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# FROM NEWTON TO BOLTZMANN: HARD SPHERES AND SHORT-RANGE POTENTIALS

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# FROM NEWTON TO BOLTZMANN: HARD SPHERES AND SHORT-RANGE POTENTIALS

**Isabelle Gallagher, Laure Saint-Raymond, Benjamin Texier**

***Abstract.*** — We provide a rigorous derivation of the Boltzmann equation as the mesoscopic limit of systems of hard spheres, or Newtonian particles interacting via a short-range potential, as the number of particles $N$ goes to infinity and the characteristic length of interaction $\varepsilon$ simultaneously goes to 0, in the Boltzmann-Grad scaling $N\varepsilon^{d-1} \equiv 1$.

The time of validity of the convergence is a fraction of the average time of first collision, due to a limitation of the time on which one can prove uniform estimates for the BBGKY and Boltzmann hierarchies.

Our proof relies on the fundamental ideas of Lanford, and the important contributions of King, Cercignani, Illner and Pulvirenti, and Cercignani, Gerasimenko and Petrina. The main novelty here is the detailed study of pathological trajectories involving recollisions, which proves the term-by-term convergence for the correlation series expansion.

# PREFACE

The appearance of irreversibility in gas dynamics, although the elementary particles obey reversible laws of motion, is a challenging issue. The first mathematical formulation of this irreversibility goes back to Boltzmann [**8**], who proposed in 1872 to represent the state of a perfect gas by some distribution function, the evolution of which is governed by the equation named after him

$$\partial_t f + v \cdot \nabla_x f = Q(f,f)$$

where the collision operator $Q$ is related to a jump process for the velocity variable. This dynamics preserves locally the mass, momentum and energy as the underlying system of particles, but it admits a Lyapunov functional referred to as the entropy, which encodes the irreversibility. It is therefore a natural question to understand in which sense the Boltzmann equation can be seen as a suitable approximation of the system of particles.

Until now, the only answer to this question is the celebrated Lanford Theorem [**34**], which gives a rigorous mathematical statement accounting for some important intuitions of Boltzmann [**9**]:

- the Boltzmann equation should be obtained as a limit when $N \to \infty$:
  *The velocity distribution of the molecules is not mathematically exact as long as the number of molecules is not assumed to be mathematically inifinitely large.*
- it predicts the most probable behavior, which does not completely exclude the occurrence of more pathological situations :
  *In nature, the tendency is to pass from the least likely state to the more likely. [....] The second principle in Thermodynamics appears therefore as a probability theorem.*
- it expresses some independence between elementary particles :
  *From now on we shall specifically assume that the motion is totally disorganized, either as an ensemble or at a molecular level, and that it remains so indefinitely.*

What is striking in Lanford's theorem is that the propagation of chaos can be rigorously established, and does not have to be assumed at all times. More precisely, it states that if one considers a system of $N$ particles interacting as hard spheres with elastic collisions, initially independent and distributed according to some smooth profile, then its distribution function converges to the solution to the Boltzmann equation in the limit when the number of particles $N$ goes to infinity and the characteristic length of interaction $\varepsilon$ simultaneously goes to 0, in the Boltzmann-Grad scaling $N\varepsilon^{d-1} \equiv 1$. The main drawback of this result is that the time of validity of the convergence is a fraction of the average time of first collision.

As we shall see, the main difference between the Boltzmann dynamics and the true dynamics of the system of particles is due to possible recollisions (which are not admissible in order that independence, also called chaos, can be propagated). The strategy of Lanford was then to decompose the dynamics in terms of collision trees and then to prove that

– with probability converging to 1, these trees have finite size,
– for trees of finite size, recollisions are of vanishing probability.

However, it seems that the arguments used to establish the second point are not entirely correct, so at some point the proof should be completed. The aim of this monograph is therefore to provide a complete self-contained proof of Lanford's theorem, and to extend this convergence result to systems of particles interacting pairwise via some compactly supported potential following the important contribution of King [**30**]. In particular we discuss in depth the notion of independence, and a precise control of all steps of the proof enables us to obtain a rate of convergence in the hard-spheres case. We insist on the fact that the strategy of the proof is by no means new. The main novelty here is the detailed study of pathological trajectories involving recollisions, which proves the termwise convergence for the correlation series expansion.

Part I presents the context in which this work is set: we discuss the notion of low density limit, recall some of the main landmarks in the vast literature concerning the Boltzmann equation, and state the main theorems proved in this monograph.

In Part II we focus on the hard-spheres situation: we first derive the BBGKY hierarchy associated with the Liouville equation and prove that it is uniformly well-posed on a short time interval. Then we turn to the notion of independence, which we describe in detail as it is in the case of independent initial data that one can recover at the limit a solution to the Boltzmann equation. Finally we give the precise convergence statement, of the BBGKY hierarchy to the Boltzmann hierarchy, of which the tensor product of solutions to the Boltzmann equation is a particular solution in the case of independent initial configurations. We finally present the salient features of the proof.

Part III is devoted to the counterpart of Part II in the case of particles interacting via a short-range potential. We first study the scattering associated to two-particle interactions, and then derive the associated BBGKY hierarchy. This turns out to be more intricate than in the hard-spheres case as many particles may interact via the potential, thus creating clusters of interacting particles. It is shown however that the main contribution to the dynamics is the first link of such a cluster, thus uniform bounds may be obtained as in the hard-spheres case. A precise statement of convergence towards the limit Boltzmann hierarchy is given, and a strategy of proof is presented.

Part IV presents the proof of both convergence results (hard-spheres and short-range potential). It turns out that the proof has been prepared in Parts II and III in such a way that both cases can be dealt with in a unified way, up to some slight variations due to the nonlocal interactions in the case of a potential. The study of pathological trajectories, which would deviate substantially from the Boltzmann trajectories, is performed in detail and we provide explicit bounds on their size (semi-explicit in the case of an interacting potential). As a consequence in the hard-spheres case we are furthermore able to obtain a precise rate of convergence. A few open problems are suggested at the end of Part IV. We emphasize here that actually the most interesting problem would be to prove the convergence for a very long time, which would validate in particular the relaxation towards equilibrium; this remains a very challenging question.

