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ON THE RADIUS OF ANALYTICITY OF SOLUTIONS TO SEMI-LINEAR PARABOLIC SYSTEMS

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Abstract. We study the radius of analyticity $R(t)$ in space, of strong solutions to systems of scale-invariant semi-linear parabolic equations. It is well-known that near the initial time, $R(t)t^{-\frac{1}{2}}$ is bounded from below by a positive constant. In this paper we prove that $\liminf_{t\to 0} R(t)t^{-\frac{1}{2}} = \infty$, and assuming higher regularity for the initial data, we obtain an improved lower bound near time zero. As an application, we prove that for any global solution u in $C([0,\infty); H^{\frac{1}{2}}(\mathbb{R}^3))$ of the Navier-Stokes equations, there holds $\liminf_{t\to\infty} R(t)t^{-\frac{1}{2}} = \infty$.

Keywords: Semi-linear parabolic systems, Radius of analyticity, Navier-Stokes equations

AMS Subject Classification (2000): 35K55

1. Introduction

We consider the following system of N equations on $\mathbb{R}^+ \times \mathbb{R}^d$:

$$\text{(SP)} \quad \begin{cases} \partial_t U - \Delta U = P(U) \quad \text{with} \\ P_j(U) \stackrel{\text{def}}{=} \sum_{\substack{\ell\in\mathbb{N}^N \\ |\ell|=k}} A_{j,\ell}(D)(U^\ell) \quad \text{for } j \text{ in } \{1,\dots,N\}, \\ U_{|t=0} = U_0\,, \end{cases}$$

where $A_{j,\ell}(D)$ are homogeneous Fourier multipliers of degree $\beta \in [0,2[$, and $U = (U_j)_{1\le j\le N}$. The order of the nonlinearity is $k \ge 2$ and we have written $U^\ell = \prod_{j=1}^N U_j^{\ell_j}$. An important property of a such a system is its scaling invariance: if a function U satisfies (SP) on a time interval $[0,T]$ with the initial data U_0, then the function U_λ defined by

$$U_\lambda(t,x) \stackrel{\text{def}}{=} \lambda^\alpha U(\lambda^2 t, \lambda x)$$

satisfies (SP) on the time interval $[0,\lambda^{-2}T]$ with the initial data $U_{0,\lambda} \stackrel{\text{def}}{=} \lambda^\alpha U_0(\lambda\,\cdot)$ for

$$\alpha \stackrel{\text{def}}{=} \frac{2-\beta}{k-1}\,\cdot$$

Note that α is positive, and in the following we shall assume that $\alpha \le d/k$. For example for the Navier-Stokes equations there holds $\beta = 1$ and $k = 2$, while for the cubic heat equation there holds $\beta = 0$ and $k = 3$. In both cases $\alpha = 1$. The scaling invariant Sobolev space for the initial data is $H^{s_{\rm crit}}(\mathbb{R}^d)$, with $s_{\rm crit} \stackrel{\rm def}{=} \frac{d}{2} - \alpha$, recalling that $H^s(\mathbb{R}^d)$ is defined by the following norm, for $s < d/2$:

$$\|f\|_{H^s(\mathbb{R}^d)} \stackrel{\rm def}{=} \Big(\int_{\mathbb{R}^d} |\xi|^{2s} |\widehat{f}(\xi)|^2 \, d\xi \Big)^{\frac12} ,$$

where $\widehat{f} = \mathcal{F} f$ is the Fourier transform of f.

The question of solving the Cauchy problem for systems such as (SP) in scale invariant spaces has been widely studied. We shall make no attempt at listing all the results on the subject but simply recall the typical so-called Kato-type theorem, which may be proved by a Banach fixed point argument (see for instance [1, 7, 9, 13, 14] among others as well as the estimates in the proof of Lemma 2.1 below).

Theorem 1.1. *Assume $\alpha \in \,]1/k, d/k]$ and define $s_{\rm crit} \stackrel{\rm def}{=} d/2 - \alpha$. Let p be given, in the interval $]\max(2/\alpha, k), \infty[$. Then, for any $\delta \in [0, \alpha[$, for any initial data U_0 belonging to $H^{s_{\rm crit}+\delta}(\mathbb{R}^d)$, a positive time T exists such that the system (SP) has a unique solution U in the Kato space K^p_T such that*

$$\|U\|_{K^p_T} \stackrel{\rm def}{=} \sup_{t \leq T} t^{\frac1p} \|U(t)\|_{H^{s_{\rm crit}+\frac2p}} < \infty \, . \tag{1.1}$$

Moreover, if δ is positve, a constant c exists such that $T \geq c \|U_0\|^{-\frac{2}{\delta}}_{H^{s_{\rm crit}+\delta}}$.

