma_{\mathrm{d}} \mathrm{d}\tau_* \leq C_d \sum_{|\zeta| \leqslant t\mathbf{V}} \int \min \left[1, \left(\frac{\varepsilon \mathbf{V}}{|(\tau_* - \tau_0^*)\Delta u| \cdot |u_i - u_k|} \right)^{d-1} \right] \mathrm{d}\tau_* \tag{D.6}$$

$$\leq C_d (t\mathbf{V})^d \left(\frac{\varepsilon \mathbf{V}}{|\Delta u| \cdot |u_i - u_k|} + (d-2) \left[\frac{\varepsilon \mathbf{V}}{|\Delta u| \cdot |u_i - u_k|} \right]^{(d-1)-(d-2)} \right)$$

$$\leq \tilde{C}_d \frac{t^d \mathbf{V}^{d+1} \cdot \varepsilon}{|\Delta u| \cdot |u_i - u_k|}. \tag{D.7}$$

Finally, if the condition (D.3) is not satisfied, then the condition $|\tau_* - \tau_0^*| \leq 2\varepsilon/|\Delta u|$ provides an even better bound integrating over τ_*, with the same singularity $|\Delta u|^{-1}$.

The case $k = j$ is treated in a similar manner, without singularity since the disjunction on the height of the cone is not necessary: the cycle must occur because of a periodic shift $\zeta \neq 0_{\mathbb{Z}^d}$, necessarily providing a height larger than 1, which yields a negligible term (of order ε^{d-1}, see for instance [8, proof of Proposition B.2]).

D.3 Non-deflected case

If none of both particles i and j has ever been deflected, we have considered the closest overlap, involving i and k at time τ_{ov}. Taking the encounter time just before τ_{ov} as a reference, call it τ_0, we will use the same notation as previously, knowing that none of the velocities changes between τ_0 and τ_{ov} and that the velocities of i and j are given by their initial velocities v_i and v_j (see Fig. 7, bottom right). Then, the overlap at time τ_{ov} and the cycle creation at time t_{c} give respectively the following conditions

$$\begin{cases} x_k(\tau_0) - x_i(\tau_0) + (\tau_{\mathrm{ov}} - \tau_0)(u_k - v_i) = \zeta_{\mathrm{ov}} + \varepsilon\omega_{\mathrm{ov}} \\ x_j(\tau_0) - x_i(\tau_0) + (t_{\mathrm{c}} - \tau_0)(v_j - v_i) = \zeta_{\mathrm{c}} + \varepsilon\omega_{\mathrm{c}}. \end{cases} \tag{D.8}$$

Since before τ_{c} all encounters are clustering, removing the two encounters t_{c} and τ_{ov} divides the past encounters involving i and j into two connected components: C_1 containing j and k and C_2 containing i. By construction, when $\vec{x}_{\mathrm{ov}}$ varies alone, both dynamics components C_1 and C_2 move rigidly one with respect to the other, and the distance $x_k(\tau_0) - x_j(\tau_0)$ remains fixed, so that

$$\vec{x}_{\mathrm{ov}} = x_k(\tau_0) - x_i(\tau_0) = [x_k(\tau_0) - x_j(\tau_0)] + [x_j(\tau_0) - x_i(\tau_0)]$$

has to belong (D.8) to two cylinders of axes $(u_k - v_i)$ and $(v_j - v_i)$ and of common width ε.

If $j \neq k$, the volume of the intersection of these cylinders is of order $\varepsilon^d |\sin(u_k - v_i, v_j - v_i)|^{-1}$ (cf. Fig. 8, **c.**), so that with the same notation as in the previous deflected case (D.6), recalling the changes of variables (5.12) and (A.2), we have

$$\int \mathbb{1}_{i \between j} \mathrm{d}\sigma_{\mathrm{ov}} \mathrm{d}\tau_{\mathrm{ov}} = \frac{\mu}{|v_i - u_k|} \int \mathbb{1}_{i \between j} \mathrm{d}\vec{x}_{\mathrm{ov}} \leq C_d \frac{(t\mathbf{V})^{2d} \cdot \varepsilon}{|\sin(u_k - v_i, v_j - v_i)| \cdot |v_i - u_k|}. \tag{D.9}$$

If $j = k$, since none of the particles has been deflected, this means that the cycle is due to the periodic conditions with $\zeta_{\mathrm{ov}} \neq \zeta_{\mathrm{c}}$. Hence, the difference of both lines of the system (D.8) provides

$$(\tau_{\mathrm{ov}} - t_{\mathrm{c}})(u_j - u_i) = (\zeta_{\mathrm{ov}} - \zeta_{\mathrm{c}}) + \varepsilon(\omega_{\mathrm{ov}} - \omega_{\mathrm{c}}),$$

so that for ε small enough $(u_j - u_i)$ must belong to a cone of opening at most 3ε, and integrating as before over v_i and v_j provides a negligible term (of order ε^{d-1}, see [8, proof of Proposition B.2] for instance).

