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A MATHEMATICAL REVIEW OF THE ANALYSIS OF THE BETAPLANE MODEL AND EQUATORIAL WAVES

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Abstract. This review paper is devoted to the presentation of recent progress in the mathematical analysis of equatorial waves. After a short presentation of the physical background, we present some of the main mathematical results related to the problem.

More precisely we are interested in the study of the shallow water equations set in the vicinity of the equator: in that situation the Coriolis force vanishes and its linearization near zero leads to the so-called betaplane model. Our aim is to study the asymptotics of this model in the limit of small Rossby and Froude numbers. We show in a first part the existence and uniqueness of bounded (strong) solutions on a uniform time, and we study their weak limit. In a second part we give a more precise account of the asymptotics by characterizing the possible defects of compactness to that limit, in the framework of weak solutions only.

These results are based on the studies [6]-[8] on the one hand, and [11] on the other.

1. **Physical background and goal of the study.**

1.1. **Physical background.** The aim of this review paper is to present a mathematical description of oceanic flows in the equatorial zone of the Earth. We are interested in domains extending over many thousands of kilometers, and on such scales the forces with dominating influence are the gravity and the Coriolis force. Our goal is to try to understand how those forces counterbalance each other to impose the so-called geostrophic constraint on the mean motion, and to describe the oscillations which are generated around this geostrophic equilibrium.

In this survey we will be concentrating on the equatorial zone. Let us however briefly discuss the situation at midlatitudes, which has been more extensively studied from a mathematical point of view, in the past years. At midlatitudes, on "small" geographical zones (namely for a small enough interval around a given latitude far enough from the equator) one can neglect the variations of the Coriolis force due to the curvature of the Earth, and this leads to a singular perturbation problem with constant coefficients. The corresponding asymptotics (in the limit of a large rotation) have been studied by a number of authors, depending on the boundary conditions, the generality of the initial data... Roughly speaking one may

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summarize the situation in that case by stating that a limiting behaviour (in the limit of a small Rossby number, meaning that the rotation of the Earth is predominant over the motion under study) can be exhibited, and some sort of convergence (weak or strong depending on the boundaries or on the initial data) can be proved. We refer for instance to the review by R. Temam and M. Ziane [21] or by the author and L. Saint-Raymond [10], or to the work by J.-Y. Chemin, B. Desjardins, the author and E. Grenier [3] for more details.

Here we are interested in a geographical zone where the variations of the Coriolis force do play a role (indeed the Coriolis force is identically zero at the equator and has a different sign on each hemisphere); we also wish to consider the interaction between the fluid and the atmosphere (more precisely we will take into account the fact that the ocean has a free surface at its upper boundary). The mathematical modelling of these various phenomena, as well as their respective importance according to the scales considered, have been studied in a rather systematic way by A. Majda [18], and R. Klein and A. Majda [15]. We refer also to the fundamental books of Gill [13] and Pedlosky [19] for a discussion of the various physical phenomena occuring in the ocean and the atmosphere which have to be included in the equations in order to provide an accurate approximation of the movement of the fluid with time.

Here we will focus on *quasigeostrophic, oceanic* flows, meaning that we will consider horizontal length scales of order 1000 kilometers and vertical length scales of order 5 kilometers, so that the shallow-water approximation is relevant (see for instance the works by D. Bresch, B. Desjardins and C.K. Lin [1] or by J.-F. Gerbeau and B. Perthame [12]). In this framework, the asymptotics of homogeneous rotating fluids (meaning with a constant, homogeneous Coriolis force) have been studied by D. Bresch and B. Desjardins [2]. For the description of equatorial flows, one has to take also into account the variations of the Coriolis force, and especially the fact that it cancels at equator. The inhomogeneity of the Coriolis force has already been studied by B. Desjardins and E. Grenier [4] and by the author and L. Saint-Raymond [11] for an incompressible fluid with rigid lid upper boundary. Here there is an additional physical effect due to the free surface.

For the sake of simplicity, we will not discuss the effects of the interaction of the fluid with the boundaries. More precisely we will assume periodicity with respect to the longitude (omitting the stopping conditions on the continents) and we will consider an infinite domain for the latitude. This last assumption is definitely not physical but we expect the exponential decay of the equatorial waves, which we will be proving in the sequel, to justify *a posteriori* this assumption; so far to our knowledge, no mathematical result in that direction has been proved.

1.2. **The shallow water model and the betaplane approximation.** Let us now present the equations we will study in this survey. We consider the ocean as an incompressible fluid with free surface submitted to gravitation, and make the following classical assumptions: we suppose that the density of the fluid is homogeneous, and that the pressure law is given by the hydrostatic approximation. Moreover we assume that the motion is essentially horizontal and does not depend on the vertical coordinate, leading to the so-called shallow water approximation. We therefore consider a so-called Saint-Venant model, which describes vertically averaged flows in three dimensional shallow domains in terms of the horizontal mean velocity field u and the depth variation h due to the free surface. Taking into account the Coriolis force, the model reads as

$$\begin{aligned} \partial_t h + \nabla\cdot (hu) = 0 \\ \partial_t(hu) + \nabla\cdot (hu \otimes u) + f(hu)^\perp + \frac{1}{\mathrm{Fr}^2} h\nabla h - \nu\Delta u = 0 \end{aligned} \tag{1}$$

where f denotes the vertical component of the earth rotation and Fr the Froude number. We have written $u^\perp$ for the vector $(u_2, -u_1)$. In the following we will restrict our study to the vicinity of the equator, in which case a linearisation of the Coriolis force (see for instance [13]) allows to denote

$$f = \beta x_2$$

where x_2 denotes the Northward coordinate (the latitude) and β is a fixed parameter. The parameter ν takes into account the viscosity of the fluid, and depending on the situation we will consider, it will be positive or zero.

1.3. **Choice of the parameters.** In this survey we are interested in the behaviour of the ocean in the equatorial zone: we expect the Froude number Fr, which is the ratio of the fluid speed to a measure of the internal wave speed, to be small: we will write $\mathrm{Fr} \sim \varepsilon$, which amounts to considering depth variations

$$h = H(1 + \varepsilon\eta)$$

where H is a constant. We will be interested in the limit when ε goes to zero, and we will assume that ε is also the order of magnitude of the Rossby number (which is the ratio of the fluid speed to the speed of rotation of the Earth). The Saint-Venant system (1) can therefore be rewritten (normalizing H to $H = 1$ for simplicity) in terms of the velocity u and the depth fluctuation η as follows:

$$\begin{gathered} \partial_t \eta + \frac{1}{\varepsilon}\nabla\cdot \Big((1+\varepsilon\eta)u\Big) = 0, \\ \partial_t\Big((1+\varepsilon\eta)u\Big) + \nabla\cdot\Big((1+\varepsilon\eta)u\otimes u\Big) + \frac{\beta x_2}{\varepsilon}(1+\varepsilon\eta)u^\perp + \frac{1}{\varepsilon}(1+\varepsilon\eta)\nabla\eta - \nu\Delta u = 0, \\ \eta_{|t=0} = \eta^0, \quad u_{|t=0} = u^0. \end{gathered} \tag{2}$$

1.4. **Aim of the survey.** The aim of mathematical studies on this system is to try to exhibit a limiting behaviour in the limit when the parameter ε goes to zero: are there bounded solutions on a uniform time interval, what is their weak (or strong) limit, etc...

From a mathematical point of view, very few results have been obtained on this question, and this survey will mainly concentrate on four recent results, three (by A. Dutrifoy and A. Majda [6]-[7] and by A. Dutrifoy, A. Majda and S. Schochet [8]) concerning the non viscous case and the behaviour of the mean flow (with in particular the proof of the existence of uniformly bounded, unique solutions), and one (by the author and L. Saint-Raymond [11]) concerning the viscous case with the consideration of the mean flow as well as the description of oscillations around the mean flow, and their resonances (this time in the case of weak, possibly non unique solutions).

The structure of the paper is the following: in the coming paragraph we will give an account of the methods developped in [6]-[8] to obtain uniform estimates in Sobolev-type spaces (which is a difficult task due to the fact that the penalization has variable coefficients), and to pass to the limit in order to describe the mean flow. The following paragraph (Section 3) deals with the study of weak solutions; in that case finding a uniform bound in the energy space is very easy, contrary to the previous paragraph, but the difficulty lies in describing all the oscillations generated by the perturbation, as well as their interactions, and in studying the full limit system. In the last part (Section 4) we present a (non exhaustive !) list of open problems.

2. Uniform bounds in the non viscous case, and study of the weak limit of strong solutions.

2.1. **Introduction.** In this section we intend to present the study of the non viscous shallow water equations (system (2) with $\nu = 0$), in the limit when ε goes to zero. This has been the object of two papers by A. Dutrifoy and A. Majda which we will now describe, as well as a more recent article by the same authors and S. Schochet (see [8]). The first paper, [6], concerns a special situation where the x_1 coordinate is assumed to vary slowly with ε: the solutions only depend on εx_1 instead of x_1, which simplifies somewhat the analysis. In the second paper, [7], this assumption is dropped and the full non viscous shallow water system presented above is studied.