The goal of this article is to analyze the instantaneous smoothing effect of (SP). Let us start by recalling some well-known facts in the case of the Navier-Stokes equations.

$$(\text{NS}) \quad \begin{cases} \partial_t u + u \cdot \nabla u - \Delta u + \nabla p = 0 \, , & (t,x) \in \mathbb{R}^+ \times \mathbb{R}^d \, , \\ \operatorname{div} u = 0 \, , \\ u|_{t=0} = u_0 \, , \end{cases}$$

where $u = (u^1, \ldots, u^d)$ denotes the velocity of the fluid and p is the scalar pressure function. in [5], the analyticity of smooth periodic solutions to the Navier-Stokes equations (NS) is proved, in the sense that if v solves (NS) then $e^{\sigma\sqrt{-t\Delta}} v(t)$ is a smooth function for some $\sigma > 0$. This result was extended in [3, 8, 11, 12] where it is proved for instance that

$$\int_{\mathbb{R}^3} |\xi| \big(\sup_{t \leq T} e^{\sqrt{t}|\xi|} |\widehat{v}(t,\xi)| \big)^2 \, d\xi + \int_0^T \int_{\mathbb{R}^3} |\xi|^3 \big(e^{\sqrt{t}|\xi|} |\widehat{v}(t,\xi)| \big)^2 \, d\xi dt < \infty \, ,$$

which shows that the radius of analyticity $R(t)$ of $v(t)$ is bounded from below by $\sqrt{t}$. Note that the above condition is equivalent to the fact that $e^{\sqrt{-t\Delta}} v(t)$ belongs to $E^\infty_T \cap E^2_T$, where E^q_T denotes the space of vector fields V such that

$$\|V\|_{E^q_T} \stackrel{\rm def}{=} \big\| 2^{j\left(\frac12 + \frac2q\right)} \|\Delta_j V\|_{L^q([0,T];L^2(\mathbb{R}^3))} \big\|_{\ell^2(\mathbb{Z})} \, .$$

This type of result is also known to hold in the more general context of (SP) (see [4, 10] for instance) and may be stated as follows.

Theorem 1.2. *The solution constructed in Theorem* 1.1 *is analytic for positive t with radius of analyticity $R(t)$ greater than $\sqrt{t}$.*

The purpose of this work is the proof of the following improved theorem.

Theorem 1.3. (a) *If δ satisfies $\max(2/\alpha, k) < 2/\delta < \infty$, the solution constructed in Theorem 1.1 satisfies*

$$\liminf_{t\to 0} \frac{R(t)}{t^{\frac12}\sqrt{-\log\big(t\|U_0\|^{\frac{2}{\delta}}_{H^{s_{\rm crit}+\delta}}\big)}} \geq \sqrt{2\delta}\,.$$

(b) *If $\delta = 0$, then, for any small enough, positive ε and for any p in $]\max(2/\alpha, k), \infty[$ we have*

$$\liminf_{t\to 0} \frac{R(t)}{t^{\frac12}\sqrt{-\log\big(\|e^{\varepsilon\tau\Delta}U_0\|_{K^p_t}\big)}} \geq 2\sqrt{1-\varepsilon}\,.$$

In particular $\displaystyle\lim_{t\to 0}\frac{R(t)}{t^{\frac12}} = \infty.$

We remark that in the case of three-dimensional incompressible Navier-Stokes system (NS), part (a) of Theorem 1.3 coincides with Theorem 1.3 of [8]. Moreover, the main idea used to prove Theorem 1.3 can be applied to investigate the radius of analyticity of any global solution of (NS). More precisely we can prove the following result.