We thank J. Bertoin, Th. Bodineau, D. Cordero-Erausquin, L. Desvillettes, F. Golse, S. Mischler, C. Mouhot and R. Strain for many helpful discussions on topics addressed in this text. We are particularly grateful to M. Pulvirenti, C. Saffirio and S. Simonella for explaining to us how condition (8.3.1) makes possible a parametrization of the collision integral by the deflection angle (see Chapter 8). Finally we thank the anonymous referee for helpful suggestions to improve the manuscript.

Paris, May 2013 *Isabelle Gallagher*

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# PART I

# INTRODUCTION

# CHAPTER 1

# THE LOW DENSITY LIMIT

We are interested in this monograph in the qualitative behavior of systems of particles with short-range interactions. We study the qualitative behaviour of particle systems with short-range binary interactions, in two cases: hard spheres, that move in uniform rectilinear motion until they undergo elastic collisions, and smooth, monotonic, compactly supported potentials.

• For hard spheres, the equations of motion are

$$\frac{dx_i}{dt} = v_i\,, \qquad \frac{dv_i}{dt} = 0\,, \tag{1.0.1}$$

for $1 \leq i \leq N$, where $(x_i, v_i) \in \mathbf{R}^d \times \mathbf{R}^d$ denote the position and velocity of particle $i$, provided that the exclusion condition $|x_i(t) - x_j(t)| > \sigma$ is satisfied, where $\sigma$ denotes the diameter of the particles. We further have to prescribe a reflection condition at the boundary : if there exists $j \neq i$ such that $|x_i - x_j| = \sigma$

$$\begin{aligned} v_i^{in} &= v_i^{out} - \nu^{i,j} \cdot (v_i^{out} - v_j^{out})\, \nu^{i,j} \\ v_j^{in} &= v_j^{out} + \nu^{i,j} \cdot (v_i^{out} - v_j^{out})\, \nu^{i,j}\,, \end{aligned} \tag{1.0.2}$$

where $\nu^{i,j} := (x_i - x_j)/|x_i - x_j|$. Note that it is not obvious to check that (1.0.1)-(1.0.2) defines global dynamics. This question is addressed in Chapter 4.

• In the case of smooth interactions, the Hamiltonian equations of motion are

$$\frac{dx_i}{dt} = v_i\,, \qquad m_i \frac{dv_i}{dt} = -\sum_{j \neq i} \nabla\Phi(x_i - x_j)\,, \tag{1.0.3}$$

where $m_i$ is the mass of particle $i$ (which we shall assume equal to 1 to simplify) and the force exerted by particle $j$ on particle $i$ is $-\nabla\Phi(x_i - x_j)$.

When the system is constituted of two elementary particles, in the reference frame attached to the center of mass, the dynamics is two-dimensional. The deflection of the particle trajectories from straight lines can then be described through explicit formulas (which are given in Chapter 8).

When the system is constituted of three particles or more, the integrability is lost, and in general the problem becomes very complicated, as already noted by Poincaré [**37**].

***Remark 1.0.1.*** — *Note that the dynamics of hard spheres is in some sense a limit of the smooth-forces case with*

$$\Phi(x) = +\infty \text{ if } |x| < \sigma\,, \qquad \Phi(x) = 0 \text{ if } |x| > \sigma\,.$$

*Nevertheless, to our knowledge, there does not exist any mathematical statement concerning these asymptotics.*

*We will however see in the sequel that the two types of systems exhibit very similar qualitative behaviours in the low density limit. Once the dynamics is defined (i.e. provided that we can discard multiple collisions), the case of hard spheres is actually simpler and we will discuss it in Part II to explain the main ideas and conceptual difficulties. We will then explain, in Part III, how to extend the arguments to the smoother case of Hamiltonian systems.*

## 1.1. The Liouville equation

In the large $N$ limit, individual trajectories become irrelevant, and our goal is to describe an average behaviour.

This average will be of course over particles which are indistiguishable, meaning that we will be only interested in some distribution related to the empirical measure

$$\mu_N\big(t, X_N(0), V_N(0)\big) := \frac{1}{N}\sum_{i=1}^N \delta_{x_i(t),v_i(t)}\,,$$

with $X_N(0) := (x_1(0),\dots,x_N(0)) \in \mathbf{R}^{dN}$ and $V_N(0) := (v_1(0),\dots,v_N(0)) \in \mathbf{R}^{dN}$, and $(x_i(t), v_i(t))$ is the state at time $t$ of particle $i$ in the system with initial configuration $\big(X_N(0), V_N(0)\big)$.

But, because we have only a vague knowledge of the state of the system at initial time, we will further average over initial configurations. At time 0, we thus start with a distribution $f_N^0(Z_N)$, where we use the following notation: for any set of $s$ particles with positions $X_s := (x_1,\dots,x_s) \in \mathbf{R}^{ds}$ and velocities $V_s := (v_1,\dots,v_s) \in \mathbf{R}^{ds}$, we write $Z_s := (z_1,\dots,z_s) \in \mathbf{R}^{2ds}$ with $z_i := (x_i, v_i) \in \mathbf{R}^{2d}$.

We then aim at describing the evolution of the distribution

$$\int \left(\frac{1}{N}\sum_{i=1}^N \delta_{z_i(t)}\right) f_N^0(Z_N)dZ_N\,.$$

We thus define the probability $f_N = f_N(t, Z_N)$, referred to as the *$N$-particle distribution function*, and we assume that it satisfies for all permutations $\sigma$ of $\{1,\dots,N\}$,

$$f_N(t, Z_{\sigma(N)}) = f_N(t, Z_N)\,, \tag{1.1.1}$$

with $Z_{\sigma(N)} = (x_{\sigma(1)}, v_{\sigma(1)},\dots,x_{\sigma(N)}, v_{\sigma(N)})$. This corresponds to the property that the particles are indistinguishable.