One of the main achievements of the study is that the authors are able to derive uniform estimates on the solution in smooth enough Sobolev-type spaces and to deduce existence and uniqueness of solutions on a uniform time interval; this is of course highly non trivial due to the fact that the penalization operator has variable coefficients, so does not disappear in energy estimates (other than L^2, which is far from enough in the non viscous case to guarantee the existence of solutions). The choice of those spaces relies on the special structure of the penalization operator, which turns out to be closely related to the harmonic oscillator. Sobolev spaces associated to the harmonic oscillator are therefore constructed, and a wellposedness result in those spaces is proved. Once a bounded family of solutions is constructed, it is natural to consider its asymptotic behaviour as the Rossby number goes to zero. That is the second main result of [7], where a linear limit system is obtained for the mean flow.

More recently, in [8], a new and simpler proof of that result was proposed (actually the result is slightly more general, as the solutions are allowed to have both a $O(\varepsilon)$ and a $O(1)$ variation in the x_1 direction). This is the result that we will describe in the following as it is both easier and more general than the previous works [6] and [7]. We will omit however in this presentation the possible $O(\varepsilon)$ variations in the direction x_1 to simplify the analysis; the interested reader can consult [8] for the more general case. It should be pointed out that the methods developped in [8] actually follow the approach of S. Klainerman and A. Majda [16] concerning the incompressible limit for periodic, compressible flows in the well prepared case (though in that case of course the vector fields used corresponded simply to Sobolev spaces, contrary to the situation considered here).

The structure of the presentation (as in [8]) is the following. In the next section we show how to reduce the study to a model problem by a change of unknowns. The following section consists in the definition of Sobolev-type spaces based on the harmonic oscillator, and in the proof of uniform estimates in those spaces. The last section consists in proving a convergence theorem, towards the solution of the mean flow equation.

2.2. **Reduction of the problem by a change of unknowns.** The first step consists in symmetrizing the system by considering the new unknown $\widetilde{\eta}$ defined by

$$\widetilde{\eta} = \frac{2}{\varepsilon}\left(\sqrt{1+\varepsilon\eta} - 1\right).$$

Then one defines

$$u_1 = \frac{r-\ell}{\sqrt{2}} \quad \text{and} \quad \widetilde{\eta} = \frac{r+\ell}{\sqrt{2}}$$

so that the vector $U = (r, \ell, u_2)$ satisfies an equation of the following type:

$$\partial_t U + A(U)\cdot\nabla U + \frac{1}{\varepsilon} MU = 0, \tag{3}$$

where A is a symmetric matrix depending smoothly on U, and M is the operator

$$M = \begin{pmatrix} \partial_1 & 0 & L_- \\ 0 & -\partial_1 & L_+ \\ L_+ & L_- & 0 \end{pmatrix}.$$

In the above matrix, the operators $L_\pm$ are the raising and lowering operators associated with the harmonic oscillator $H = \partial_2^2 - \beta^2 x_2^2$, namely

$$H = L_-L_+ + L_+L_-, \quad \text{with} \quad L_\pm = \frac{1}{\sqrt{2}}(\partial_2 \mp \beta x_2).$$

It is easy to see that M is skew-symmetric. In order to analyze the equation, the main idea is then to find a vector field which commutes with the matrix M, in order to make the (a priori unbounded) skew-symmetric term disappear from the energy estimates.

It turns out that the matrix

$$D = \begin{pmatrix} H-2\beta & 0 & 0 \\ 0 & H+2\beta & 0 \\ 0 & 0 & H \end{pmatrix}$$

satisfies $[D, M] = 0$. The difficulty here is that D is not a scalar operator. However its higher order part is indeed scalar, which is sufficient to obtain an energy estimate. That is the object of the coming section.

2.3. **Adapted function spaces.** In this section we will define the function spaces which are well adapted to the problem, in the sense that they involve the operator D introduced above. We will then show how to obtain an energy estimate in those spaces.

2.3.1. *Definition of the function spaces.* Considering the form of the matrix D introduced in the previous paragraph, it is natural to define the following norm, for any scalar function u:

$$\|u\|_{W_{2n}} \stackrel{\text{def}}{=} \left(\sum_{k+2p\le 2n} \|\partial_1^k H^p u\|^2_{L^2(\mathbf{T}\times\mathbf{R})} \right)^{\frac{1}{2}},$$

which can be shown to satisfy

$$\sum_{j+m+k\le 2n} \|\partial_1^k x_2^j \partial_2^m u\|^2_{L^2(\mathbf{T}\times\mathbf{R})} \le C_n \|u\|^2_{W_{2n}},$$

due to the commutation relations $[H,x_2]=2\partial_2$, $[H,\partial_2]=2\beta^2 x_2$, and $[\partial_2,x_2]=1$. It is moreover easy to see that bounded sets of W_{2n} are precompact in $W_{2(n-1)}$ (since x_1 belongs to $\mathbf{T}$ and due to the factor x_2 in the definition of the W_{2n} norm).

For any vector field U, controling the W_{2n} norm of its components will amount to controling the L^2 norm of $\partial_1^k D^p U$ for $0 \leq k+2p \leq 2n$. In turn this amount to writing an energy estimate on (3) using that vector field, which commutes with M.

2.3.2. *Uniform bounds.* The interest of the introduction of the spaces W_{2n} lies in the following proposition.

Proposition 2.1. *If the initial data $U_{0,\varepsilon}=(r_{0,\varepsilon},\ell_{0,\varepsilon},u_{0,\varepsilon,2})$ has components uniformly bounded in the space W_{2n} for some $n\geq 2$ and for ε in $]0,1[$, then there is a positive time T, independent of ε, such that there is a unique solution to (3), uniformy bounded in $L^\infty([0,T];W_{2n})$.*

The proof of the result consists in applying the operator $\partial_1^k D^p$ to Equation (3), for $0\leq k+2p\leq 2n$. Denoting $U_{kp}=\partial_1^k D^p U$ one finds

$$\partial_t U_{kp} + A(U)\cdot\nabla U_{kp} + \frac{1}{\varepsilon} M U_{kp} = [\partial_1^k D^p, A(U)]\cdot\nabla U.$$

One can prove that

$$\|[\partial_1^k D^p, A(U)]\cdot\nabla U\|_{L^2} \leq F(\|U\|_{W_{2n}}^2),$$

for some smooth function F, by analyzing precisely the commutator: the difficulty lies in the fact that the commutator between two matrix operators is not necessarily of lower order than the sum of their orders; however as pointed out above, in this situation the term of highest order is scalar, so that allows to obtain the estimate. Using the embedding of W_{2n} into H^3 allows to control $A(D)U$ in Lipschitz norm, which finally gives an estimate of the type

$$\frac{d}{dt}\|U\|_{W_{2n}} \leq F(\|U\|_{W_{2n}})$$

which allows to conclude the proof.

2.4. **The convergence result.** The convergence theorem is the following.

Theorem 2.2. *Let $(\eta_{0,\varepsilon},u_{0,\varepsilon})$ be bounded in W_{2n} for some $n\geq 3$, and suppose they satisfy*

$$(\beta x_2 u_{0,\varepsilon,2}+\partial_1\eta_{0,\varepsilon}, -\beta x_2 u_{0,\varepsilon,1}+\partial_2\eta_{0,\varepsilon}, \nabla\cdot u_{0,\varepsilon}) = O(\varepsilon) \quad in \quad W_{2(n-1)}.$$

Then there is a time $T>0$, independent of ε, such that there is a unique solution $(u_\varepsilon,\eta_\varepsilon)$ bounded in $C^0([0,T];W_{2n})\cap C^1([0,T];W_{2(n-1)})$, converging towards the limit system

$$\begin{gathered}\partial_t u_1 - \beta x_2 V = 0, \quad \partial_t\eta + \partial_2 V = 0\\ u_2 = 0, \quad -\beta x_2 u_1 + \partial_2\eta = 0, \quad \partial_1 u_1 = \partial_1\eta = 0,\end{gathered}$$

where V is a Lagrange multiplier associated with the constraints on u_1 and η.

Note that the assumptions made on the initial data amount to a “well-prepared” assumption.

The uniform existence was proved in the previous section (see Proposition 2.1), so we will concentrate here on the convergence result. Actually in order to prove that result we will again work with the formulation in terms of the vector field U_ε. It is easy to see that the assumptions on the initial data imply that

$$\|\partial_1 r_{0,\varepsilon} + L_- u_{0,\varepsilon,2}\|_{W_{2(n-1)}} + \|\partial_1 \ell_{0,\varepsilon} - L_+ u_{0,\varepsilon,2}\|_{W_{2(n-1)}} \\ + \|L_+ r_{0,\varepsilon} + L_- \ell_{0,\varepsilon}\|_{W_{2(n-1)}} \leq C\varepsilon,$$

so using the equation satisfied by U_ε, one gets that $\partial_t U_{\varepsilon|t=0}$ is bounded in $W_{2(n-1)}$. Then using similar arguments to the previous section (namely an energy estimate on $\partial_t U_\varepsilon$ using the vector fields $\partial_1^k D^p$), one can show that $\partial_t U_\varepsilon$ is bounded in $L^\infty([0,T];W_{2(n-1)})$. A compactness argument then implies that up to the extraction of a subsequence, there is strong convergence of U_ε in $C^0([0,T];W_{2(n-1)} \cap H^{2n-\delta}_{loc})$ for any positive δ, to some limit $U = (r, \ell, u_2)$.