Corollary 1.1. *Let $u \in C([0,\infty); H^{\frac12}(\mathbb{R}^3))$ be a global solution of (NS). Then one has*

$$\liminf_{t\to\infty}\frac{R(t)}{t^{\frac12}} = \infty\,. \tag{1.2}$$

2. Proof of Theorem 1.3

We shall perform all our computations on the approximated system

$$(\mathrm{SP}_n)\quad \begin{cases} \partial_t U - \Delta U = P_n(U)\\ U_{|t=0} = U_0\,, \end{cases} \qquad \text{with}\quad P_{n,j}(U) \stackrel{\rm def}{=} \sum_{\substack{\ell\in\mathbb{N}^N\\ |\ell|=k}} \mathbf{1}_{B(0,n)}(D)A_{j,\ell}(D)(U^\ell)$$

for j in $\{1,\dots,N\}$, and where we have written $\mathbf{1}_{B(0,n)}$ for the characteristic function of the ball $B(0,n) \stackrel{\rm def}{=} \Big\{\xi\in\mathbb{R}^d;\ |\xi|\leq n\Big\}$. The system (SP_n) is an ordinary differential equation in all Sobolev spaces. All the quantities we shall write are defined in this case, and we neglect the index n in all that follows. We also skip the final stage of passing to the limit when n tends to infinity.

Let us consider three positive real numbers T, λ and ε which will be chosen later on in the proof. Motivated by [8], we define

$$U_a(t,x) \stackrel{\rm def}{=} \mathcal{F}^{-1}\big(e^{-\frac{\lambda^2}{4(1-\varepsilon)}\frac{t}{T}+\lambda\frac{t}{\sqrt{T}}|\xi|}|\widehat{U}(t,\xi)|\big)\,. \tag{2.1}$$

The main point is that the function U_a behaves like a solution of a modified system (SP) where the viscosity is ε instead of 1 and the non-linear term has a factor $e^{\frac{\lambda^2(k-1)}{4(1-\varepsilon)}}$. We shall make this idea more precise in what follows.

The key ingredient used to prove Theorem 1.3 will be the following lemma.

Lemma 2.1. *Let U_a be defined by* (2.1). *Then for any p in* $]\max(2/\alpha, k), \infty[$, *there exists a positive constant $C_{k,\varepsilon}$ such that*

$$\|U_a\|_{K_T^p} \leq \|e^{\varepsilon t\Delta}U_0\|_{K_T^p} + C_{k,\varepsilon}\Big(e^{\frac{\lambda^2}{4(1-\varepsilon)}}\|U_a\|_{K_T^p}\Big)^{k-1}\|U_a\|_{K_T^p}\,. \tag{2.2}$$

Proof. A solution of (SP_n) satisfies

$$|\widehat{U}(t,\xi)| \leq e^{-t|\xi|^2}|\widehat{U}_0(\xi)| + C\int_0^t e^{-(t-t')|\xi|^2}|\xi|^\beta \big(\underbrace{|\widehat{U}(t')|\star\cdots\star|\widehat{U}(t')|}_{k \text{ times}}\big)(\xi)dt'\,. \tag{2.3}$$

Let us observe that

$$-\frac{\lambda^2}{4(1-\varepsilon)}\frac{1}{T} + \frac{\lambda}{\sqrt{T}}|\xi| - |\xi|^2 \leq -\varepsilon|\xi|^2.$$

Thus by definition (2.1), we infer from (2.3) that

$$\begin{aligned}\widehat{U}_a(t,\xi) \leq e^{-\varepsilon t|\xi|^2}|\widehat{U}_0(\xi)| + C\int_0^t & e^{-\varepsilon(t-t')|\xi|^2}e^{-\frac{\lambda^2}{4(1-\varepsilon)}\frac{t'}{T}+\lambda\frac{t'}{\sqrt{T}}|\xi|}\\ & \times |\xi|^\beta\big(\underbrace{|\widehat{U}(t')|\star\cdots\star|\widehat{U}(t')|}_{k \text{ times}}\big)(\xi)dt'.\end{aligned}$$

Notice that

$$\big(\underbrace{|\widehat{U}(t')|\star\cdots\star|\widehat{U}(t')|}_{k \text{ times}}\big)(\xi) = \int_{\sum_{\ell=1}^k \xi_\ell=\xi}\Big(\prod_{\ell=1}^k|\widehat{U}(t',\xi_\ell)|\Big)d\xi_1\ldots d\xi_k\,,$$

and using that, for any $(\xi_j)_{1\leq j\leq k}$ in $(\mathbb{R}^d)^k$ such that $\displaystyle\sum_{j=1}^k \xi_j = \xi$, there holds $e^{|\xi|} \leq \displaystyle\prod_{j=1}^k e^{|\xi_j|}$, we infer that

$$\begin{aligned}\widehat{U}_a(t,\xi) \leq e^{-\varepsilon t|\xi|^2}|\widehat{U}_0(\xi)| + Ce^{\frac{\lambda^2(k-1)}{4(1-\varepsilon)}}\int_0^t & e^{-\varepsilon(t-t')|\xi|^2}|\xi|^\beta\\ & \times\big(\underbrace{\widehat{U}_a(t')\star\cdots\star\widehat{U}_a(t')}_{k \text{ times}}\big)(\xi)dt'\,.\end{aligned} \tag{2.4}$$