D.4 Handling the singularities

Now, we are left with some singularities to handle in our bounds (D.7) and (D.9). To do so, we will either use some of the relative velocities appearing in product over the edges to cancel them, use a previous deflection to integrate them, or integrate them using the initial velocities $\underline{v}_{r+k}$ in the case where there is no such deflection. First, the singularity $|u_i - u_k|^{-1}$ in (D.7) and (D.9) appears in the product over the edges in the integral (D.1) that we want to bound, so that it cancels out. The remaining singularities above hence consist in

$$\frac{1}{|u_j - u_i|} + \frac{1}{|u_j - u_k|} + \frac{1}{|u_j - u_l|} + \frac{1}{|\sin(u_k - v_i, v_j - v_i)|}.$$

For one of the three first singularities, of the form, $|u_j - u_n|^{-1}$, we will discriminate on the history of j and n.

If j or n has been deflected by a particle m at a closest time $\tau_{\tilde{\mathrm{d}}}$, we will integrate over this collision's parameters. If this deflection is between j and n, by the scattering identity $|u_j - u_n| = |u_j^\star - u_n^\star|$ between post and pre-collisional velocities, the singularity cancels out in the product over the edges. Otherwise, let us say by symmetry that j is colliding $m \neq n$, so that in particular u_n do not depend on $\sigma_{\tilde{\mathrm{d}}}$. Using the deflection equation (A.1), we write

$$\begin{aligned} \int \frac{\mathrm{d}\sigma_{\tilde{\mathrm{d}}} \mathrm{d}\tau_{\tilde{\mathrm{d}}}}{|u_j - u_n|} &= \int \mathrm{d}\sigma_{\tilde{\mathrm{d}}} \mathrm{d}t_{\tilde{\mathrm{d}}} \left| \frac{u_j^\star + u_m^\star}{2} - u_n + \frac{|u_j^\star - u_m^\star|}{2} \sigma_{\tilde{\mathrm{d}}} \right|^{-1} \\ &= \frac{2}{|u_j^\star - u_m^\star|} \int \mathrm{d}\sigma_{\tilde{\mathrm{d}}} \mathrm{d}\tau_{\tilde{\mathrm{d}}} \left| \frac{u_j^\star + u_m^\star - 2u_n}{|u_j^\star - u_m^\star|} + \sigma_{\tilde{\mathrm{d}}} \right|^{-1}. \end{aligned}$$

We are brought back to studying an integral of the following form, computed in hyperspherical coordinates (cf. Section A) for $w = (1, 0, \dots, 0)$, its maximum value being reached for any $|w| = 1$,

$$\int \frac{\mathrm{d}\sigma}{|w + \sigma|} = C_d \int_0^\pi \frac{\sin^{d-2}\theta}{\sqrt{1 - \cos\theta}} \mathrm{d}\theta = \tilde{C}_d.$$

The remaining singularity $|u_j^\star - u_m^\star|^{-1}$, due to the collision, appears in the product over the edges in (D.1) and hence get cancelled like $|u_i - u_k|^{-1}$.

Nevertheless, to be able to integrate over these parameters $(\sigma_{\tilde{d}}, \tau_{\tilde{d}})$, we first need to dispose of the velocities that are impacted by them: as we did for K in the beginning of this appendix, we fix the corresponding set J of edges, and sum over them beforehand (along with the choice of K, this corresponds to a factor 3^{c}).

If neither j nor n has ever been deflected before τ_{d}, we integrate directly the singularity over the velocity v_j at time 0, using part of the exponential decay $e^{-\frac{\beta}{2}|v_j|^2}$, the singularity being locally integrable in dimension $d > 1$.

The sine singularity only appears in the non-deflected case, and is so integrated in the same way over v_i and v_j, using for example once again hyperspherical coordinates

$$\begin{aligned}\int \frac{e^{-{v_j}^2-{v_i}^2}}{|\sin(u_k - v_i, v_j - v_i)|}\mathrm{d}v_j\mathrm{d}v_i &= \int \left(\int \frac{e^{-(v+v_i)^2-{v_i}^2}}{|\sin(u_k - v_i, v)|}\mathrm{d}v \right) \mathrm{d}v_i \\ &\le \int \left(C_d \int r^{d-1} e^{-r^2+2|v_i|r} \frac{\sin^{d-2}\theta}{|\sin\theta|}\mathrm{d}\theta \mathrm{d}r \right) e^{-{v_i}^2}\mathrm{d}v_i \le \tilde{C}_d.\end{aligned}$$