It is classical to see that the limit must be in the kernel of M, which implies easily that it must satisfy the following constraints:

$$u_2 = \partial_1 r = \partial_1 \ell = 0.$$

The last step consists in looking for the equation satisfied by r and ℓ. One can start by taking the mean in x_1 in (3), since the limit does not depend on x_1. Then one notices that the nonlinear term disappears in the limit (for algebraic reasons), and the rest of the proof follows from the bounds obtained on U_ε. We refer to [8] for more details.

3. The viscous case: study of the oscillations and of the full limit system.

3.1. Introduction.

In this section we will consider the shallow water model (2), in the limit when ε goes to zero, in the viscous case: we assume $\nu > 0$. The existence of uniformly bounded weak solutions is simply due to the usual theory of the isentropic Navier-Stokes equations (see for instance [17]). In particular if $(\eta^0, u^0) \in L^2$ and $(\eta^0_\varepsilon, u^0_\varepsilon)$ are such that

$$\frac{1}{2}\int \left(|\eta^0_\varepsilon|^2 + (1+\varepsilon\eta^0_\varepsilon)|u^0_\varepsilon|^2\right) dx \leq \mathcal{E}^0 \quad \text{and} \quad (\eta^0_\varepsilon, u^0_\varepsilon) \to (\eta^0, u^0) \text{ in } L^2,$$

then for all $\varepsilon > 0$, (2) has at least one weak solution $(\eta_\varepsilon, u_\varepsilon) \in L^\infty(\mathbf{R}^+, L^2)$ satisfying

$$\frac{1}{2}\int (\eta_\varepsilon^2 + (1+\varepsilon\eta_\varepsilon)|u_\varepsilon|^2)(t,x)dx + \int_0^t \int \nu|\nabla u_\varepsilon|^2(s,x)dxds \leq \mathcal{E}^0.$$

In particular, there exist $\eta \in L^\infty(\mathbf{R}^+;L^2)$ and $u \in L^\infty(\mathbf{R}^+;L^2) \cap L^2(\mathbf{R}^+, \dot H^1)$ such that, up to extraction of a subsequence, $(\eta_\varepsilon, u_\varepsilon)$ converges weakly to (η, u) in the space $L^2_{loc}(\mathbf{R}^+ \times \mathbf{T} \times \mathbf{R})$.

We are interested in understanding more precisely the convergence above, in particular in terms of possible defects of compactness (we will identify the oscillations responsible for the lack of strong convergence), and in computing the equation satisfied by u (as well as by the oscillating profiles). First we will present the various waves induced by the singular perturbation defined by

$$L : (\eta, u) \in L^2(\mathbf{T} \times \mathbf{R}) \mapsto (\nabla\cdot u, \beta x_2 u^\perp + \nabla\eta). \tag{4}$$

This will enable us, in the spirit of [20], to study the "filtered" shallow-water system and to present its limit. A wellposedness result on that limit system (Theorem 3.2) will then allow us to finally state the main, strong convergence result (Theorem 3.4). An easier, weak convergence result is also provided in Theorem 3.3.

3.2. **Description of the geostrophic constraint and of the waves.**

3.2.1. *The geostrophic constraint.* As in the previous part (see Section 2.4), it is easy to see that the weak limit must belong to the kernel of the singular perturbation, and thus must satisfy

$$\begin{gathered} u_2 = \partial_1 u_1 = \partial_1 \eta = 0 \\ -\beta x_2 u_1 + \partial_2 \eta = 0. \end{gathered}$$

3.2.2. *Precise description of the oscillations.* The description of the eigenmodes of L can be achieved using the Fourier transform with respect to x_1 and the decomposition on the Hermite functions $(\psi_n)_{n\in\mathbf{N}}$ with respect to x_2. We recall that the Hermite functions $(\psi_n)_{n\in\mathbf{N}}$ satisfy

$$H\psi_n = -\psi_n'' + \beta^2 x_2^2 \psi_n = \beta(2n+1)\psi_n,$$

and constitute a Hermitian basis of $L^2(\mathbf{R})$. In the following we will denote by $\widehat{f}(n,k)$ the components of any function f in the Hermite-Fourier basis $(2\pi)^{-1/2}\psi_n(x_2)e^{ikx_1}$. In other words we have

$$\forall (n,k) \in \mathbf{N}\times\mathbf{Z}, \quad \widehat{f}(n,k) = \frac{1}{\sqrt{2\pi}} \int_{\mathbf{T}\times\mathbf{R}} \psi_n(x_2) e^{-ikx_1} f(x_1,x_2)\, dx_1 dx_2,$$

along with the inversion formula $f(x_1,x_2) = \dfrac{1}{\sqrt{2\pi}} \displaystyle\sum_{\substack{n\in\mathbf{N}\\ k\in\mathbf{Z}}} \psi_n(x_2) e^{ikx_1} \widehat{f}(n,k)$.

Let us look for L^2 solutions to $L(\eta,u) = i\tau(\eta,u)$, with u_2 non identically zero. As a necessary condition one finds that the Fourier transform of u_1 with respect to x_1 (denoted by $\mathcal{F}_1 u_1$) satisfies

$$(\mathcal{F}_1 u_1)'' + \left(\tau^2 - k^2 + \frac{\beta k}{\tau} - \beta^2 x_2^2\right) \mathcal{F}_1 u_1 = 0,$$

from which we deduce that $\mathcal{F}_1 u_1$ is proportional to some ψ_n and that

$$\tau^3 - (k^2 + \beta(2n+1))\tau + \beta k = 0, \tag{5}$$

for some $n \in \mathbf{N}$. Elementary algebraic computations show that for any given $n \in \mathbf{N}\setminus\{0\}$ and $k \in \mathbf{Z}$, this polynomial has three distinct roots in $\mathbf{R}$, denoted

$$\tau(n,k,-1) < \tau(n,k,0) < \tau(n,k,1).$$

The case $n = 0$ is somewhat special and can be dealt with directly (see below).

• If $k \neq 0$ and $n \neq 0$, (5) admits three solutions, and one can check that these solutions are eigenvalues of L associated to the following unitary eigenvectors: we have $\Psi_{n,k,j} = C_{n,k,j} e^{ikx_1} \widetilde{\Psi}_{n,k,j}(x_2)$, with

$$\widetilde{\Psi}_{n,k,j} = \begin{pmatrix} \dfrac{i}{k-\tau(n,k,j)}\sqrt{\dfrac{\beta n}{2}}\psi_{n-1}(x_2) + \dfrac{i}{\tau(n,k,j)+k}\sqrt{\dfrac{\beta(n+1)}{2}}\psi_{n+1}(x_2) \\ \dfrac{i}{k-\tau(n,k,j)}\sqrt{\dfrac{\beta n}{2}}\psi_{n-1}(x_2) - \dfrac{i}{\tau(n,k,j)+k}\sqrt{\dfrac{\beta(n+1)}{2}}\psi_{n+1}(x_2) \\ \psi_n(x_2) \end{pmatrix}, \tag{6}$$

and

$$C_{n,k,j} = (2\pi)^{-1/2}\left(\frac{\beta n}{(\tau(n,k,j)+k)^2} + \frac{\beta(n+1)}{(\tau(n,k,j)-k)^2} + 1\right)^{-1/2}.$$

The modes corresponding to $\tau(n,k,-1)$ and $\tau(n,k,1)$ are called *Poincaré modes*, and satisfy

$$\tau(n,k,\pm 1) \sim \pm\sqrt{k^2+\beta(2n+1)} \text{ as } |k| \text{ or } n \to \infty,$$

which are the frequencies of the gravity waves. The modes corresponding to values $\tau(n,k,0)$ are called *Rossby modes*, and satisfy

$$\tau(n,k,0) \sim \frac{\beta k}{k^2+\beta(2n+1)} \text{ as } |k| \text{ or } n \to \infty.$$

• If $k=0$ and $n \neq 0$, the three distinct solutions to (5) are the two Poincaré modes $\tau(n,0,\pm 1) = \pm\sqrt{\beta(2n+1)}$ and the *non-oscillating mode* $\tau(n,0,0)=0$. The corresponding eigenvectors of L are given by (6) if $j \neq 0$ and by

$$\Psi_{n,0,0} = C_{n,0,0}\begin{pmatrix} \sqrt{\dfrac{\beta(n+1)}{2}}\psi_{n-1}(x_2) + \sqrt{\dfrac{\beta n}{2}}\psi_{n+1}(x_2) \\ \sqrt{\dfrac{\beta(n+1)}{2}}\psi_{n-1}(x_2) - \sqrt{\dfrac{\beta n}{2}}\psi_{n+1}(x_2) \\ 0 \end{pmatrix}. \tag{7}$$