Let us recall the following result on products in Sobolev spaces: for any positive real number s, smaller than $d/2$ and greater than $d/2 - d/k$, there holds

$$\Big\|\prod_{\ell=1}^k a_k\Big\|_{H^{ks-(k-1)\frac{d}{2}}} \leq C_k\prod_{\ell=1}^k\|a_k\|_{H^s}\,. \tag{2.5}$$

Now let us choose p in $[1,\infty]$ such that

$$0 < \frac{2}{p} < \alpha \quad\text{and set}\quad s_p \stackrel{\text{def}}{=} s_{\text{crit}} + \frac{2}{p} = \frac{d}{2} + \frac{2}{p} - \alpha\,. \tag{2.6}$$

Notice that

$$\underbrace{\widehat{U}_a(t')\star\cdots\star\widehat{U}_a(t')}_{k \text{ times}} = (2\pi)^{kd}\mathcal{F}(U_a^k(t'))\,.$$

The assumption that $\alpha \leq \frac{d}{k}$ implies that $\alpha < \frac{d}{k} + \frac{2}{p}$ and $s_p > \frac{d}{2} - \frac{d}{k}$, so (2.5) ensures that

$$\big(\underbrace{\widehat{U}_a(t') \star \cdots \star \widehat{U}_a(t')}_{k \text{ times}}\big)(\xi) \leq C^k |\xi|^{-\left(s_p + (k-1)\left(\frac{2}{p} - \alpha\right)\right)} f(t', \xi) \|U_a(t')\|^k_{H^{s_p}}$$

$$\text{with} \quad \|f(t')\|_{L^2(\mathbb{R}^d)} = 1\,.$$

As $\alpha(k-1) = 2 - \beta$, plugging the above inequality in (2.4) gives

$$\widehat{U}_a(t,\xi) \leq e^{-\varepsilon t|\xi|^2} |\widehat{U}_0(\xi)| + C^k e^{\frac{\lambda^2(k-1)}{4(1-\varepsilon)}} \int_0^t e^{-\varepsilon(t-t')|\xi|^2} \times |\xi|^{-s_p + 2 - (k-1)\frac{2}{p}} f(t',\xi) \|U_a(t')\|^k_{H^{s_p}} dt'.$$

By multiplication of this inequality by $t^{\frac{1}{p}} |\xi|^{s_p}$ and by definition of the norm $\|\cdot\|_{K^p_T}$ in (1.1), we get, for any t in the interval $[0,T]$,

$$t^{\frac{1}{p}} |\xi|^{s_p} \widehat{U}_a(t,\xi) \leq t^{\frac{1}{p}} |\xi|^{s_p} e^{-\varepsilon t|\xi|^2} |\widehat{U}_0(\xi)| + C^k e^{\frac{\lambda^2(k-1)}{4(1-\varepsilon)}} \|U_a\|^k_{K^p_T} t^{\frac{1}{p}} \int_0^t e^{-\varepsilon(t-t')|\xi|^2} \frac{1}{(t')^{\frac{k}{p}}} |\xi|^{2\left(1 - \frac{k-1}{p}\right)} f(t',\xi) dt'.$$

If we assume that p is greater than k, the function $y \in [0,\infty] \mapsto y^{1-\frac{k-1}{p}} e^{-\varepsilon y}$ is bounded, we infer that for any t in the interval $[0,T]$,

$$t^{\frac{1}{p}} |\xi|^{s_p} \widehat{U}_a(t,\xi) \leq t^{\frac{1}{p}} |\xi|^{s_p} e^{-\varepsilon t|\xi|^2} |\widehat{U}_0(\xi)| + C_{k,\varepsilon} e^{\frac{\lambda^2(k-1)}{4(1-\varepsilon)}} \|U_a\|^k_{K^p_T} t^{\frac{1}{p}} \int_0^t \frac{1}{(t-t')^{1-\frac{k-1}{p}}} \frac{1}{t'^{\frac{k}{p}}} f(t',\xi) dt'.$$