D.5 Conclusion

In the end, we have obtained enough smallness and there is no more singularity, so that we can eventually sum over all the remaining edges, and conclude the domination exactly as in Section 6.2, yielding the same bound as (6.7). The choice of the disjoint sets $K, J \subset E_\Upsilon$ gives a factor 3^c, which becomes $3^{r+k}/2$ with the sum over c, so that we get the following final bound (the constant C stemming from the different cases)

$$\begin{aligned}\sum_{\Upsilon \in \mathcal{T}^{\mathrm{d}}_{R,k}} \int \mathbb{1}_{T^{\mathrm{d}}_R = \Upsilon} \mathrm{d}\underline{v}_{r+k} \mathrm{d}\underline{\sigma}_{E_\Upsilon} \mathrm{d}\underline{\tau}_{E_\Upsilon} \prod_{e \in E_\Upsilon} s_e |v_m(\tau_e{}^+) - v_{m'}(\tau_e{}^+)| e^{-\beta \|\underline{v}_{r+k}\|^2} \times \mathbb{1}^{\between}_R \qquad \text{(D.10)} \\ \le C(r+k)^7 3^{r+k} \frac{t^{r+k-1}}{(r+k-1)!} \int \mathrm{d}\underline{v}_{r+k} e^{-\beta \mathbf{V}^2} (r+k)^{r-1} \left(\sqrt{(r+k)\mathbf{V}^2} \right)^{r+k-1} \cdot \mathbf{V}^{2d+1} \varepsilon \\ \le \varepsilon \cdot (\tilde{C}t)^{r+k-1} (r+k)^{r-1}.\end{aligned}$$

This bound is similar to (6.7), possibly for different constants, with an additional factor ε. Hence, the same computation as in Section 6.2 leads to Proposition 7.2.1. Note that more precisely, τ_{d} or τ_{ov} has been integrated before, and v_i and v_j also might have been integrated before with the singularities, yet it only changes constants.

To sum up the strategy, we fixed a configuration of cycle $[i, j, k, l, (\tau_c, \tau_{\mathrm{d}}, \tau_{\mathrm{ov}}), J, K]$, summed over all the edges in K whose associated velocities would have been impacted by the collision angle σ_{d}, so that we could integrate over the collision parameters σ_{d}, and τ_{d} or τ_{ov}. This made appear some singularities that we have handled by summing first over the edges of J, to free some collision parameters or velocities, eventually integrating over them. □

E Linear Boltzmann equations

In this section, we prove the well-posedness of the system of linear Boltzmann-like equations (7.9) that corresponds to the Hamilton–Jacobi system presented in Section 7.5. Before stating this result in Proposition E.2.1, we start with a general study of the linear Rayleigh–Boltzmann equation.

E.1 Linear Rayleigh–Boltzmann equation

We proceed to a similar decomposition of this equation as in [21], and compute the integrals in hyperspherical coordinates. The linear Rayleigh–Boltzmann equation (2.7) might indeed be written

$$\partial_t \varphi + v \cdot \nabla_x \varphi = \mathcal{K}\varphi(v) - \nu_\beta(v)\varphi(v)$$

where the gain operator is

$$\mathcal{K}\varphi(v) \doteq \int \langle v_c - v, \omega\rangle_+ M_\beta(v_c)\varphi(v')\mathrm{d}\omega\mathrm{d}v_c, \tag{E.1}$$

and the loss factor is

$$\nu_\beta(v) \doteq \int \langle v_c - v, \omega\rangle_+ M_\beta(v_c)\mathrm{d}\omega\mathrm{d}v_c. \tag{E.2}$$

The loss factor being bounded below in v, it will provide some decay of the solution, while the gain operator will rather make it grow. We will prove that it can be made nonetheless bounded, starting with the following lemma.

Lemma E.1.1 (Integral kernel of the gain part)**.** *The gain operator $\mathcal{K}$ defined above* (E.1) *has the following kernel integral formula*

$$\mathcal{K}\varphi(v) = \sqrt{\frac{\beta}{2\pi}} \int \frac{\varphi(\eta)}{|\eta - v|^{d-2}} \exp\left(-\beta\left[\frac{|\eta - v|^2}{8} + \frac{|\eta|^2 - |v|^2}{4} + \frac{\left(|\eta|^2 - |v|^2\right)^2}{8|\eta - v|^2}\right]\right)\mathrm{d}\eta.$$

Proof. We start by the translated change of variable $V \doteq v_c - v$. Additionally, changing the angle ω into $-\omega$ merely changes the collision kernel to $|\langle v_c - v, \omega\rangle_-|$, so that up to dividing the integral over ω by two, it can be made on the whole sphere $\mathbb{S}^{d-1}$. Eventually, at fixed angle ω, we isolate the part of $V = v_c - v$ supported on ω, whose norm appears in the collision kernel: we write

$$V \doteq V_\omega\ \omega + U_\perp\ , \qquad \text{with } U_\perp \cdot \omega = 0.$$