• If $n=0$, the three solutions to (5) are the two Poincaré and *mixed Poincaré-Rossby modes* $\tau(0,k,\pm 1) = -\frac{k}{2} \pm \frac{1}{2}\sqrt{k^2+4\beta}$ with asymptotic behaviours

$$\tau(0,k,-\operatorname{sign}(k)) \sim -k \quad \text{and} \quad \tau(0,k,\operatorname{sign}(k)) \sim \frac{\beta}{k} \text{ as } |k| \to \infty,$$

and the *Kelvin mode* $\tau(0,k,0)=k$. The corresponding eigenvectors of L are given by (6) if $j \neq 0$ and by

$$\Psi_{0,k,0} = \frac{1}{\sqrt{4\pi}} e^{ikx_1}\begin{pmatrix} -\psi_0(x_2) \\ \psi_0(x_2) \\ 0 \end{pmatrix}. \tag{8}$$

We recall that the functions ψ_n are defined by $\psi_n(x_2) = e^{-\frac{\beta x_2^2}{2}} P_n(x_2)$, where P_n is a polynomial of degree n. We therefore have an exponential decay far from the equator. It can be shown that this provides a Hilbertian basis of eigenvectors, which satisfy

$$\|\Psi_{n,k,j}\|_{W^{s,\infty}(\mathbf{T}\times\mathbf{R})} \leq C_s(1+|k|^2+n)^{s/2},$$
$$C_s^{-1}(1+|k|^2+n)^{s/2} \leq \|\Psi_{n,k,j}\|_{H^s(\mathbf{T}\times\mathbf{R})} \leq C_s(1+|k|^2+n)^{s/2}.$$

Moreover the eigenspace associated with any non zero eigenvalue is of finite dimension.

In the following we will write $\Pi_{n,k,j}$ for the orthogonal projection on the eigenmode $\Psi_{n,k,j}$ of L, and Π_λ for the orthogonal projection on the eigenspace associated with the eigenvalue $i\lambda$ of L.

3.2.3. *The filtering operator and the formal limit system.* Let $\mathcal{L}$ be the semi-group generated by L: $\mathcal{L}(t) = \exp(-tL)$. Then, for any three component vector field Φ in $L^2(\mathbf{T}\times\mathbf{R})$, we have, denoting by $\mathfrak{S}$ the set of eigenvalues of L,

$$\mathcal{L}(t)\Phi = \sum_{i\lambda\in\mathfrak{S}} e^{-it\lambda}\Pi_\lambda\Phi. \tag{9}$$

Now let $(\eta_\varepsilon, u_\varepsilon)$ be a weak solution to (2) and let us define $\Phi_\varepsilon = \mathcal{L}\left(-\frac{t}{\varepsilon}\right)(\eta_\varepsilon, u_\varepsilon)$. Conjugating formally equation (2) by the semi-group gives

$$\partial_t\Phi_\varepsilon + \mathcal{L}\left(-\frac{t}{\varepsilon}\right) Q\left(\mathcal{L}\left(\frac{t}{\varepsilon}\right)\Phi_\varepsilon, \mathcal{L}\left(\frac{t}{\varepsilon}\right)\Phi_\varepsilon\right) - \nu\mathcal{L}\left(-\frac{t}{\varepsilon}\right)\Delta'\mathcal{L}\left(\frac{t}{\varepsilon}\right)\Phi_\varepsilon = R_\varepsilon, \tag{10}$$

where Δ' and Q are the linear and symmetric bilinear operator defined by

$$\Delta'\Phi = (0, \Delta\Phi') \quad \text{and} \quad Q(\Phi,\Phi) = (\nabla\cdot(\Phi_0\Phi'), (\Phi'\cdot\nabla)\Phi') \tag{11}$$

and where $R_\varepsilon = \mathcal{L}\left(-\frac{t}{\varepsilon}\right)(0, -\nu\dfrac{\varepsilon\eta_\varepsilon}{1+\varepsilon\eta_\varepsilon}\Delta u_\varepsilon)$. We have defined, for any three component vector field $\Phi = (\Phi_0, \Phi_1, \Phi_2)$, the two-component vector field $\Phi' = (\Phi_1, \Phi_2)$.

We therefore expect to get a bound on the time derivative of Φ_ε in some space of distributions. A formal passage to the limit in (10) as ε goes to zero (based on a nonstationary phase argument) leads then to

$$\partial_t\Phi + Q_L(\Phi,\Phi) - \nu\Delta'_L\Phi = 0, \tag{12}$$

where Δ'_L and Q_L denote the linear and symmetric bilinear operator defined by

$$\Delta'_L\Phi = \sum_{i\lambda\in\mathfrak{S}} \Pi_\lambda\Delta'\Pi_\lambda\Phi \quad \text{and} \quad Q_L(\Phi,\Phi) = \sum_{\substack{i\lambda, i\mu, i\tilde\mu\in\mathfrak{S}\\ \lambda=\mu+\tilde\mu}} \Pi_\lambda Q(\Pi_\mu\Phi, \Pi_{\tilde\mu}\Phi).$$

3.3. **Interactions between equatorial waves.** Let us state the following important result, which indicates what type of nonlinearity remains at the limit.

Proposition 3.1. *Except for a countable number of β and with the notation of Section 3.2.2, the following condition of non resonance holds for all $n, n^*, m \in \mathbf{N}$, all $k, k^* \in \mathbf{Z}$ and all $j, j^*, \ell \in \{-1, 0, 1\}$:*

$$\tau(n,k,j) + \tau(n^*,k^*,j^*) = \tau(m, k+k^*, \ell)$$

implies

$$\text{either } \tau(n,k,j) = 0 \text{ or } \tau(n^*,k^*,j^*) = 0 \text{ or } \tau(m,k+k^*,\ell) = 0,$$
$$\text{or } \tau(n,k,j), \tau(n^*,k^*,j^*), \tau(m,k+k^*,\ell) \in \mathbf{Z}^*,$$

meaning that, among the ageostrophic modes, only three Kelvin waves may interact.

Let us describe the main ideas of the proof, without giving all the details of the computations.

Let us start by noticing that by definition of Kelvin waves, Kelvin resonances necessarily take place simply because they correspond to convolution in Fourier space. The crucial argument leading to Proposition 3.1 is then that the eigenvalues of the penalization operator L are defined as the roots of a countable number of polynomials whose coefficients depend (linearly) on the ratio β.

In particular, for fixed $n, n^*, m \in \mathbf{N}$ and $k, k^* \in \mathbf{Z}$, the occurence of a resonant triad $\tau(n,k,j) + \tau(n^*,k^*,j^*) = \tau(m,k+k^*,\ell)$ is controlled by the cancellation of some polynomial $P_{n,n^*,m,k,k^*}(\beta)$. Therefore, either this polynomial has a finite number of zeros, or it is identically zero. To eliminate the second possibility we use the asymptotics $\beta \to \infty$ as well as $\beta \to 0$. In the case when $n = 0$ or $n^* = 0$, the previous argument needs a refinement, by introducing an auxiliary polynomial. We refer to [11] for details.

In the special case of Ker L, it is not too difficult to prove that for every $n \in \mathbf{N}$ and every smooth vector fields Φ and Φ^*, one has $(\Psi_{n,0,0}|Q_L(\Phi,\Phi^*))_{L^2(\mathbf{T}\times\mathbf{R})} = 0$. In particular, the projection of the limit system (12) onto Ker L can be formally written

$$\partial_t \Pi_0 \Phi - \nu \Delta'_L \Pi_0 \Phi = 0.$$

We recover the fact that the (weak) limit system is linear.

3.4. **The envelope equations.** In this section we shall analyze the system (12) obtained formally above as the limit of the filtered system (10) as $\varepsilon \to 0$.

It can be shown that weak solutions exist for all times, and that a unique, strong solution (on a short time interval) exists also, similarly to the situation of the 3D Navier-Stokes system: although the setting is two dimensional, we need to work in specially adapted function spaces, similarly to the results [6]-[8] presented in the previous section, for which product rules are unfortunately the same as in 3D rather than in 2D.

However the study of resonances above showed that in fact except for a countable number of values of β, there are extremely few non linear terms in the limit system (and these are essentially 1D since they correspond to Kelvin waves). So the analysis will be much simplified in that case, and we will restrict our study to that situation. The interested reader can consult [11] for the general case.