Taking the L^2 norm with respect to the variable ξ gives, for any t in the interval $[0,T]$,

$$t^{\frac{1}{p}} \|U_a(t)\|_{H^{s_p}} \leq t^{\frac{1}{p}} \|e^{\varepsilon t\Delta} U_0\|_{H^{s_p}} + C_{k,\varepsilon} e^{\frac{\lambda^2(k-1)}{4(1-\varepsilon)}} \|U_a\|^k_{K^p_T}.$$

Taking the supremum with respect to t in the interval $[0,T]$ gives (2.2). □

Let us now turn to the proof of Theorem 1.3.

Proof of Theorem 1.3. Let U and U_a be determined respectively by (SP_n) and (2.1). We make the following induction hypothesis where p is any real number in $]\max(2/\alpha, k), \infty[$

$$\|U_a\|_{K^p_T} \leq c_{k,\varepsilon} e^{-\frac{\lambda^2}{4(1-\varepsilon)}} \quad \text{with} \quad c_{k,\varepsilon} \stackrel{\text{def}}{=} \frac{1}{(4C_{k,\varepsilon})^{\frac{1}{k-1}}} \tag{2.7}$$

with $C_{k,\varepsilon}$ being determined by Lemma 2.1. As long as this induction hypothesis is satisfied, Inequality (2.2) becomes

$$\|U_a\|_{K^p_T} \leq \frac{4}{3} \|e^{\varepsilon t\Delta} U_0\|_{K^p_T}\,. \tag{2.8}$$

Now let us distinguish the case when U_0 belongs to the space $H^{s_{\text{crit}}+\delta}$ from the case when U_0 belongs only to the critical space $H^{s_{\text{crit}}}$.

(a) The case when U_0 belongs to the space $H^{s_{\text{crit}}+\delta}$.

We first observe that $p \leq \frac{2}{\delta}$

$$\|e^{\varepsilon t\Delta}U_0\|_{K^p_T} \leq C_{\delta,p} T^{\frac{\delta}{2}} \|U_0\|_{H^{s_{\rm crit}+\delta}} \tag{2.9}$$

Let us define

$$T_\varepsilon(U_0) \overset{\rm def}{=} \eta_\varepsilon \|U_0\|^{-\frac{2}{\delta}}_{H^{s_{\rm crit}+\delta}} \quad \text{with} \quad \eta_\varepsilon \overset{\rm def}{=} \left(\frac{c_{k,\varepsilon}}{2C_{\delta,p}}\right)^{\frac{2}{\delta}}.$$

By definition of $T_\varepsilon(U_0)$, we have that for any $T \leq T_\varepsilon(U_0)$,

$$2C_{\delta,p} T^{\frac{\delta}{2}} \|U_0\|_{H^{s_{\rm crit}+\delta}} \leq c_{k,\varepsilon}.$$

Now let us define

$$\lambda_T \overset{\rm def}{=} \sqrt{2\delta(1-\varepsilon)}\log^{\frac{1}{2}}\left(\frac{\eta_\varepsilon}{T\|U_0\|^{\frac{2}{\delta}}_{H^{s_{\rm crit}+\delta}}}\right).$$

Then for $T \leq T_\varepsilon(U_0)$, we deduce from the Ineqalities (2.8) and (2.9) that

$$\|U_a\|_{K^p_T} \leq \frac{4}{3} C_{\delta,p} T^{\frac{\delta}{2}} \|U_0\|_{H^{s_{\rm crit}+\delta}} < 2C_{\delta,p} T^{\frac{\delta}{2}} \|U_0\|_{H^{s_{\rm crit}+\delta}} = c_{k,\varepsilon} e^{-\frac{\lambda_T^2}{4(1-\varepsilon)}}.$$

This in turn shows that (2.7) holds for $T \leq T_\varepsilon(U_0)$. Furthermore, according to (1.1) and (2.1), there holds

$$T^{\frac{1}{p}} \left\|e^{\lambda_T\sqrt{T}|D|}U(T)\right\|_{H^{s_{\rm crit}+\frac{2}{p}}} \leq c_{k,\varepsilon}.$$