The distribution we are interested in is therefore nothing else than the first marginal $f_N^{(1)}$ of the distribution function $f_N$, defined by

$$f_N^{(1)}(t, Z_1) := \int f_N(t, Z_N)\, dz_2 \dots dz_N\,.$$

Since $f_N$ is an invariant of the particle system, the *Liouville equation* relative to the particle system (1.0.3) is

$$\partial_t f_N + \sum_{i=1}^N v_i \cdot \nabla_{x_i} f_N - \sum_{i=1}^N \sum_{\substack{j=1\\ j\neq i}}^N \nabla_x \Phi\,(x_i - x_j) \cdot \nabla_{v_i} f_N = 0\,. \tag{1.1.2}$$

For hard spheres, provided that we can prove that the dynamics is well defined for almost all initial configurations, we find the Liouville equation

$$\partial_t f_N + \sum_{i=1}^N v_i \cdot \nabla_{x_i} f_N = 0 \tag{1.1.3}$$

on the domain

$$\mathcal{D}_N := \Big\{ Z_N \in \mathbf{R}^{2dN} \,/\, \forall i \neq j,\ |x_i - x_j| > \sigma \Big\}$$

with the boundary condition $f_N(t, Z_N^{in}) = f_N(t, Z_N^{out})$, meaning that on the part of the boundary such that $|x_i - x_j| = \sigma$

$$f_N(t, \dots, x_i, v_i^{in}, \dots x_j, v_j^{in}, \dots) = f_N(t, \dots, x_i, v_i^{out}, \dots x_j, v_j^{out}, \dots)$$

where the ingoing and outgoing velocities are related by (1.0.2).

## 1.2. Mean field versus collisional dynamics

In this framework, in order for the average energy per particle to remain bounded, one has to assume that the energy of each pairwise interaction is small. In other words, one has to consider a rescaled potential $\Phi_\varepsilon$ obtained

- either by scaling the strength of the force,
- or by scaling the range of potential.

According to the scaling chosen, we expect to obtain different asymptotics.

• In the case of a weak coupling, i.e. when the strength of the individual interaction becomes small (of order $1/N$) but the range remains macroscopic, the convenient scaling in order for the macroscopic dynamics to be sensitive to the coupling is:

$$\partial_t f_N + \sum_{i=1}^N v_i \cdot \nabla_{x_i} f_N - \frac{1}{N} \sum_{i=1}^N \sum_{\substack{j=1\\ j\neq i}}^N \nabla \Phi\,(x_i - x_j) \cdot \nabla_{v_i} f_N = 0\,.$$

Then each particle feels the effect of the force field created by all the (other) particles

$$F_N(x) = -\frac{1}{N} \sum_{j=1}^N \nabla_x \Phi\,(x - x_j) \sim -\iint \nabla \Phi(x-y) f_N^{(1)}(t,y,v)\,dy\,dv\,.$$

In particular, the dynamics seems to be stable under small perturbations of the positions or velocities of the particles.

In the limit $N \to \infty$, we thus get a *mean field approximation*, that is an equation of the form

$$\partial_t f + v \cdot \nabla_x f + F \cdot \nabla_v f = 0$$

for the first marginal, where the coupling arises only through some average

$$F := -\nabla_x \Phi * \int f dv \, .$$

An important amount of literature is devoted to such asymptotics, but this is not our purpose here. We refer to [**11, 41**] for pioneering results, to [**25**] for a recent study and to [**21**] for a review on that topic.

• The scaling we shall deal with in the present work corresponds to a strong coupling, i.e. to the case when the amplitude of the potential remains of size $O(1)$, but its range becomes small.

Introduce a small parameter $\varepsilon > 0$ corresponding to the typical interaction length of the particles. For hard spheres, $\varepsilon$ is simply the diameter of particles. In the case of Hamiltonian systems, $\varepsilon$ will be the range of the interaction potential. We shall indeed assume throughout this text the following properties for $\Phi$ (a *short-range* potential).

***Assumption 1.2.1.*** — *The potential $\Phi : \mathbf{R}^d \to \mathbf{R}$ is a radial, nonnegative, nonincreasing function supported in the unit ball of $\mathbf{R}^d$, of class $C^2$ in $\{x \in \mathbf{R}^d \, , 0 < |x| < 1\}$. Moreover it is assumed that $\Phi$ is unbounded near zero, goes to zero at $|x| = 1$ with bounded derivatives, and that $\nabla\Phi$ vanishes only on $|x| = 1$.*

Then in the macroscopic spatial and temporal scales, the Hamiltonian system becomes

$$\frac{dx_i}{dt} = v_i \, , \qquad \frac{dv_i}{dt} = -\frac{1}{\varepsilon} \sum_{j \neq i} \nabla\Phi \left( \frac{x_i - x_j}{\varepsilon} \right) , \tag{1.2.1}$$

and the Liouville equation takes the form

$$\partial_t f_N + \sum_{i=1}^N v_i \cdot \nabla_{x_i} f_N - \sum_{i=1}^N \sum_{\substack{j=1 \\ j \neq i}}^N \frac{1}{\varepsilon} \nabla_x \Phi \left( \frac{x_i - x_j}{\varepsilon} \right) \cdot \nabla_{v_i} f_N = 0 \, . \tag{1.2.2}$$

With such a scaling, the dynamics is very sensitive to the positions of the particles.

Situations 1 and 2 on Figure 1 are differ by a spatial translation of $O(\varepsilon)$ only. However in Situation 1, particles will interact and be deviated from their free motion, while in Situation 2, they will evolve under free flow.

### 1.3. The Boltzmann-Grad limit

Particles move with uniform rectilinear motion as long as they remain at a distance greater than $\varepsilon$ to other particles. In the limit $\varepsilon \to 0$, we thus expect trajectories to be almost polylines.

Deflections are due to elementary interactions

- which occur when two particles are at a distance smaller than $\varepsilon$ (exactly $\varepsilon$ in the case of hard spheres),
- during a time interval of order $\varepsilon$ (if the relative velocity is not too small) or even instantaneously in the case of hard spheres,

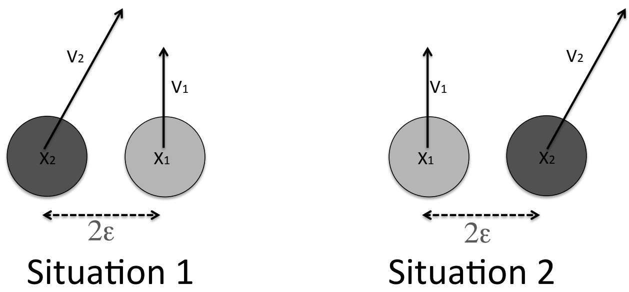


FIGURE 1. Instability

– which involve generally only two particles : the probability that a third particle enters a security ball of radius $\varepsilon$ should indeed tend to 0 as $\varepsilon \to 0$ in the convenient scaling. We are therefore brought back to the case of the two-body system, which is completely integrable (see Chapter 8).