3.4.1. *Definition of suitable functional spaces.* Similarly to the study of the previous part, we introduce weighted Sobolev spaces associated with some derivation-like operator which acts separately on each eigenmode of L: for any nonnegative real number s, we define the space H^s_L as the subspace of $(L^2(\mathbf{T}\times\mathbf{R}))^3$ given by the following norm:

$$\|\Phi\|_{H^s_L} \stackrel{\text{def}}{=} \left(\sum_{(n,k,j)\in S} (1+n+|k|^2)^s \|\Pi_{n,k,j}\Phi\|^2_{L^2(\mathbf{T}\times\mathbf{R})} \right)^{\frac{1}{2}},$$

where we have defined $S = \mathbf{N} \times \mathbf{Z} \times \{-1,0,1\}$. Due to the definition of the eigenvectors of L seen above, one can prove that

$$\|\Phi\|_{H^s_L} \sim \|(\mathrm{Id} - \Delta + \beta^2 x_2^2)^{s/2}\Phi\|_{L^2(\mathbf{T}\times\mathbf{R})}.$$

Moreover we have

$$\forall \Phi \in (\mathrm{Ker}\ L)^\perp, \quad \|\Phi\|_{H^s_L} \sim \left(\sum_{i\lambda \in \mathfrak{S}\setminus\{0\}} \|\Pi_\lambda \Phi\|^2_{H^s(\mathbf{T}\times\mathbf{R})} \right)^{\frac{1}{2}}.$$

3.4.2. *Wellposedness results.* Let us denote by $\Pi_\perp\Phi$ the orthogonal projection of Φ onto $(\operatorname{Ker} L)^\perp$, and by Π_P (resp Π_K) the projection onto Poincaré-type eigenvectors (resp Kelvin-type). We recall that for any three component vector field Φ, we denote by Φ' its two last components.

Theorem 3.2. *There is a constant C and a countable subset $\mathcal{N}$ of $\mathbf{R}^+$ such that for any β in $\mathbf{R}^+\setminus\mathcal{N}$, the following result holds. Let $\Phi^0\in L^2(\mathbf{T}\times\mathbf{R})$ be given. Then (12) is globally wellposed, in the sense that there is a unique, global solution Φ in $L^\infty(\mathbf{R}^+;L^2(\mathbf{T}\times\mathbf{R}))$ such that $\Pi_\perp\Phi$ belongs to the space $L^2(\mathbf{R}^+;H^1_L)$, and which satisfies the energy inequality*

$$\frac{1}{2}\|\Phi(t)\|^2_{L^2}+\nu\int_0^t\|\nabla(\Pi_0\Phi)'(t')\|^2_{L^2}dt'+\frac{\nu}{C}\int_0^t\|\nabla(\Pi_\perp\Phi)(t')\|^2_{L^2}dt'\leq\frac{1}{2}\|\Phi^0\|^2_{L^2}.$$

• *if we further assume that $\Pi_\perp\Phi^0$ belongs to H^s_L, for $0\leq s\leq 1$, then $\Pi_\perp\Phi$ belongs to the space $L^\infty_{loc}(\mathbf{R}^+,H^s_L)\cap L^2_{loc}(\mathbf{R}^+,H^{s+1}_L)$.*
• *finally if $(\Pi_P+\Pi_K)\Phi^0$ belongs to H^α_L for $\alpha>1/2$, then for all $t\in\mathbf{R}^+$*

$$\int_0^t\|\nabla\cdot\Phi'(t')\|_{L^\infty(\mathbf{R}\times\mathbf{T})}dt'<+\infty.$$

These results are based on a precise study of the structure of (12), and in particular of the ageostrophic part of that equation, meaning its projection onto $(\operatorname{Ker} L)^\perp$. One can prove in particular that the ageostrophic part of (12) is in fact fully parabolic: for any $s\geq 0$, there is a constant C such that for any $\Phi\in(\operatorname{Ker} L)^\perp$, we have

$$\|\Phi\|^2_{H^{s+1}_L}\leq C(\Phi|-\Delta'_L\Phi)_{H^s_L}.$$

That is due to the fact that for each eigenmode of L, the first and second components of the eigenvectors (corresponding to η and u_1) have very similar behaviours, and thus controlling the regularity of the last two components is sufficient to have an estimate on $\Pi_\perp\Phi$ in H^1_L.

The global existence of weak solutions then follows the lines of the classical proof of the Leray theorem, which we will not detail. To prove the uniqueness of those solutions, we notice that for almost all β the limit system is a linear equation on all modes except the Kelvin modes. But those modes are essentially one dimensional, so there is enough smoothness to prove without difficulty uniqueness for that equation.

The propagation of regularity and the estimate on the divergence are obtained along the same lines.

3.5. **Convergence results.** In Section 3.5.1 we describe the weak limit of weak solutions to (2) as ε goes to zero, which is proved to satisfy the geostrophic equation, i.e., the projection of (12) onto Ker L. The statement is given in Theorem 3.3 below. Then we prove in Section 3.5.2 (Theorem 3.4) the strong convergence of the filtered sequence of solutions towards the unique solution of (12).

3.5.1. *Weak convergence.*

Theorem 3.3. *Let $(\eta^0,u^0)\in L^2(\mathbf{T}\times\mathbf{R})$ and $(\eta^0_\varepsilon,u^0_\varepsilon)$ be such that*

$$\frac{1}{2}\int\left(|\eta^0_\varepsilon|^2+(1+\varepsilon\eta^0_\varepsilon)|u^0_\varepsilon|^2\right)dx\leq\mathcal{E}^0,\quad\textit{and}\quad(\eta^0_\varepsilon,u^0_\varepsilon)\to(\eta^0,u^0)\ \textit{in}\ L^2(\mathbf{T}\times\mathbf{R}).$$

For all $\varepsilon > 0$*, denote by* $(\eta_\varepsilon, u_\varepsilon)$ *a solution of (2) with initial data* $(\eta^0_\varepsilon, u^0_\varepsilon)$*. Then up to the extraction of a subsequence,* $(\eta_\varepsilon, u_\varepsilon)$ *converges weakly in* $L^2_{loc}(\mathbf{R}^+ \times \mathbf{T} \times \mathbf{R})$ *to the solution* $(\eta, u) \in L^\infty(\mathbf{R}^+, L^2(\mathbf{R}))$*, with* $u \in L^2(\mathbf{R}^+, \dot{H}^1(\mathbf{R}))$*, of the following linear equation (given in weak formulation)*

$$\begin{aligned} u_2 = \partial_1 u_1 = \partial_1 \eta = 0 \\ -\beta x_2 u_1 + \partial_2 \eta = 0. \end{aligned} \tag{13}$$

and for all $(\eta^*, u^*) \in L^2 \times H^1(\mathbf{R})$ *satisfying (13),*

$$\int (\eta\eta^* + u_1 u_1^*)(t,x)\,dx + \nu \int_0^t \int \nabla u_1 \cdot \nabla u_1^*(t',x)\,dx\,dt' = \int (\eta_0 \eta^* + u_{0,1} u_1^*)(x)\,dx. \tag{14}$$

This theorem shows that the system satisfied by the weak limits of η_ε and u_ε is linear. There is therefore no convective term in the mean flow: system (13, 14) actually corresponds to the projection of (12) onto Ker L: that projection can indeed be formally written

$$\begin{aligned} \partial_t(\eta, u_1, 0) - \nu \Pi_0(0, \Delta u_1, 0) = 0, \\ (\eta, u)(t) = \Pi_0(\eta, u)(t)\ \forall t \geq 0, \\ (\eta, u)_{|t=0} = \Pi_0(\eta^0, u^0), \end{aligned}$$

where Π_0 denotes the orthogonal projection onto the kernel of L. Note moreover that (η^0, u^0) do not necessarily satisfy the constraints (13), so in general $(\eta, u)_{|t=0}$ is not equal to (η^0, u^0).

One can also notice that the study of the waves induced by L, in Section 3.4, revealed the presence of trapped equatorial waves, which however do not appear in the mean flow described by Equation (14): no constructive interferences take place in the limiting process, in other words the fast oscillating modes decouple from the mean flow, without creating any additional term in the limit system (that feature was already observed in [9] in the case of inhomogeneous rotating fluid equations, modelling the ocean or the atmosphere at midlatitudes).

We will not prove this theorem here; let us simply note that it is not difficult to prove that the weak limit is necessarily in the kernel of L; the main difficulty of the proof relies in taking weak limits in the nonlinear term, and that consists in a compensated compactness argument, in the spirit of [5] or [9].

3.5.2. *Strong convergence.*

Theorem 3.4. *There is a countable subset* $\mathcal{N}$ *of* $\mathbf{R}^+$ *such that for any* $\beta \in \mathbf{R}^+ \setminus \mathcal{N}$*, the following result holds. Let* $\Phi^0 \in L^2(\mathbf{T} \times \mathbf{R})$ *be given, and consider a family* $((\eta^0_\varepsilon, u^0_\varepsilon))_{\varepsilon > 0}$ *such that*

$$\begin{aligned} \frac{1}{2} \int \left(|\eta^0_\varepsilon|^2 + (1 + \varepsilon \eta^0_\varepsilon) |u^0_\varepsilon|^2 \right) dx \leq \mathcal{E}^0 \quad \textit{and} \\ \frac{1}{2} \int \left(|\eta^0_\varepsilon - \eta^0|^2 + (1 + \varepsilon \eta^0_\varepsilon) |u^0_\varepsilon - u^0|^2 \right) dx \to 0 \ \textit{as}\ \varepsilon \to 0. \end{aligned} \tag{15}$$

For all $\varepsilon > 0$ *denote by* $(\eta_\varepsilon, u_\varepsilon)$ *a solution of (2) with initial data* $(\eta^0_\varepsilon, u^0_\varepsilon)$*. Finally suppose that* $\Pi_P \Phi^0$ *and* $\Pi_K \Phi^0$ *belong to* H^α_L *for some* $\alpha > 1/2$*. Then the sequence of filtered solutions* $\Phi_\varepsilon = \mathcal{L}\left(-\frac{t}{\varepsilon}\right)(\eta_\varepsilon, u_\varepsilon)$ *converges strongly towards* Φ *in*

$L^2_{loc}(\mathbf{R}^+; L^2(\mathbf{T}\times\mathbf{R}))$*, where* Φ *is the unique solution of (12) constructed in Theorem 3.2.*

Note that the strong compactness of (Φ_ε) in $L^2_{loc}(\mathbf{R}^+, L^2(\mathbf{T}\times\mathbf{R}))$ cannot be obtained directly using some a priori estimates, since we have a priori no uniform regularity on η_ε with respect to the space variable. The proof of convergence is actually based on a stability property of (2). The idea is therefore to construct a smooth approximate solution to Φ_ε, writing an asymptotic expansion in ε whose first term is Φ. Then a weak-strong stability method allows to conclude.