As δ is less than α, taking $p = \dfrac{2}{\delta}$ ensures that

$$\forall T \leq T_\varepsilon(U_0)\,,\ R(T) \geq \sqrt{2\delta(1-\varepsilon)}T^{\frac{1}{2}} \log^{\frac{1}{2}}\left(\frac{\eta_\varepsilon}{T\|U_0\|^{\frac{2}{\delta}}_{H^{s_{\rm crit}+\delta}}}\right).$$

This inequality means exactly that

$$\liminf_{T\to 0} \frac{R(T)}{T^{\frac{1}{2}}\sqrt{-\log\big(T\|U_0\|^{\frac{2}{\delta}}_{H^{s_{\rm crit}+\delta}}\big)}} \geq \sqrt{2\delta(1-\varepsilon)}\,.$$

Due the fact that ε is arbitrary, we conclude the proof of part (a) of Theorem 1.3.

(b) The case when U_0 belongs to the critical space $H^{s_{\rm crit}}$.

Let us use the fact that in this case

$$\lim_{T\to 0} \|e^{\varepsilon t\Delta}U_0\|_{K^p_T} = 0\,. \tag{2.10}$$

Then we consider $T_\varepsilon(U_0)$ such that

$$\|e^{\varepsilon t\Delta}U_0\|_{K^p_{T_\varepsilon(U_0)}} \leq c_{k,\varepsilon}\,. \tag{2.11}$$

For any $T \leq T_\varepsilon(U_0)$, let us define

$$\lambda_T \overset{\rm def}{=} 2(1-\varepsilon)^{\frac{1}{2}} \log^{\frac{1}{2}}\left(\frac{c_{k,\varepsilon}}{2\|e^{\varepsilon t\Delta}U_0\|_{K^p_T}}\right).$$

Then it follows from Inequality (2.8) that

$$\|U_a\|_{K^p_T} \leq \frac{4}{3}\|e^{\varepsilon t\Delta}U_0\|_{K^p_T} < 2\|e^{\varepsilon t\Delta}U_0\|_{K^p_T} = c_{k,\varepsilon} e^{-\frac{\lambda_T^2}{4(1-\varepsilon)}}.$$

This shows that (2.7) indeed holds for $T \leq T_\varepsilon(U_0)$. Furthermore, according to (1.1) and (2.1), there holds

$$T^{\frac{1}{p}} \left\| e^{\lambda_T \sqrt{T}|D|} U(T) \right\|_{H^{s_{\rm crit}+\frac{2}{p}}} \leq c_{k,\varepsilon}\,.$$

By definition of λ_T this means in particular that

$$\forall T \leq T_\varepsilon(U_0)\,,\ R(T) \geq 2(1-\varepsilon)^{\frac{1}{2}} T^{\frac{1}{2}} \log^{\frac{1}{2}} \left(\frac{c_{k,\varepsilon}}{2\|e^{\varepsilon t\Delta} U_0\|_{K^p_T}} \right).$$

This inequality means exactly that for any small strictly positive ε, we have

$$\liminf_{T\to 0} \frac{R(T)}{T^{\frac{1}{2}} \sqrt{-\log^{\frac{1}{2}}\big(\|e^{\varepsilon t\Delta} U_0\|_{K^p_T}\big)}} \geq \lim_{T\to 0} 2(1-\varepsilon)^{\frac{1}{2}} \left(1 - \frac{\log c_{k,\varepsilon}}{\log\big(2\|e^{\varepsilon t\Delta} U_0\|_{K^p_T}\big)} \right)^{\frac{1}{2}},$$

which together with (2.10) ensures part (b) of of Theorem 1.3. This completes the proof of the theorem. □

3. Proof of Corollary 1.1

Let $u \in C([0,\infty); H^{\frac{1}{2}}(\mathbb{R}^3))$ be a global solution of the Navier-Stokes system (NS) with initial data u_0. Then it follows from [2] that this solution is unique, so that applying Theorem 2.1 of [6] yields

$$\lim_{t\to\infty} \|u(t)\|_{H^{\frac{1}{2}}} = 0\,. \tag{3.1}$$

Moreover, for any $t_0 > 0$, u verifies

$$(NS_{t_0}) \quad \begin{cases} \partial_t u + u\cdot\nabla u - \Delta u + \nabla p = 0, & (t,x) \in]t_0,\infty[\times\mathbb{R}^3, \\ \operatorname{div} u = 0, \\ u|_{t=t_0} = u(t_0). \end{cases}$$