Then, recalling the scattering formula (2.2) in which the projection of V on ω also appears, we have

$$\mathcal{K}\varphi(v) = \frac{1}{2}\int |V_\omega| M_\beta(v + V)\varphi(v + V_\omega\omega)\mathrm{d}\omega\mathrm{d}V.$$

The key computation is now to relax the angle ω, and to introduce the variable

$$\xi \doteq V_\omega\omega,$$

going twice across $\mathbb{R}^d$, and satisfying $U_\perp \cdot \xi = 0$. We perform the change of variable

$$\begin{aligned}\mathrm{d}\omega\mathrm{d}V &= \mathrm{d}\omega\ \mathrm{d}V_\omega\ \mathrm{d}U_\perp\\ &= 2\frac{\mathrm{d}\xi}{|\xi|^{d-1}}\ \mathrm{d}U_\perp.\end{aligned}$$

Eventually, we perform the translated change of variable $\xi \mapsto v + \xi \doteq \eta$, so that

$$\begin{aligned}\mathcal{K}\varphi(v) &= \int |\xi| M_\beta(v + \xi + U_\perp)\varphi(v + \xi)\frac{\mathrm{d}\xi}{|\xi|^{d-1}}\mathrm{d}U_\perp\\ &= \int \frac{1}{|\eta - v|^{d-2}} M_\beta(\eta + U_\perp)\varphi(\eta)\ \mathrm{d}\eta\ \mathrm{d}U_\perp. \end{aligned}\tag{E.3}$$

To compute the integral over $U^\perp$, it is useful to harness a vector whose scalar product with $\xi = \eta - v$ is easy to calculate: we set

$$\alpha \doteq \frac{v + \eta}{2}.$$

At lign (E.3), the Maxwellian M_β depends on the square of the velocities (2.6), and so makes appear

$$\begin{aligned}|\eta + U_\perp + \alpha - \alpha|^2 &= |U_\perp + \alpha|^2 + |\eta - \alpha|^2 + 2\langle U_\perp + \alpha, \eta - \alpha\rangle\\ &= |U_\perp + \alpha|^2 + \frac{|\eta - v|^2}{4} + \frac{1}{2}\langle 2U_\perp + \eta + v, \eta - v\rangle,\end{aligned}$$

so that using the fact that $\langle U_\perp, \eta - v\rangle = 0$, we have

$$|\eta + U_\perp|^2 = |U_\perp + \alpha|^2 + \frac{|\eta - v|^2}{4} + \frac{|\eta|^2 - |v|^2}{2}.$$

Decomposing α according to ξ as

$$\alpha = \alpha_1 + \alpha_\perp \in \mathrm{Span}(\xi) \oplus \xi^\perp,$$

it all boils down to writing

$$\begin{aligned}\int M_\beta(\eta + U_\perp)\mathrm{d}U_\perp &= \left(\frac{\beta}{2\pi}\right)^{\frac{d}{2}} e^{-\frac{\beta}{2}\left[|\alpha_1|^2 + \frac{|\eta-v|^2}{4} + \frac{|\eta|^2-|v|^2}{2}\right]} \int e^{-\frac{\beta}{2}|\alpha_\perp + U_\perp|^2}\mathrm{d}U_\perp \\ &= \sqrt{\frac{\beta}{2\pi}} e^{-\frac{\beta}{2}\left[|\alpha_1|^2 + \frac{|\eta-v|^2}{4} + \frac{|\eta|^2-|v|^2}{2}\right]}.\end{aligned}$$

The result follows observing that

$$|\alpha_1|^2 = \left|\frac{v+\eta}{2} \cdot \frac{\xi}{|\xi|}\right|^2 = \frac{\left(|\eta|^2 - |v|^2\right)^2}{4|\eta - v|^2}.$$

□

To get existence results on large times, we would like this kernel to be bounded, yet in this form it is not. To exhibit this, we start denoting $u \doteq v - \eta$, so that

$$|\eta|^2 - |v|^2 = |v - u|^2 - |v|^2 = |u|^2 - 2u \cdot v.$$

Thus, in hyperspherical coordinates (see Appendix A) with respect to v, denoting $\mathrm{d}J(\underline{\theta})$ the Jacobian corresponding to the measure on the unit sphere, with $\theta = \theta_{d-2} \doteq \widehat{(u,v)}$, one has

$$\begin{aligned}\int \frac{\mathrm{d}\eta}{|\eta - v|^{d-2}} e^{-\beta\left[\frac{|\eta-v|^2}{8} + \frac{|\eta|^2-|v|^2}{4} + \frac{\left(|\eta|^2-|v|^2\right)^2}{8|\eta-v|^2}\right]} &= \int \mathrm{d}r\mathrm{d}J(\underline{\theta}_{d-1}) r e^{-\beta\left[\frac{r^2}{8} + \frac{r^2 - 2r|v|\cos\theta}{4} + \frac{\left(r - 2|v|\cos\theta\right)^2}{8}\right]} \\ &= \int \mathrm{d}r\mathrm{d}J(\underline{\theta}_{d-1}) r e^{-\frac{\beta}{2}(r - |v|\cos\theta)^2} \geq C_\beta |v|.\end{aligned}$$