Let us give some details. We consider the solution Φ of the limit system constructed in the previous paragraph, and we truncate that solution in the following way:

$$\Phi_N = J_N \Pi_\perp \Phi + \Pi_0 \Phi_N,$$

where J_N is the spectral truncation defined by $J_N = \displaystyle\sum_{i\lambda\in\mathfrak{S}_N} \Pi_\lambda$, with $\mathfrak{S}_N = \left\{ i\tau(n,k,j) \in \mathfrak{S} \big/ n\leq N,\ |k|\leq N \right\}$. Finally $\Pi_0\Phi_N$ solves

$$\partial_t \Pi_0\Phi_N - \nu \Pi_0 \Delta' \Pi_0 \Phi_N = 0, \quad \Pi_0 \Phi_{N|t=0} = \sum_{0\leq n\leq N} \Pi_{n,0,0}\Phi^0,$$

where $\Pi_{n,0,0}$ denotes the projection onto the eigenvector $\Psi_{n,0,0}$ of Ker L. We know that for all fixed $N\in\mathbf{N}$, $\Pi_0\Phi_N$ belongs to $L^\infty(\mathbf{R}^+; H^\sigma_L)$, for all nonnegative σ. Moreover by the stability of the limit system (which is essentially linear, up to the one-dimensional Kelvin modes) we have

$$\lim_{N\to\infty} \|\Pi_0\Phi_N - \Pi_0\Phi\|_{L^\infty([0,T];L^2(\mathbf{T}\times\mathbf{R}))} = 0,\ \forall T>0, \tag{16}$$

We have moreover, for all $T>0$, and as $N\to\infty$,

$$\Big(\|\Pi_\perp(\Phi-\Phi_N)\|_{L^\infty([0,T];L^2(\mathbf{R}\times\mathbf{T}))} + \|(\Pi_K+\Pi_P)(\Phi-\Phi_N)\|_{L^\infty([0,T];H^\alpha_L)} \Big) \to 0 \tag{17}$$

and

$$\Big(\|\Pi_\perp(\Phi-\Phi_N)\|_{L^2([0,T];H^1_L)} + \|(\Pi_K+\Pi_P)(\Phi-\Phi_N)\|_{L^2([0,T];H^{\alpha+1}_L)} \Big) \to 0. \tag{18}$$

Finally since J_N commutes with Δ'_L, the vector field Φ_N satisfies

$$\begin{aligned}
&\partial_t \left(\mathcal{L}\left(\frac{t}{\varepsilon}\right)\Phi_N\right) + \frac{1}{\varepsilon} L\left(\mathcal{L}\left(\frac{t}{\varepsilon}\right)\Phi_N\right) + Q\left(\mathcal{L}\left(\frac{t}{\varepsilon}\right)\Phi_N, \mathcal{L}\left(\frac{t}{\varepsilon}\right)\Phi_N\right)\\
&- \nu\Delta'\mathcal{L}\left(\frac{t}{\varepsilon}\right)\Phi_N = (Q-Q_L)\left(\mathcal{L}\left(\frac{t}{\varepsilon}\right)\Phi_N, \mathcal{L}\left(\frac{t}{\varepsilon}\right)\Phi_N\right) - \nu(\Delta'-\Delta'_L)\mathcal{L}\left(\frac{t}{\varepsilon}\right)\Phi_N\\
&+ (Id-J_N)Q_L\left(\mathcal{L}\left(\frac{t}{\varepsilon}\right)\Phi, \mathcal{L}\left(\frac{t}{\varepsilon}\right)\Phi\right) + Q_L\left(\mathcal{L}\left(\frac{t}{\varepsilon}\right)(\Phi_N-\Phi), \mathcal{L}\left(\frac{t}{\varepsilon}\right)(\Phi_N+\Phi)\right).
\end{aligned}$$

Because of (16) and (17), the last term in the right-hand side is expected to be small when N is large, uniformly in ε, and similarly for the third term, using the stability of the limit system. So we are left with the first two terms, which as usual cannot be dealt with so easily since they do not converge strongly towards zero. However they are fast oscillating terms, and will be treated (in the spirit of [14],

[20]) by introducing a small quantity $\varepsilon\phi_N$ (which will be small when ε goes to zero, for each fixed N), so that

$$(\partial_t + \frac{1}{\varepsilon}L)\left(\mathcal{L}\left(\frac{t}{\varepsilon}\right)\varepsilon\phi_N\right) \sim -(Q-Q_L)\left(\mathcal{L}\left(\frac{t}{\varepsilon}\right)\Phi_N, \mathcal{L}\left(\frac{t}{\varepsilon}\right)\Phi_N\right) \\ +\nu(\Delta' - \Delta'_L)\mathcal{L}\left(\frac{t}{\varepsilon}\right)\Phi_N.$$

We define

$$\phi_N = -\sum_{\substack{\lambda\neq\mu+\tilde\mu \\ i\lambda\in\mathfrak{S}, i\mu, i\tilde\mu\in\mathfrak{S}_N}} \frac{e^{i\frac{t}{\varepsilon}(\lambda-\mu-\tilde\mu)}}{i(\lambda-\mu-\tilde\mu)}\Pi_\lambda Q(\Pi_\mu\Phi_N, \Pi_{\tilde\mu}\Phi_N) \\ +\nu\sum_{\substack{\lambda\neq\mu, \\ i\lambda\in\mathfrak{S}, i\mu\in\mathfrak{S}_N}} \frac{e^{i\frac{t}{\varepsilon}(\lambda-\mu)}}{i(\lambda-\mu)}\Pi_\lambda\Delta'\Pi_\mu\Phi_N,$$

and consider $\Phi_{\varepsilon,N} = \Phi_N + \varepsilon\phi_N$. The main result is the following.

Proposition 3.5. *There is a countable subset $\mathcal{N}$ of $\mathbf{R}^+$ such that for any $\beta \in \mathbf{R}^+\setminus\mathcal{N}$, the following result holds. Consider a vector field $\Phi^0 = (\eta^0, u^0) \in L^2(\mathbf{T}\times\mathbf{R})$, with $(\Pi_P + \Pi_K)\Phi^0$ in H^α_L for some $\alpha > 1/2$. Denote by Φ the associate solution of (12). Then there exists a family $(\eta_{\varepsilon,N}, u_{\varepsilon,N}) = \mathcal{L}\left(\frac{t}{\varepsilon}\right)\Phi_{\varepsilon,N}$, bounded in the space $L^\infty_{loc}(\mathbf{R}^+, L^2)\cap L^2_{loc}(\mathbf{R}^+, H^1)$, such that $(\Pi_P+\Pi_K)(\eta_{\varepsilon,N}, u_{\varepsilon,N})$ is uniformly bounded in the space $L^\infty_{loc}(\mathbf{R}^+, H^\alpha_L)\cap L^2_{loc}(\mathbf{R}^+, H^{\alpha+1}_L)$, and satisfying the following properties:*

- *$\Phi_{\varepsilon,N}$ converges towards Φ as $\varepsilon\to 0$ and $N\to\infty$, in $L^\infty([0,T]; L^2(\mathbf{T}\times\mathbf{R}))$;*
- *for all $N\in\mathbf{N}$, $(\eta_{\varepsilon,N}, u_{\varepsilon,N})$ is smooth: for all $T>0$ and all $Q\in\mathbf{R}[X]$,*

$$Q(x_2)(\eta_{\varepsilon,N}, u_{\varepsilon,N}) \text{ is bounded in } L^\infty([0,T]; C^\infty(\mathbf{T}\times\mathbf{R})), \text{ uniformly in } \varepsilon; \tag{19}$$

- *$(\eta_{\varepsilon,N}, u_{\varepsilon,N})$ satisfies the uniform regularity estimate*

$$\forall T>0, \quad \sup_{N\in\mathbf{N}}\lim_{\varepsilon\to 0}\|\nabla\cdot u_{\varepsilon,N}\|_{L^1([0,T];L^\infty(\mathbf{T}\times\mathbf{R}))} \le C_T; \tag{20}$$

- *$U_{\varepsilon,N} = (\eta_{\varepsilon,N}, u_{\varepsilon,N})$ satisfies approximatively (2):*

$$\partial_t U_{\varepsilon,N} + \frac{1}{\varepsilon}LU_{\varepsilon,N} + QU_{\varepsilon,N} - \nu\Delta' U_{\varepsilon,N} = R_{\varepsilon,N} \tag{21}$$

where $R_{\varepsilon,N}$ goes to 0 as $\varepsilon\to 0$ then $N\to\infty$: for all $T>0$,

$$\lim_{N\to\infty}\lim_{\varepsilon\to 0}\left(\|R_{\varepsilon,N}\|_{L^1([0,T];L^2(\mathbf{T}\times\mathbf{R}))} + \varepsilon\|R_{\varepsilon,N}\|_{L^\infty([0,T]\times\mathbf{T}\times\mathbf{R})}\right) = 0. \tag{22}$$

Let us give some indications of the proof of that result.