In order for the interactions to have a macroscopic effect on the dynamics, each particle should undergo a finite number of collisions per unit of time. A scaling argument, giving the mean free path in terms of $N$ and $\varepsilon$, then shows that $N\varepsilon^{d-1} = O(1)$: indeed a particle travelling at speed bounded by $R$ covers in unit time an area of size $R\varepsilon^{d-1}$, and there are $N$ such particles. This is the Boltzmann-Grad scaling (see [**24**]).

The Boltzmann equation, which is the master equation in collisional kinetic theory [**15, 46**], is expected to describe such a dynamics.

# CHAPTER 2

# THE BOLTZMANN EQUATION

## 2.1. Transport and collisions

As mentioned in the previous chapter, the state of the system in the low density limit should be described (at the statistical level) by the kinetic density, i.e. by the probability $f \equiv f(t, x, v)$ of finding a particle with position $x$ and velocity $v$ at time $t$.

This density is expected to evolve under both the effects of transport and binary elastic collisions, which is expressed in the Boltzmann equation (introduced by Boltzmann in [**8**]-[**9**]) :

$$\underbrace{\partial_t f + v \cdot \nabla_x f}_{\text{free transport}} \quad = \quad \underbrace{Q(f,f)}_{\text{localized binary collisions}} \tag{2.1.1}$$

The Boltzmann collision operator, present in the right-hand side of (2.1.1), is the quadratic form, acting on the velocity variable, associated with the bilinear operator

$$Q(f,f) = \iint [f' f_1' - f f_1] \, b(v - v_1, \omega) \, dv_1 d\omega \tag{2.1.2}$$

where we have used the standard abbreviations

$$f = f(v)\,, \quad f' = f(v')\,, \quad f_1' = f(v_1')\,, \quad f_1 = f(v_1)\,,$$

with $(v', v_1')$ given by

$$v' = v + \omega \cdot (v_1 - v)\, \omega\,, \quad v_1' = v_1 - \omega \cdot (v_1 - v)\, \omega\,.$$

One can easily show that the quadruple $(v, v_1, v', v_1')$ parametrized by $\omega \in \mathbf{S}_1^{d-1}$ (where $\mathbf{S}_\rho^{d-1}$ denotes the sphere of radius $\rho$ in $\mathbf{R}^d$) provides the family of all solutions to the system of $d+1$ equations

$$\begin{aligned} v + v_1 &= v' + v_1'\,, \\ |v|^2 + |v_1|^2 &= |v'|^2 + |v_1'|^2\,, \end{aligned} \tag{2.1.3}$$

which, at the kinetic level, express the fact that collisions are elastic and thus conserve momentum and energy. Notice that the transformation $(v, v_1, \omega) \mapsto (v', v_1', -\omega)$ is an involution.

The Boltzmann collision operator can therefore be split, at least formally, into a gain term and a loss term (see [**13, 46**])

$$Q(f,f) = Q^+(f,f) - Q^-(f,f).$$

The loss term counts all collisions in which a given particle of velocity $v$ will encounter another particle, of velocity $v_1$, and thus will change its velocity leading to a loss of particles of velocity $v$, whereas the

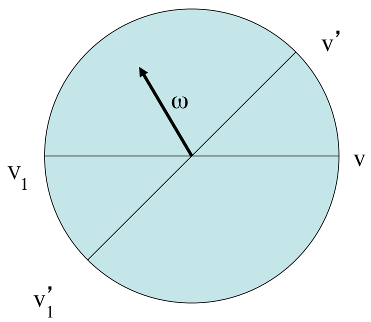


FIGURE 2. Parametrization of the collision by the deflection angle $\omega$

gain term measures the number of particles of velocity $v$ which are created due to a collision between particles of velocities $v'$ and $v_1'$.

The collision kernel $b = b(w, \omega)$ is a measurable function positive almost everywhere, which measures the statistical repartition of post-collisional velocities $(v, v_1)$ given the pre-collisional velocities $(v', v_1')$. Its precise form depends crucially on the nature of the microscopic interactions, and will be discussed in more details in the sequel. Note that, due to the Galilean invariance of collisions, it only depends on the magnitude of the relative velocity $|w|$ and on the deviation angle $\theta$, or deflection (scattering) angle, defined by $\cos\theta = k \cdot \omega$ where $k = w/|w|$.

## 2.2. Boltzmann's H theorem and irreversibility

From (2.1.3) and using the well-known facts (see [**13**]) that transforming $(v, v_1) \mapsto (v_1, v)$ and $(v, v_1, \omega) \mapsto (v', v_1', \omega)$ merely induces mappings with unit Jacobian determinants, one can show that formally

$$\int Q(f,f)\varphi dv = \frac{1}{4}\iiint [f'f_1' - ff_1](\varphi + \varphi_1 - \varphi' - \varphi_1')\, b(v - v_1, \omega)\, dv dv_1 d\omega\,. \tag{2.2.1}$$

In particular,

$$\int Q(f,f)\varphi dv = 0$$

for all $f$ regular enough, if and only if $\varphi(v)$ is a collision invariant, i.e. $\varphi(v)$ is a linear combination of $\{1, v_1, \ldots, v_d, |v|^2\}$. Thus, successively multiplying the Boltzmann equation (2.1.1) by the collision invariants and then integrating in velocity yields formally the local conservation laws

$$\partial_t \int_{\mathbf{R}^d} f \begin{pmatrix} 1 \\ v \\ \frac{|v|^2}{2} \end{pmatrix} dv + \nabla_x \cdot \int_{\mathbf{R}^d} f \begin{pmatrix} v \\ v \otimes v \\ \frac{|v|^2}{2} v \end{pmatrix} dv = 0\,, \tag{2.2.2}$$

which provides the link to a macroscopic description of the gas.

The other very important feature of the Boltzmann equation comes also from the symmetries of the collision operator. Disregarding integrability issues, we choose $\varphi = \log f$ and use the properties of the logarithm, to find

$$\begin{aligned} D(f) &\equiv - \int Q(f,f) \log f dv \\ &= \frac{1}{4} \int_{\mathbf{R}^d \times \mathbf{R}^d \times \mathbf{S}_1^{d-1}} b(v - v_1, \omega)(f' f_1' - f f_1) \log \frac{f' f_1'}{f f_1} \, dv dv_1 d\omega \geq 0\,. \end{aligned} \tag{2.2.3}$$

The so-defined entropy dissipation is therefore a nonnegative functional.