The last equality above merely results from developing and refactorizing the term in brackets. The final lower bound follows from recognizing a non-normalized Gaussian centered around $|v|\cos\theta$, then integrating over θ, which provides a constant depending on the dimension.

It is hence necessary to compute a change of unknown in the linear Rayleigh-Boltzmann equation (2.7), to better distribute the exponential decay and get a bounded operator. Setting

$$R = M_\beta^{\frac{1}{2}} \varphi,$$

the Rayleigh–Boltzmann equation becomes

$$\partial_t R + v \cdot \nabla_x R = \hat{\mathcal{K}} R(v) - \nu_\beta(v) R(v), \tag{E.4}$$

for the new operator

$$\hat{\mathcal{K}} R(v) \doteq \int \langle v_c - v, \omega\rangle_+ M_\beta(v_c) M_\beta^{\frac{1}{2}}(v) M_\beta^{-\frac{1}{2}}(v') R(v') \mathrm{d}\omega \mathrm{d}v_c. \tag{E.5}$$

Lemma E.1.2 (Integral kernel of the modified gain part)**.** *The new gain operator $\hat{\mathcal{K}}$ defined above* (E.5) *has the following kernel integral formula*

$$\hat{\mathcal{K}}R(v) = \sqrt{\frac{\beta}{2\pi}} \int \frac{R(\eta)}{|\eta - v|^{d-2}} \exp\left(-\beta\left[\frac{|\eta-v|^2}{8} + \frac{\left(|\eta|^2 - |v|^2\right)^2}{8|\eta-v|^2}\right]\right) \mathrm{d}\eta,$$

and is bounded on $\mathbb{L}^\infty(\mathcal{D})$ as a consequence of the following estimate on its integral kernel

$$\sqrt{\frac{\beta}{2\pi}} \int \frac{R(\eta)}{|\eta - v|^{d-2}} \exp\left(-\beta\left[\frac{|\eta-v|^2}{8} + \frac{\left(|\eta|^2 - |v|^2\right)^2}{8|\eta-v|^2}\right]\right) \mathrm{d}\eta \leq \frac{17|\mathbb{S}^{d-1}|}{\beta|v|}.$$

Proof. The only difference between both operators is the factor $M_\beta^{\frac{1}{2}}(v)M_\beta^{-\frac{1}{2}}(v') = \exp(\beta(|v|^2 - |\eta|^2)/4)$ in the change of variable of the proof of Lemma E.1.1, proving the new kernel formula.

Using once again the hyperspherical coordinates (see Appendix A and above), we have

$$\int \frac{\mathrm{d}\eta}{|\eta - v|^{d-2}} e^{-\beta\left[\frac{|\eta-v|^2}{8} + \frac{\left(|\eta|^2-|v|^2\right)^2}{8|\eta-v|^2}\right]} = \int \mathrm{d}r \mathrm{d}J(\underline{\theta}_{d-1}) r e^{-\frac{\beta}{8}\left[r^2 + (r - 2|v|\cos\theta)^2\right]}.$$

Now, one can integrate

$$\int_0^\infty r e^{-\frac{\beta}{8}\left[r^2+(r-2|v|\cos\theta)^2\right]}\mathrm{d}r = \left[\frac{-2}{\beta} e^{-\frac{\beta}{8}\left[r^2+(r-2|v|\cos\theta)^2\right]}\right]_0^\infty + |v|\cos\theta \int_0^\infty r e^{-\frac{\beta}{8}\left[r^2+(r-2|v|\cos\theta)^2\right]}\mathrm{d}r.$$

In the second integral, part of the weight concentrates around 0, and the rest around $2|v|\cos\theta$, so it can be split into

$$\begin{aligned}\int_0^{|v|\cos\theta} e^{-\frac{\beta}{8}\left[r^2+(r-2|v|\cos\theta)^2\right]}\mathrm{d}r &\leq \int_{|v|\cos\theta}^\infty e^{-\frac{\beta}{8}u^2}\mathrm{d}u \\ &\leq \frac{8}{\beta|v|\cos\theta} e^{-\frac{\beta}{8}|v|^2\cos^2\theta}\end{aligned}$$