The convergence and regularity results are due to the definition of ϕ_N and to the asymptotic behaviour of the eigenvalues of L and will not be proved here. It remains then to establish the equation satisfied by $(\eta_{\varepsilon,N}, u_{\varepsilon,N})$. A direct computation provides

$$\begin{aligned}\varepsilon\partial_t\phi_N = &-\sum_{\substack{\lambda\neq\mu+\tilde\mu\\ i\lambda\in\mathfrak{S},i\mu,i\tilde\mu\in\mathfrak{S}_N}} e^{i\frac{t}{\varepsilon}(\lambda-\mu-\tilde\mu)}\Pi_\lambda Q(\Pi_\mu\Phi_N,\Pi_{\tilde\mu}\Phi_N)\\ &+\nu\sum_{\substack{\lambda\neq\mu,\\ i\lambda\in\mathfrak{S},i\mu\in\mathfrak{S}_N}} e^{i\frac{t}{\varepsilon}(\lambda-\mu)}\Pi_\lambda\Delta'\Pi_\mu\Phi_N\\ &-2\varepsilon\sum_{\substack{\lambda\neq\mu+\tilde\mu\\ i\lambda\in\mathfrak{S},i\mu,i\tilde\mu\in\mathfrak{S}_N}} \frac{e^{i\frac{t}{\varepsilon}(\lambda-\mu-\tilde\mu)}}{i(\lambda-\mu-\tilde\mu)}\Pi_\lambda Q(\Pi_\mu\partial_t\Phi_N,\Pi_{\tilde\mu}\Phi_N)\\ &+\varepsilon\nu\sum_{\substack{\lambda\neq\mu,\\ i\lambda\in\mathfrak{S},i\mu\in\mathfrak{S}_N}} \frac{e^{i\frac{t}{\varepsilon}(\lambda-\mu)}}{i(\lambda-\mu)}\Pi_\lambda\Delta'\Pi_\mu\partial_t\Phi_N.\end{aligned}$$

It is easy to see that $\partial_t\Phi_N$ is smooth and rapidly decaying, and thus the last two terms go to zero as $\varepsilon\to 0$ (for all fixed N). We have therefore

$$\begin{aligned}\varepsilon\partial_t\phi_N = -\mathcal{L}\left(-\frac{t}{\varepsilon}\right)(Q-Q_L)\left(\mathcal{L}\left(\frac{t}{\varepsilon}\right)\Phi_N,\mathcal{L}\left(\frac{t}{\varepsilon}\right)\Phi_N\right)\\ +\nu\mathcal{L}\left(-\frac{t}{\varepsilon}\right)(\Delta'-\Delta'_L)\mathcal{L}\left(\frac{t}{\varepsilon}\right)\Phi_N+r_{\varepsilon,N}\end{aligned}$$

where

$$\forall T>0,\forall k\in\mathbf{N},\forall N\in\mathbf{N},\forall Q\in\mathbf{R}[X],\quad \lim_{\varepsilon\to0}\|Q(x_2)r_{\varepsilon,N}\|_{L^\infty([0,T];C^k(\mathbf{T}\times\mathbf{R}))}=0.$$

Finally

$$\begin{aligned}&\partial_t(\eta_{\varepsilon,N},u_{\varepsilon,N})+\frac{1}{\varepsilon}L(\eta_{\varepsilon,N},u_{\varepsilon,N})+Q\left((\eta_{\varepsilon,N},u_{\varepsilon,N}),(\eta_{\varepsilon,N},u_{\varepsilon,N})\right)-\nu\Delta'(\eta_{\varepsilon,N},u_{\varepsilon,N})\\ &=(Id-J_N)Q_L\left(\mathcal{L}\left(\frac{t}{\varepsilon}\right)\Phi,\mathcal{L}\left(\frac{t}{\varepsilon}\right)\Phi\right)+Q_L\left(\mathcal{L}\left(\frac{t}{\varepsilon}\right)(\Phi_N-\Phi),\mathcal{L}\left(\frac{t}{\varepsilon}\right)(\Phi_N+\Phi)\right)\\ &+\varepsilon Q\left(\mathcal{L}\left(\frac{t}{\varepsilon}\right)\phi_N,\mathcal{L}\left(\frac{t}{\varepsilon}\right)(2\Phi_N+\varepsilon\phi_N)\right)-\varepsilon\nu\Delta'\left(\mathcal{L}\left(\frac{t}{\varepsilon}\right)\phi_N\right)+r_{\varepsilon,N}.\end{aligned}$$

The regularity estimates on Φ_N and ϕ_N allow to prove that the two last explicit terms in the right-hand side go to zero as $\varepsilon\to0$ (for all fixed N), and therefore to incorporate them into the remainder $r_{\varepsilon,N}$. Stability arguments (which are left out here) on the limit filtered system then enable us to prove that the other terms on the right-hand side go to zero, and therefore that $(\eta_{\varepsilon,N},u_{\varepsilon,N})$ satisfies the expected approximate equation (21), where $R_{\varepsilon,N}$ satisfies the expected estimate (22).

The result is proved.

Equipped with that result, we are now ready to prove the strong convergence theorem. The method relies on a weak-strong stability method which we shall now detail. We are going to prove that

$$\forall T>0,\quad \lim_{N\to\infty}\lim_{\varepsilon\to0}\|(\eta_\varepsilon,u_\varepsilon)-(\eta_{\varepsilon,N},u_{\varepsilon,N})\|_{L^2([0,T]\times\mathbf{T}\times\mathbf{R})}=0, \tag{23}$$

where $(\eta_{\varepsilon,N},u_{\varepsilon,N})$ is the approximate solution defined in Proposition 3.5. Note that combining this estimate with the fact that $(\eta_{\varepsilon,N},u_{\varepsilon,N})$ is close to $\mathcal{L}\left(\frac{t}{\varepsilon}\right)\Phi$ provides the expected convergence, namely the fact that

$$\forall T>0,\quad \lim_{\varepsilon\to0}\left\|(\eta_\varepsilon,u_\varepsilon)-\mathcal{L}\left(\frac{t}{\varepsilon}\right)\Phi\right\|_{L^2([0,T]\times\mathbf{T}\times\mathbf{R})}=0.$$

The key to the proof of (23) lies in the following proposition.

Proposition 3.6. *There is a constant C such that the following property holds. Let (η^0, u^0) and $(\eta^0_\varepsilon, u^0_\varepsilon)$ satisfy assumption (15), and let $T > 0$ be given. For all $\varepsilon > 0$, denote by $(\eta_\varepsilon, u_\varepsilon)$ a solution of (2) with initial data (η^0, u^0). For any couple of vector fields $(\overline{\eta}, \overline{u})$ belonging to $L^\infty([0,T]; C^\infty(\mathbf{T}\times\mathbf{R}))$ and rapidly decaying with respect to y, define*

$$\mathcal{E}_\varepsilon(t) = \frac{1}{2}\int \left((\eta_\varepsilon - \overline{\eta})^2 + (1+\varepsilon\eta_\varepsilon)|u_\varepsilon - \overline{u}|^2\right)(t,x)dx + \nu\int_0^t\int |\nabla(u_\varepsilon - \overline{u})|^2(t',x)dxdt'.$$

Then the following stability inequality holds for all $t \in [0,T]$:

$$\begin{aligned}
&\mathcal{E}_\varepsilon(t) \le C\mathcal{E}_\varepsilon(0)\exp\left(\chi(t)\right) + \omega_\varepsilon(t)\\
&+ C\int_0^t e^{\chi(t)-\chi(t')}\int \left(\partial_t\overline{\eta} + \frac{1}{\varepsilon}\nabla\cdot\overline{u} + \nabla\cdot(\overline{\eta}\overline{u})\right)(\overline{\eta} - \eta_\varepsilon)(t',x)dxdt'\\
&+ C\int_0^t e^{\chi(t)-\chi(t')}\int (1+\varepsilon\eta_\varepsilon)\left(\partial_t\overline{u} + \frac{1}{\varepsilon}(\beta x_2\overline{u}^\perp + \nabla\overline{\eta}) + (\overline{u}\cdot\nabla)\overline{u} - \nu\Delta\overline{u}\right)\\
&\qquad\qquad\qquad\qquad\qquad\qquad\cdot(\overline{u} - u_\varepsilon)(t',x)dxdt',
\end{aligned}$$

where $\omega_\varepsilon(t)$ depends on $\overline{u}$ and goes to zero with ε, uniformly in time, and where

$$\chi(t) = C\int_0^t \left(\|\nabla\cdot\overline{u}\|_{L^\infty(\mathbf{T}\times\mathbf{R})} + \|\nabla\overline{u}\|^2_{L^2(\mathbf{T}\times\mathbf{R})}\right)(t')dt'.$$