This leads to Boltzmann's H theorem, also known as second principle of thermodynamics, stating that the entropy is (at least formally) a Lyapunov functional for the Boltzmann equation.

$$\partial_t \int_{\mathbf{R}^d} f \log f dv + \nabla_x \cdot \int_{\mathbf{R}^d} f \log f v dv \leq 0\,. \tag{2.2.4}$$

As to the equation $Q(f,f) = 0$, it is possible to show that it is only satisfied by the so-called Maxwellian distributions $M_{\rho,u,\theta}$, which are defined by

$$M_{\rho,u,\theta}(v) := \frac{\rho}{(2\pi\theta)^{\frac{d}{2}}} e^{-\frac{|v-u|^2}{2\theta}}\,, \tag{2.2.5}$$

where $\rho \in \mathbf{R}_+$, $u \in \mathbf{R}^d$ and $\theta \in \mathbf{R}_+$ are respectively the macroscopic density, bulk velocity and temperature, under some appropriate choice of units. The relation $Q(f,f) = 0$ expresses the fact that collisions are no longer responsible for any variation in the density and so, that the gas has reached statistical equilibrium. In fact, it is possible to show that if the density $f$ is a Maxwellian distribution for some $\rho(t,x)$, $u(t,x)$ and $\theta(t,x)$, then the macroscopic conservation laws (2.2.2) turn out to constitute the compressible Euler system.

More generally, the H-theorem (2.2.4) together with the conservation laws (2.2.2) constitute the key elements of the study of hydrodynamic limits.

***Remark 2.2.1***. — *Note that the irreversibility inherent to the Boltzmann dynamics seems at first sight to contradict the possible existence of a connection with the microscopic dynamics which is reversible and satisfies the Poincaré recurrence theorem (while the Boltzmann dynamics predict some relaxation towards equilibrium).*

*That irreversibility will actually appear in the limiting process as an arbitrary choice of the time direction (encoded in the distinction between pre-collisional and post-collisional particles), and more precisely as an arbitrary choice of the initial time, which is the only time for which one has a complete information on the correlations. The point is that the joint probability of having particles of velocity $(v', v_1')$ (respectively of velocities $(v, v_1)$) before the collision is assumed to be equal to $f(t,x,v')f(t,x,v_1')$ (resp. to $f(t,x,v)f(t,x,v_1)$), meaning that particles should be independent before collision.*

## 2.3. The Cauchy problem

Let us first describe briefly the most apparent problems in trying to construct a general, good Cauchy theory for the Boltzmann equation. In the full, general situation, known a priori estimates for the Boltzmann equation are only those which are associated with the basic physical laws, namely the formal conservation of mass and energy, and the bounds on entropy and entropy dissipation. Note that, when the physical space is unbounded, the dispersive properties of the free transport operator allow to further expect some control on the moments with respect to $x$-variables. Yet the Boltzmann collision integral is a quadratic operator that is purely local in the position and time variables, meaning that it acts as a convolution in the $v$ variable, but as a pointwise multiplication in the $t$ and $x$ variables : thus, with the only a priori estimates which seem to hold in full generality, the collision integral is even not a well-defined distribution with respect to $x$-variables. This major obstruction is one of the reasons why the Cauchy problem for the Boltzmann equation is so tricky, another reason being the intricate nature of the Boltzmann operator.

For the sake of simplicity, we shall consider here only bounded collision cross-sections $b$. A huge literature is devoted to the study of more singular cross-sections insofar as the presence of long range interactions always creates singularities associated to grazing collisions. However, at the present time, there is no extension of Lanford's convergence result in this framework.

### 2.3.1. Short time existence of continuous solutions. — 

The easiest way to construct local solutions to the Boltzmann equation is to use a fixed point argument in the space of continuous functions.

Remarking that the free transport operator preserves weighted $L^\infty$ norms

$$\Big\| f_0(x-vt,v)\exp\left(\frac{\beta}{2}|v|^2\right)\Big\|_{L^\infty} = \Big\| f_0(x,v)\exp(\frac{\beta}{2}|v|^2)\Big\|_{L^\infty}\,,$$

and that the following continuity property holds for the collision operator

$$\Big\| Q(f,f)(v)\exp(\frac{\beta}{2}|v|^2)\Big\|_{L^\infty} \leq C_\beta \Big\| f(v)\exp(\frac{\beta}{2}|v|^2)\Big\|^2_{L^\infty}\,,$$

we get the existence of continuous solutions, the lifespan of which is inversely proportional to the norm of the initial data.

***Theorem 1.*** — *Let $f_0 \in C^0(\mathbf{R}^d \times \mathbf{R}^d)$ such that*

$$\Big\| f_0 \exp(\frac{\beta}{2}|v|^2)\Big\|_{L^\infty} < +\infty \tag{2.3.1}$$

*for some $\beta > 0$.*

*Then, there exists $C_\beta > 0$ (depending only $\beta$) such that the Boltzmann equation* (2.1.1) *with initial data $f_0$ has a unique continuous solution on* $[0,T]$ *with*

$$T = \frac{C_\beta}{\Big\| f_0 \exp(\frac{\beta}{2}|v|^2)\Big\|_{L^\infty}}\,\cdot$$

Note that the weigthed $L^\infty$ norm controls in particular the macroscopic density

$$\rho(t,x) := \int f(t,x,v)dv \leq C_\beta \| f(t,x,v)\exp(\frac{\beta}{2}|v|^2)\|_\infty\,,$$

therefore the possible concentrations for which the collision process can become very pathological. This restriction, even coming from a very rough analysis, has therefore a physical meaning.

**2.3.2. Fluctuations around some global equilibrium.** — Historically the first global existence result for the spatially inhomogeneous Boltzmann equation is due to S. Ukai [**43, 44**], who considered initial data that are fluctuations around a global equilibrium, for instance around the reduced centered Gaussian $M := M_{1,0,1}$ with notation (2.2.5):

$$f_0 = M(1+g_0)\,.$$

He proved the global existence of a solution to the Cauchy poblem for (2.1.1) under the assumption that the initial perturbation $g_0$ is smooth and small enough in a norm that involves derivatives and weights so as to ensure decay for large $v$.