(bounding by standard estimates on the Gaussian cumulative distribution function), and similarly

$$\int_{|v|\cos\theta}^\infty e^{-\frac{\beta}{8}\left[r^2+(r-2|v|\cos\theta)^2\right]}\mathrm{d}r \leq \frac{8}{\beta|v|\cos\theta} e^{-\frac{\beta}{8}|v|^2\cos^2\theta}.$$

In the end, we get

$$\int_0^\infty r e^{-\frac{\beta}{8}\left[r^2+(r-2|v|\cos\theta)^2\right]}\mathrm{d}r \leq \frac{2}{\beta} e^{-\frac{\beta}{8}[2|v|\cos\theta]^2} + \frac{16}{\beta} e^{-\frac{\beta}{8}|v|^2\cos^2\theta}.$$

It remains to integrate over θ as

$$\int_0^\pi \sin\theta e^{-\frac{\beta}{8}|v|^2\cos^2\theta}\mathrm{d}\theta = \int_{-1}^1 e^{-\frac{\beta}{8}|v|^2x^2}\mathrm{d}x \leq \frac{1}{|v|}\sqrt{\frac{8\pi}{\beta}}.$$

The contribution of the integral over θ to the volume of the sphere is 2, so that the integral over the other angles yields $|\mathbb{S}^{d-1}|/2$. In the end,

$$\int_0^\infty r e^{-\frac{\beta}{8}\left[r^2+(r-2|v|\cos\theta)^2\right]}\mathrm{d}r \leq \frac{|\mathbb{S}^{d-1}|}{2}\left(\frac{2}{\beta}\cdot\frac{1}{|v|}\sqrt{\frac{2\pi}{\beta}} + \frac{16}{\beta}\cdot\frac{1}{|v|}\sqrt{\frac{8\pi}{\beta}}\right),$$

which concludes the proof when multiplying by the factor $\sqrt{\beta(2\pi)^{-1}}$. The kernel is also bounded for small values of $|v|$ in dimensions $d \geqslant 2$, so that the operator is bounded from $\mathbb{L}^\infty(\mathcal{D})$ to itself. □

E.2 Boltzmann–Hamilton–Jacobi system

To gain boundedness of the gain operator appearing in the Rayleigh–Boltzmann equation, we computed a change of unknown that distributed differently the exponential weights. However, doing so, we caused the new gain operator to be greater than the loss factor for small velocities, so that the latter do not compensate totally the gain operator. Because of this, they will not immediately provide a maximum principle.

In the specific case of the system (7.9) that we study here, we have no hope of a maximum principle, because of the gain terms $-\theta_t\chi_t$ and $\theta_t\eta_t$ that depend on the observable θ_t. We recall the system

$$\begin{cases} (\partial_s - v\cdot\nabla_x)\chi = -\theta\chi + \displaystyle\int dv_2 d\omega \langle v_1 - v_2, \omega\rangle_+ \Big(M_\beta(v_2')\chi(z_1') - M_\beta(v_2)\chi(z_1)\Big) \\ (\partial_s - v\cdot\nabla_x)\eta = +\theta\eta - \displaystyle\int dv_2 d\omega \langle v_1 - v_2, \omega\rangle_+ \Big(M_\beta(v_2')\eta(z_1') - M_\beta(v_2)\eta(z_1)\Big), \end{cases} \tag{E.6}$$

of unknowns (χ,η), with the boundary conditions

$$\begin{cases} \chi(0) = M_\beta\varphi_0 \\ \eta(t) = \gamma(t). \end{cases}$$

Recall the definition (3.16) of the functional space $\mathcal{F}_{1,\beta}$, embedded with the $\mathbb{L}^\infty(\mathcal{D})$-norm with a weight M_β. It thus corresponds to functions decaying faster than the $\beta-$Gaussian in velocities.

Proposition E.2.1 (Well-posedness of the Boltzmann–Hamilton–Jacobi system)**.** *For any time $t > 0$, for a bounded gain weight $\theta \in \mathbb{L}^\infty(\mathcal{D})$, and boundary conditions*

$$\begin{cases} \chi(0) = M_\beta\varphi_0 \in \mathcal{F}_{1,\beta/2} \\ \eta(t) = \gamma(t) \in \mathcal{F}_{1,\beta/2}, \end{cases}$$

the system (E.6) *has a unique global positive solution* $(\chi,\eta) \in \mathbb{L}^\infty\left([0,t],\ \mathcal{F}_{1,\beta/2}\right)^2$.