Let us omit the proof of that result, and end the proof of Theorem 3.4. We apply that proposition to $(\overline{\eta}, \overline{u}) = (\eta_{\varepsilon,N}, u_{\varepsilon,N})$, where $(\eta_{\varepsilon,N}, u_{\varepsilon,N})$ is the approximate solution given by Proposition 3.5. We will denote by $\chi_{\varepsilon,N}$ and $\mathcal{E}_{\varepsilon,N}$ the quantities defined in Proposition 3.6, where $(\overline{\eta}, \overline{u})$ has been replaced by $(\eta_{\varepsilon,N}, u_{\varepsilon,N})$. Because of the uniform regularity estimates on $(\eta_{\varepsilon,N}, u_{\varepsilon,N})$, we have

$$\forall T > 0, \quad \sup_N \lim_{\varepsilon\to 0}\left(\|\nabla u_{\varepsilon,N}\|^2_{L^2([0,T];L^2(\mathbf{T}\times\mathbf{R}))} + \|\nabla\cdot u_{\varepsilon,N}\|_{L^1([0,T];L^\infty(\mathbf{R}\times\mathbf{T}))}\right) \le C_T,$$

so we get a uniform bound on $\chi_{\varepsilon,N}$: $\sup_N \lim_{\varepsilon\to 0}\|\chi_{\varepsilon,N}\|_{L^\infty([0,T])} \le C_T$. Then, from the initial convergence (15) we obtain that

$$\forall N \in \mathbf{N}, \quad \mathcal{E}_{\varepsilon,N}(0)\exp\left(\chi_{\varepsilon,N}(t)\right) \to 0 \text{ as } \varepsilon \to 0 \text{ in } L^\infty([0,T]).$$

Moreover by Proposition 3.6 we have

$$\begin{aligned}
\partial_t(\eta_{\varepsilon,N}, u_{\varepsilon,N}) + \frac{1}{\varepsilon}L(\eta_{\varepsilon,N}, u_{\varepsilon,N}) + Q((\eta_{\varepsilon,N}, u_{\varepsilon,N}), (\eta_{\varepsilon,N}, u_{\varepsilon,N}))&\\
- \nu\Delta'(\eta_{\varepsilon,N}, u_{\varepsilon,N}) = R_{\varepsilon,N}.& \quad (24)
\end{aligned}$$

Finally it is not too difficult to estimate the contribution of the remainder term and to prove that

$$\begin{aligned}
&\int_0^t e^{\chi_{\varepsilon,N}(t)-\chi_{\varepsilon,N}(t')}\int R_{\varepsilon,N}\cdot\left((\eta_{\varepsilon,N} - \eta_\varepsilon), (1+\varepsilon\eta_\varepsilon)(u_{\varepsilon,N} - u_\varepsilon)\right)(t',x)dxdt'\\
&\le \frac{1}{2}(\|\eta_{\varepsilon,N} - \eta_\varepsilon\|^2_{L^\infty([0,T];L^2)} + \|\sqrt{1+\varepsilon\eta_\varepsilon}(u_{\varepsilon,N} - u_\varepsilon)\|^2_{L^\infty([0,T];L^2)}) + \omega_{\varepsilon,N}(t),
\end{aligned}$$

where

$$\lim_{N\to\infty}\lim_{\varepsilon\to 0}\|\omega_{\varepsilon,N}(t)\|_{L^\infty([0,T])} = 0.$$

We now recall that by Proposition 3.6, using (24), we have

$$\mathcal{E}_{\varepsilon,N}(t) \leq C\mathcal{E}_{\varepsilon,N}(0)\exp\left(\chi_{\varepsilon,N}(t)\right) + \omega_\varepsilon(t)$$
$$+ C\int_0^t e^{\chi_{\varepsilon,N}(t)-\chi_{\varepsilon,N}(t')}\int R_{\varepsilon,N}\cdot\left((\eta_{\varepsilon,N}-\eta_\varepsilon),(1+\varepsilon\eta_\varepsilon)(u_{\varepsilon,N}-u_\varepsilon)\right)(t',x)dxdt'$$

where $\mathcal{E}_{\varepsilon,N}(t)$ is equal to

$$\frac{1}{2}\left(\|(\eta_\varepsilon-\eta_{\varepsilon,N})(t)\|_{L^2}^2 + \|\sqrt{1+\varepsilon\eta_\varepsilon}(u_\varepsilon-u_{\varepsilon,N})(t)\|_{L^2}^2\right)$$
$$+\nu\int_0^t \|\nabla(u_\varepsilon-u_{\varepsilon,N})(t')\|_{L^2}^2(t')dt'.$$

Putting together the previous results we get that $\lim_{N\to\infty}\lim_{\varepsilon\to0}\mathcal{E}_{\varepsilon,N}(t) = 0$ uniformly on $[0,T]$, hence that

$$\lim_{N\to\infty}\lim_{\varepsilon\to0}\|\eta_{\varepsilon,N}-\eta_\varepsilon\|_{L^\infty([0,T];L^2(\mathbf{T}\times\mathbf{R}))} = 0,$$
$$\lim_{N\to\infty}\lim_{\varepsilon\to0}\|\sqrt{1+\varepsilon\eta_\varepsilon}(u_{\varepsilon,N}-u_\varepsilon)\|_{L^\infty([0,T];L^2(\mathbf{T}\times\mathbf{R}))} = 0,$$
$$\lim_{N\to\infty}\lim_{\varepsilon\to0}\|u_{\varepsilon,N}-u_\varepsilon\|_{L^2([0,T];\dot H^1(\mathbf{T}\times\mathbf{R}))} = 0.$$

By interpolation we therefore find that

$$\lim_{N\to\infty}\lim_{\varepsilon\to0}\left(\|\eta_{\varepsilon,N}-\eta_\varepsilon\|_{L^\infty([0,T];L^2(\mathbf{T}\times\mathbf{R}))} + \|u_{\varepsilon,N}-u_\varepsilon\|_{L^2([0,T];H^1(\mathbf{T}\times\mathbf{R}))}\right) = 0,$$

hence (23) is proved, as well as Theorem 3.4.

4. **Open problems.** In this survey we have presented two different types of results, concerning the shallow-water model in $\mathbf{T}\times\mathbf{R}$ under the betaplane approximation, in the limit of small Rossby and Froude numbers.

On the one hand the existence of uniformly bounded solutions was investigated. In the framework of unique solutions, such a result was obtained by writing energy estimates using appropriate (non scalar) vector fields which commute with the penalization operator. In the framework of weak, possibly non unique solutions, that was an easy consequence of the theory of compressible Navier-Stokes equations, along with the skew-symmetry of the penalization operator in L^2.

On the other hand the asymptotics of those solutions was studied, in a weak sense for strong solutions, and in a strong sense for weak solutions.

It seems that the analysis of the strong asymptotics of strong solutions should be obtained as well, and should come from putting together both types of techniques.

Let us now present some other questions which seem important to understand. First, the boundary conditions considered in this paper are of course not realistic: first of all we have neglected the presence of shores which lead to lateral boundary conditions (instead of periodic boundary conditions), and second the latitude was chosen in $\mathbf{R}$. In the case of straight lateral boundaries the analysis should not be too different to the one carried out here, but realistic ones seem much more difficult since the spectral structure of the penalization operator has to be analysed again in that new situation, and will ceratinly not be as explicit. Similarly the variable x_2 was taken in $\mathbf{R}$, which is of course not physically reasonable since x_2 stands for the latitude. The justification of that choice was that the strong decay of the modes associated wtih L far from the equator should imply that boundaries far enough from the equator do not affect the asymptotics. This of course needs to be justified.

This discussion actually leads to a more general problem of studying a "full" model which would incorporate both the betaplane approximation near the equator, and an f-plane type approximation at midlatitudes (in such regions the Coriolis force may be considered as a constant; such models, in simplified geometries, are mathematically quite well understood and documented). In fact this means more generally writing the equation on the full globe, and trying to understand the coupling between the different effects.

It should finally be emphasized that there exists to our knowledge no mathematical justification of the shallow water model studied in this paper, and commonly used in Physics. Indeed this model originates from a vertical averaging of the three dimensional Navier-Stokes equations with free surface, assuming that vertical variations are much smaller than in the other directions. On the one hand, one should try to justify in some rigorous way this assumption, and on the other hand supposing it holds, a proper mathematical derivation of the limit (even in the case of a simplified geometry) is lacking.

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