The convenient functional space to be considered is indeed

$$H_{\ell,k} = \{g \equiv g(x,v) \,/\, \|g\|_{\ell,k} := \sup_v (1+|v|^k)\|M^{1/2}g(\cdot,v)\|_{H^\ell_x} < +\infty\}\,.$$

***Theorem 2*** **([43, 44]).** — *Let $g_0 \in H_{\ell,k}$ for $\ell > d/2$ and $k > d/2+1$ such that*

$$\|g_0\|_{\ell,k} \leq a_0 \tag{2.3.2}$$

*for some $a_0$ sufficiently small.*

*Then, there exists a unique global solution $f = M(1+g)$ with $g \in L^\infty(\mathbf{R}^+, H_{\ell,k}) \cap C(\mathbf{R}^+, H_{\ell,k})$ to the Boltzmann equation* (2.1.1) *with initial data*

$$g_{|t=0} = g_0\,.$$

Such a global existence result is based on Duhamel's formula and on Picard fixed point theorem. It requires a very precise study of the linearized collision operator $\mathcal{L}_M$ defined by

$$\mathcal{L}_M g := -\frac{2}{M}Q(M, Mg)\,,$$

and more precisely of the semi-group $U$ generated by

$$v\cdot\nabla_x + \mathcal{L}_M\,.$$

The main disadvantage inherent to that strategy is that one cannot expect to extend such a result to classes of initial data with less regularity.

**2.3.3. Renormalized solutions.** — The theory of renormalized solutions goes back to the late 80s and is due to R. DiPerna and P.-L. Lions [**18**]. It holds for physically admissible initial data of arbitrary sizes, but does not yield solutions that are known to solve the Boltzmann equation in the usual weak sense.

Rather, it gives the existence of a global weak solution to a class of formally equivalent initial-value problems.

***Definition 2.3.1.*** — *A renormalized solution of the Boltzmann equation* (2.1.1) *relatively to the global equilibrium $M$ is a function $f \in C(\mathbf{R}^+, L^1_{loc}(\mathbf{R}^d \times \mathbf{R}^d))$ such that*

$$H(f|M)(t) := \iint\int \left(f\log\frac{f}{M} - f + M\right)(t,x,v)\;dv\,dx < +\infty\,,$$

*which satisfies in the sense of distributions*

$$M\Big(\partial_t + v\cdot\nabla_x\Big)\Gamma\left(\frac{f}{M}\right) = \Gamma'\left(\frac{f}{M}\right)Q(f,f) \qquad \textit{on } \mathbf{R}^+\times\mathbf{R}^d\times\mathbf{R}^d\,,$$
$$f_{|t=0} = f_0 \geq 0 \qquad \textit{on } \mathbf{R}^d\times\mathbf{R}^d\,. \tag{2.3.3}$$

*for any* $\Gamma\in C^1(\mathbf{R}^+)$ *such that* $|\Gamma'(z)|\leq C/\sqrt{1+z}$.

With the above definition of renormalized solution relatively to $M$, the following existence result holds :

***Theorem 3*** **([18]).** — *Given any initial data* $f_0$ *satisfying*

$$H(f_0|M) = \int\!\!\int\left(f_0\log\frac{f_0}{M} - f_0 + M\right)(x,v)\,dv\,dx < +\infty\,, \tag{2.3.4}$$

*there exists a renormalized solution* $f\in C(\mathbf{R}^+, L^1_{loc}(\mathbf{R}^d\times\mathbf{R}^d))$ *relatively to* $M$ *to the Boltzmann equation* (2.1.1) *with initial data* $f_0$.

*Moreover,* $f$ *satisfies*

*- the continuity equation*

$$\partial_t\int f dv + \nabla_x\cdot\int f v dv = 0\,; \tag{2.3.5}$$

*- the momentum equation with defect measure*

$$\partial_t\int f v dv + \nabla_x\cdot\int f v\otimes v dv + \nabla_x\cdot m = 0 \tag{2.3.6}$$

*where* $m$ *is a Radon measure on* $\mathbf{R}^+\times\mathbf{R}^d$ *with values in the nonnegative symmetric matrices;*

*- the entropy inequality*

$$H(f|M)(t) + \int \mathrm{trace}(m)(t) + \int_0^t\int D(f)(s,x)dsdx \leq H(f_0|M) \tag{2.3.7}$$

*where* trace$(m)$ *is the trace of the nonnegative symmetric matrix* $m$*, and the entropy dissipation* $D(f)$ *is defined by* (2.2.3).

The weak stability of approximate solutions is inherited from the entropy inequality. In order to take limits in the renormalized Boltzmann equation, we have further to obtain some strong compactness. The crucial idea here is to use the velocity averaging lemma due to F. Golse, P.-L. Lions, B. Perthame and R. Sentis [**22**], stating that the moments in $v$ of the solution to some transport equation are more regular than the function itself.

***Remark 2.3.2.*** — *As we will see, the major weakness of the convergence theorem describing the Boltzmann equation as the low density limit of large systems of particles is the very short time on which it holds. However, the present state of the art regarding the Cauchy theory for the Boltzmann equation makes it very difficult to improve.*

*Because of the scaling of the microscopic interactions, the conditioning on energy surfaces (see Chapter* 6*) introduces strong spatial oscillations in the initial data. We therefore do not expect to get regularity so that we could take advantage of the perturbative theory of S. Ukai* [**43, 44**]*. A coarse* *graining argument would be necessary to retrieve spatial regularity on the kinetic distribution, but we are not aware of any breakthrough in this direction.*

*As for using the DiPerna-Lions theory* [**18**]*, the first step would be to understand the counterpart of renormalization at the level of the microscopic dynamics, which seems to be also a very challenging problem.*

# CHAPTER 3

# MAIN RESULTS

## 3.1. Lanford and King's theorems

The main goal of this monograph is to prove the two following statements. We give here compact, and somewhat informal, statements of our two main results. Precise statements are given in Chapters 6 and 11 (see Theorem 8 page 51 for the hard-spheres case, and 11 page 91 for the potential case).