Proof. We will compute a different change of unknown to go from the operators of the Boltzmann–Hamilton–Jacobi system (E.6) to the bounded operator $\hat{\mathcal{K}}$ defined in (E.5). Considering $\chi(0) \in \mathcal{F}_{1,\beta/2}$, we will look at $R(0) \doteq M_\beta^{-\frac{1}{2}}\chi(0) \in \mathbb{L}^\infty(\mathcal{D})$. The existence of a global solution for the first equation of the system (E.6) above, on χ in $\mathcal{F}_{1,\beta/2}$, is thus equivalent to the global existence in $\mathbb{L}^\infty(\mathcal{D})$ of the modified Rayleigh–Boltzmann equation

$$\partial_t R + v\cdot\nabla_x R = \hat{\mathcal{K}}R(v) - \theta R - \nu_\beta(v)R(v),$$

with the modified gain operator $\hat{\mathcal{K}}$, which is shown to be bounded in $\mathbb{L}^\infty(\mathcal{D})$ (Lemma E.1.2). Hence, the operator $R \mapsto \hat{\mathcal{K}}R - \theta R$ is also bounded for $\theta \in \mathbb{L}^\infty(\mathcal{D})$. Classical results in kinetic theory states that there exists a global unique positive solution to this equation, with the bound, for any time $t \geqslant 0$,

$$\|R(t)\|_{\mathbb{L}^\infty} \le \|R(0)\|_{\mathbb{L}^\infty} e^{Ct},$$

where the constant C depends on the operators' kernels as

$$C = \sup_{z\in\mathcal{D}} \left[\int_{\mathbb{R}^d} \frac{\sqrt{\beta(2\pi)^{-1}}}{|\eta - v|^{d-2}} \cdot e^{-\beta\left[\frac{|\eta-v|^2}{8} + \frac{\left(|\eta|^2-|v|^2\right)^2}{8|\eta-v|^2}\right]} d\eta \ - \theta(z) - \nu_\beta(v) \right].$$

This concludes the proof, returning to the variable $\chi \in \mathcal{F}_{1,\beta/2}$. The same computation holds backwards in time for η. Note that even in the case $\theta = 0$, one can check that the constant C is positive because of small velocities. In this case nevertheless, one can still check a maximum principle using more elaborate methods (see for example [19, Section 3.1]). $\square$

Notation index

t	Time horizon	2.1
ε	Particles' diameter	2.1
$\underline{z}_N = (\underline{x}_N, \underline{v}_N) \in \mathcal{D}^N$	Particles' state vector	2.1
$\mathcal{X}_N^\varepsilon, \mathcal{D}_N^\varepsilon$	Hard sphere domains (in position, in phase space)	(2.1)
${v_i}', {v_j}', {v_i}^\star, {v_j}^\star$	Post-collisional, pre-collisional velocities	(2.2)
ω, σ	Scattering angles	(2.3)
$M_\beta(x, v)$	Maxwellian equilibrium	(2.6)
$W_N^\varepsilon(t, \underline{z}_n, \underline{\ell}_n)$	Canonical particles' density in phase space	2.2
φ, φ_0	Solution to Rayleigh–Boltzmann equation, initial perturbation	(2.7)
λ, μ	Chemical potentials (tagged particles, equilibrium particles)	$(S_{\varepsilon,\mu,\lambda})$
$p_\mu = \lambda\mu^{-1}$	Tagged particle ratio	3.2
$\underline{\ell}_n \in \Lambda_n,\ \lvert\underline{\ell}_n\rvert$	Tags' vector, its cardinality	2.3, 3.2
$\varphi_0^{\otimes \underline{\ell}_n}(\underline{z}_{\underline{\ell}_n})$	Initial perturbation of tagged particles	2.3
$F_n^\varepsilon(t, \underline{z}_n, \underline{\ell}_n)$	Correlation functions	2.3
$\mathcal{Z}_\mu$	Grand canonical partition function	(2.9)
$H_n, H^{\otimes n}$	Smooth observables	2.3
$\mathcal{C}_n^\ell$	Collision operators	(2.12)
$\pi_t^\varepsilon[H], \tilde{\pi}_t^\varepsilon[H]$	Empirical measure (non-tagged one, tagged one)	(3.2)(3.3)
$Q_{n,\underline{\ell}_k^*}(t), Q^{\lim}_{n,\underline{0}_k}(t)$	Successive-collision operators	(3.7)(3.14)
$T_k(t)$	Set of ordered times	(3.8)
$\mathcal{M}_{n,k}$	Set of admissible ordered added particles	3.2
$\chi_k \doteq (\underline{m}_k, \underline{\ell}_k^*, \underline{s}_k) \in \mathcal{H}_{n,k}$	History vector of a pseudo-trajectory	(3.9)
$\underline{z}_n^{[t]}, \underline{\zeta}_n^{[t]}$	Pseudo-trajectory at time t, limit version	(3.10)(3.15)
$G_n = M_\beta^{\otimes n}\varphi^{\otimes \underline{\ell}_n}$	Limit correlation function	3.2
$\lVert f_k \rVert_{k,\beta},\ \mathcal{F}_{k,\beta}$	Sub-Gaussian norm and space	(3.16)
C_0	Initial density bound	(3.17)
K	Tree pruning parameter	3.6
$\underline{h}_K, t_k^{\mathrm{p}}$	Time cutting intervals and steps	(3.24)
$R_n^{[K]}(t)$	Pruned-out term	(3.27)
$\widehat{G}_n = G_n - R_n^{[K],\lim}$	Pruned limit correlation function	3.6
$\mathbf{V}, \delta$	Truncation parameters, in velocities and time separation	3.8
$\mathrm{Traj}([0,t], \mathcal{M}(\mathcal{D}))$	Set of trajectories on $\mathcal{M}(\mathcal{D})$	4.1
ζ_t^ε	Fluctuation field of the tagged empirical measure	(4.2)
$\{h, \mathfrak{m}\}_t$	Filtered mean of the observable h against measure $\mathfrak{m}$	(4.4)
$\mathcal{H}(q, p)$	Hamiltonian governing the limit cumulants	(4.5)
$\mathcal{I}(t, h), \boldsymbol{\Lambda}(t, \mathfrak{v})$	Limit cumulant generating function, its Legendre transform	(7.3)
$\mathfrak{G}_\varepsilon^{[t]}[H]$	Cumulant generating function	(4.15)
$G_{[\sigma]} = \prod_{i=1}^{\lvert\sigma\rvert} G_{\lvert\sigma_i\rvert}$	Decomposition of G_n along the partition $\sigma \in \mathcal{P}_n$	(4.11)
f_n^ε, f_n	Cumulants of the correlation functions, limit version	(4.11)