Similar to (2.1) we denote

$$u_{a,t_0}(t,x) \stackrel{\text{def}}{=} \mathcal{F}^{-1}\big(e^{-\frac{\lambda^2}{4(1-\varepsilon)}\frac{t-t_0}{T} + \lambda\frac{t-t_0}{\sqrt{T}}|\xi|} |\widehat{u}(t,\xi)|\big), \tag{3.2}$$

and

$$\|u\|_{K^p_{t_0,T}} \stackrel{\text{def}}{=} \sup_{t\in[t_0,t_0+T]} \Big((t-t_0)^{\frac{1}{p}} \|u(t)\|_{H^{\frac{1}{2}+\frac{2}{p}}} \Big).$$

Then along the same lines as the proof of Lemma 2.1, we deduce that for $p \in]\max(2/\alpha,k),\infty[$

$$\|u_{a,t_0}\|_{K^p_{t_0,T}} \leq \|e^{\varepsilon(t-t_0)\Delta} u(t_0)\|_{K^p_{t_0,T}} + C_\varepsilon e^{\frac{\lambda^2}{4(1-\varepsilon)}} \|u_{a,t_0}\|^2_{K^p_{t_0,T}}\,. \tag{3.3}$$

By (3.1), we can choose t_0 so large that

$$\|u(t_0)\|_{H^{\frac{1}{2}}} \leq \frac{c_\varepsilon}{2K_\varepsilon}$$

with K_ε being determined by $\|e^{\varepsilon t\Delta} u_0\|_{K^p_\infty} \leq K_\varepsilon \|u_0\|_{H^{\frac{1}{2}}}$.

Let us make the induction assumption that

$$\|u_{a,t_0}\|_{K^p_{t_0,T}} \leq c_\varepsilon e^{-\frac{\lambda^2}{4(1-\varepsilon)}} \quad \text{with} \quad c_\varepsilon \stackrel{\text{def}}{=} \frac{1}{4C_\varepsilon}\,. \tag{3.4}$$

Then as long as the induction assumption is satisfied, we infer from (3.3) that

$$\|u_{a,t_0}\|_{K^p_{t_0,T}} \leq \frac{4}{3}\|e^{\varepsilon(t-t_0)\Delta}u(t_0)\|_{K^p_{t_0,T}} < 2\|e^{\varepsilon(t-t_0)\Delta}u(t_0)\|_{K^p_{t_0,T}} \leq 2K_\varepsilon\|u(t_0)\|_{H^{\frac12}} \leq c_\varepsilon.$$

Then defining for any $T > 0$

$$\lambda_T \stackrel{\text{def}}{=} 2(1-\varepsilon)^{\frac12}\log^{\frac12}\Bigl(\frac{c_\varepsilon}{2K_\varepsilon\|u(t_0)\|_{H^{\frac12}}}\Bigr),$$

we have

$$\|u_{a,t_0}\|_{K^p_{t_0,T}} < 2K_\varepsilon\|u(t_0)\|_{H^{\frac12}} \leq c_\varepsilon e^{-\frac{\lambda_T^2}{4(1-\varepsilon)}}.$$

This in turn shows that (3.4) holds for any $T > 0$, and (3.4) in particular implies that

$$T^{\frac1p}\bigl\|e^{\lambda_T\sqrt{T}|D|}u(t_0+T)\bigr\|_{H^{\frac12+\frac2p}} \leq c_\varepsilon\,.$$

As a result, there holds

$$\forall T\,,\ R(t_0+T) \geq 2(1-\varepsilon)^{\frac12}T^{\frac12}\log^{\frac12}\Bigl(\frac{c_\varepsilon}{2K_\varepsilon\|u(t_0)\|_{H^{\frac12}}}\Bigr),$$

from which we infer that

$$\liminf_{T\to\infty}\frac{R(t_0+T)}{\sqrt{t_0+T}} = \liminf_{T\to\infty}\frac{R(t_0+T)}{\sqrt{T}} \geq 2(1-\varepsilon)^{\frac12}\log^{\frac12}\Bigl(\frac{c_\varepsilon}{2K_\varepsilon\|u(t_0)\|_{H^{\frac12}}}\Bigr).$$

This together with (3.1) ensures (1.2). This finishes the proof of Corollary 1.1. □

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