The following statement concerns the case of hard spheres dynamics, and the main ideas behind its proof go back to the fundamental work of Lanford [**34**].

***Theorem 4.*** — *Let $f_0 : \mathbf{R}^{2d} \mapsto \mathbf{R}^+$ be a continuous density of probability such that*

$$\left\| f_0(x,v) \exp(\frac{\beta}{2}|v|^2) \right\|_{L^\infty(\mathbf{R}^{2d})} < +\infty$$

*for some $\beta > 0$.*

*Consider the system of $N$ hard spheres of diameter $\varepsilon$, initially distributed according to $f_0$ and "independent", governed by the system* (1.0.1)-(1.0.2)*. Then, in the Boltzmann-Grad limit $N \to \infty$, $N\varepsilon^{d-1} = 1$, its distribution function converges to the solution to the Boltzmann equation* (2.1.1) *with the cross-section $b(w,\omega) := (\omega \cdot w)_+$ and with initial data $f_0$, in the sense of observables.*

The next theorem concerns the Hamiltonian case (with a repulsive potential), and important steps of the proof can be found in the thesis of King [**30**].

***Theorem 5.*** — *Assume that the repulsive potential $\Phi$ satisfies Assumption* 1.2.1 *as well as the technical assumption* (8.3.1)*. Let $f_0 : \mathbf{R}^{2d} \mapsto \mathbf{R}^+$ be a continuous density of probability such that*

$$\left\| f_0 \exp(\frac{\beta}{2}|v|^2) \right\|_{L^\infty} < +\infty$$

*for some $\beta > 0$.*

*Consider the system of $N$ particles, initially distributed according to $f_0$ and "independent", governed by the system* (1.2.1)*. Then, in the Boltzmann-Grad limit $N \to \infty$, $N\varepsilon^{d-1} = 1$, its distribution function converges to the solution to the Boltzmann equation* (2.1.1) *with a bounded cross-section, depending on $\Phi$ implicitly, and with initial data $f_0$, in the sense of observables.*

***Remark 3.1.1***. — *Convergence in the sense of observables means that, for any test function* $\varphi$ *in* $C^0_c(\mathbf{R}^d_v)$, *the corresponding observable*

$$\phi_\varepsilon(t,x) := \int f_\varepsilon(t,x,v)\varphi(v)dv \longrightarrow \phi(t,x) := \int f(t,x,v)\varphi(v)dv$$

*uniformly in* $t$ *and* $x$. *We indeed recall that the kinetic distribution cannot be measured, only averages can be reached by physical experiments : this accounts for the terminology "observables".*

*In mathematical terms, this means that we establish only weak convergence with respect to the* $v$*-variable. Such a convergence result does not exclude the existence of pathological behaviors, in particular dynamics obtained by reversing the arrow of time and which are predicted by the (reversible) microscopic system. We shall only prove that these behaviors have negligible probability in the limit* $\varepsilon \to 0$.

***Remark 3.1.2***. — *The initial independence assumption has to be understood also asymptotically. It will be discussed with much details in Chapter* 6 *(see also Chapter* 11 *in the case of a potential): it is actually related to some coarse-graining arguments which are rather not intuitive at first sight.*

*For hard spheres, the exclusion obviously prevents independence for fixed* $\varepsilon$, *but we expect to retrieve this independence as* $\varepsilon \to 0$ *if we consider a fixed number* $s$ *of particles. The question is to deal with an infinite number of such particles.*

*The case of the smooth Hamiltonian system could seem to be simpler insofar as particles can occupy the whole space. Nevertheless, in order to control the decay at large energies, we need to introduce some conditioning on energy surfaces, which is very similar to exclusion.*

***Remark 3.1.3***. — *The technical assumption* (8.3.1) *will be made explicit in Chapter* 8 *: it ensures that the deviation angle is a suitable parametrization of the collision, and more precisely that we can retrieve the impact parameter from both the ingoing velocity and the deviation angle. What we will use is the fact that the jacobian of this change of variables is bounded at least locally.*

*Such an assumption is not completely compulsory for the proof. We can imagine of splitting the integration domain in many subdomains where the deviation angle is a good parametrization of the collision, but then we have to extend the usual definition of the cross-section. The important point is that the deviation angle cannot be a piecewise constant function of the impact parameter.*

### 3.2. Background and references

The problem of asking for a rigorous derivation of the Boltzmann equation from the Hamiltonian dynamics goes back to Hilbert [**27**], who suggested to use the Boltzmann equation as an intermediate step between the Hamiltonian dynamics and fluid mechanics, and who described this axiomatization of physics as a major challenge for mathematicians of the twentieth century.

We shall not give an exhaustive presentation of the studies that have been carried out on this question but indicate some of the fundamental landmarks, concerning for most of them the case of hard spheres. First one should mention N. Bogoliubov [**6**], M. Born, and H. S. Green [**10**], J. G. Kirkwood [**31**] and J. Yvan [**47**], who gave their names to the BBGKY hierarchy on the successive marginals, which we shall be using extensively in this study. H. Grad was able to obtain in [**23**] a differential equation on the first marginal which after some manipulations converges towards the Boltzmann equation.

The first mathematical result on this problem goes back to C. Cercignani [**12**] and O. Lanford [**34**] who proved that the propagation of chaos should be established by a careful study of trajectories of a hard spheres system, and who exhibited – for the first time – the origin of irreversibility. The proof, even though incomplete, is therefore an important breakthrough. The limits of their methods, on which we will comment later on – especially regarding the short time of convergence – are still challenging questions.

The argument of O. Lanford was then revisited and completed in several works. Let us mention especially the contributions of K. Uchiyama [**42**], C. Cercignani, R. Illner and M. Pulvirenti [**15**] and H. Spohn [**40**] who introduced a mathematical formalism, in particular to get uniform a priori estimates for the solutions to the BBGKY hierarchy which turns out to be a theory in the spirit of the Cauchy-Kowalewskaya theorem.

The term-by-term convergence of the hierarchy in the Boltzmann-Grad scaling was studied in more details by C. Cercignani, V. I. Gerasimenko and D. I. Petrina [**14**] : they provide for the first time quantative estimates on the set of "pathological trajectories", i.e. trajectories for which the Boltzmann equation does not provide a good approximation of the dynamics. What is not completely clear in this approach is the stability of the estimates under microscopic spatial translations.