ϕ_n	Cumulants of the exclusion condition $\mathbb{1}_{\mathcal{X}_n^\varepsilon}$	(4.11)
$\Psi_n = (k, \chi_k, \underline{t}_k, \underline{\omega}_k, \underline{v}_k^*)$	Pseudo-trajectory parametrizing vector	5.1
$\mathrm{d}\nu_{[t]}(\Psi_n)$	Pseudo-trajectory measure	(5.1)
$\mathrm{d}\nu_{[0,t]}^{[H]}(\Psi_n)$	Pseudo-trajectory H-weighted measure	(5.7)
$F_n^\varepsilon[H](t, \underline{z}_n, \underline{\ell}_n)$	H-weighted correlation function	(5.3)
$f_n^\varepsilon[H](t, \underline{z}_n, \underline{\ell}_n)$	Cumulants of the H-weighted correlation functions	(5.9)
$P(A)$	Particles stemming from A	5.2
$\mathbb{1}_{A \not\sim A'}$	Indicator of non-interaction	5.2
$\mathrm{agg}(A)$	Indicator of aggregation	5.2
$\aleph_{\lvert\rho\rvert}$, Γ_p	Set of clusters, set of particles	5.3
$\phi_{S,k+l}$	Cumulants of the cluster exclusion	5.3
$\mathcal{C}_A$	Connected graphs on A	5.3
$T_A^{\mathrm{c}} = T^{\mathrm{c}}(\underline{z}_{\Psi_A}^{[0,t]}), \tau_i^{\mathrm{c}}, \omega_i^{\mathrm{c}}$	Recollision tree, recollision times and angles	5.4
$\mathfrak{S}_{(p_e,q_e)_\Upsilon}[A]$	Indicator of clustering compatibility	5.4
$\overline{x}_A$	Barycenter of the positions $\underline{x}_A$	5.4
$T_R^{\mathrm{ov}} = T^{\mathrm{ov}}(\underline{z}_{\Psi_R}^{[0,t]}), \tau_i^{\mathrm{ov}}, \omega_i^{\mathrm{ov}}$	Overlap tree, overlap times and angles	5.4
$\mathcal{C}_{\lvert\kappa\rvert}(\Upsilon)$	Connected graphs containing the tree Υ	5.4
$\phi_R = \phi_R^{[\mathrm{tree}]} + \phi_R^{[\mathrm{cycle}]}$	Decomposition of the cumulant in tree and cycle part	5.4
$T_R^\star, \tau_i^\star, \omega_i^\star$	Clustering tree, clustering times and angles	5.4
$\mathfrak{CC}^\star(\Upsilon)$	Set of decorated connected components	5.4
T_R^{d}	Dynamics tree	5.4
$f_n^\varepsilon[H]^{[\mathrm{tree}]}(t), f_n^\varepsilon[H]^{[\mathrm{cycle}]}(t)$	Decomposition of the cumulant in tree and cycle part	(5.14)
$I_n^{[\rho]}[H](t, \underline{\ell}_n)$	ρ-contribution to the tree cumulant integral	(5.14)
$\mathbb{B}_{T,\beta}$	Functional space of the large deviation observables	7.4
$\mathcal{S}_\lambda$	Random set of tagged particles	8.1

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