The method of proof was then extended

- to the case when the initial distribution is close to vacuum, in which case global in time results may be proved [**15, 28, 29**];
- to the case when interactions are localized but not pointwise [**30**]. Because multiple collisions are no longer negligible, this requires a careful study of clusters of particles.

Many review papers deal with those different results, see [**19, 38, 46**] for instance.

Let us now summarize the strategy of the proofs. Their are two main steps:

(i) a short time bound for the series expansion expressing the correlations of the system of $N$ particles and the corresponding quantities of the Boltzmann equation;
(ii) the term by term convergence.

In the case of hard spheres, point (i) is just a matter of explicit estimates, while point (ii) is usually considered as almost obvious (but deep). Among experts in the field the hard sphere case is therefore considered to be completely solved. However, we could not find a proof for the measure zero estimates (i.e. the control of recollisions) in the litterature. It might be that to experts in the field such an estimate is easy, but from our point of view it turned out to be quite delicate.

- For the Boltzmann dynamics, it seems to be correct that a zero measure argument allows to control recollisions inasmuch as particles are pointwise.
- For fixed $\varepsilon$, we will see that the set of velocities leading to recollisions (even in the case of three particles) is small but not zero : this cannot be obtained by a straightforward thickening argument without any **geometrical information** on the limiting zero measure set.
- For the microscopic system of $N$ particles, collisional particles are at a distance $\varepsilon$ from each other, we thus expect that even "good trajectories" deviate from trajectories associated to the Boltzmann dynamics. We shall therefore need some **stability of "pathological sets"** of velocities with respect to microscopic spatial translations, to be able to iterate the process.

### 3.3. New contributions

Our goal here is to provide a self-contained presentation, which includes all the details of the proofs, especially concerning term-by-term convergence which to our knowledge is not completely written anywhere, even in the hard-spheres case.

Part II is a review of known results in the case of hard spheres. Following Lanford's strategy, we shall establish the starting hierarchy of equations, providing a short time, uniform estimate. Note that, because the dynamics of hard spheres is singular, the definition of collision integrals in this hierarchy is rather subtle. This point, which was missed in the first version as well as in the existing literature, is dealt with in details at the beginning of Chapter 5.

We focus especially on the **definition of functional spaces**: we shall see that the short time estimate is obtained as an analytical type result, meaning that we control all correlation functions together. The functional spaces we consider are in some sense natural from the point of view of statistical physics, since they involve two parameters $\beta$ and $\mu$ (related to the inverse temperature and chemical potential) to control the growth of energy and of the number of particles. Nevertheless, instead of usual $L^1$ norms, we use $L^\infty$ norms, which are needed to control collision integrals (see Remark 2.3.2).

The second point we discuss in details is the **notion of independence**. As noted in Remark 3.1.2, for any fixed $\varepsilon > 0$, because of the exclusion, particles cannot be independent. In the $2Nd$-dimensional phase-space, we shall see actually that the Gibbs measure has support on only a very small set. Careful estimates on the partition function show however that the marginal of order $s$ (for any fixed $s$) converges to some tensorized distribution, meaning that independence is recovered at the limit $\varepsilon \to 0$.

Part III deals with the case of the Hamiltonian system, with a repulsive potential. It basically follows King's thesis [**30**], filling in some gaps.

In the limit $\varepsilon \to 0$ with $N\varepsilon^{d-1} \equiv 1$, we would like to obtain a kind of homogeneization result : we want to average the motion over the small scales in $t$ and $x$, and replace the localized interactions by pointwise collisions as in the case of hard spheres. We therefore introduce an **artificial boundary** (following [**30**]) so that

– on the exterior domain, the dynamics reduces to free transport,
– on the interior domain, the dynamics can be integrated in order to compute outwards boundary conditions in terms of the incoming flux. Note that such a scattering operator is relevant only if we can guarantee that there is no other particle involved in the interaction.

An important point is therefore to control multiple collisions, which - contrary to the case of hard spheres - could happen for a non zero set of initial data. We however expect that they become negligible in the Boltzmann-Grad limit (as the probability of finding three particles having approximately the same position tends to zero). **Cluster estimates**, based on suitable partitions of the $2Nd$-dimensional phase-space and symmetry arguments, give the required asymptotic bound on multiple collisions.

Part IV is the heart of our contribution, where we establish the term-by-term convergence. Note that the arguments work in the same way in both situations (hard spheres and potential case), up to some minor technical points due to the fact that, for the $N$-particle Hamiltonian system, pre-collisional and post-collisional configurations differ by their velocities but also by their microscopic positions and by some microscopic shift in time.

However the two main difficulties are exactly the same:

- describing geometrically the set of "pathological" velocities and deflection angles leading to possible recollisions, in order to get a **quantitative estimate** of its measure;
- proving that this set is **stable under small translations** of positions.

Note that the estimates we establish depend only on the scattering operator, so that we have a rate of convergence which can be made explicit for instance in the case of hard spheres.

To control the set of recolliding trajectories by means of explicit estimates, we make use of properties of the cross-section which are not guaranteed a priori for a generic repulsive potential. Assumption (8.3.1) guarantees that these conditions are satisfied.

# PART II

# THE CASE OF HARD SPHERES

# CHAPTER 4

# MICROSCOPIC DYNAMICS AND BBGKY HIERARCHY

In this chapter we define the $N$-particle flow for hard spheres (introduced in Chapter 1), and write down the associated BBGKY hierarchy. Finally we present a formal derivation of the Boltzmann hierarchy, and the Boltzmann equation of hard spheres. This chapter follows the classical approaches of [**1**], [**14**], [**15**], [**34**], among others.

## 4.1. The $N$-particle flow

We consider $N$ particles in the space $\mathbf{R}^d$, the motion of which is described by $N$ positions $(x_1, \dots, x_N)$ and $N$ velocities $(v_1, \dots, v_N)$, each in $\mathbf{R}^d$. Denoting by $Z_N := (z_1, \dots, z_N)$ the set of particles, each particle $z_i := (x_i, v_i) \in \mathbf{R}^{2d}$ is submitted to free flow

$$\forall 1 \leq i \leq N\,, \quad \frac{dx_i}{dt} = v_i\,, \quad \frac{dv_i}{dt} = 0 \tag{4.1.1}$$

on the domain

$$\mathcal{D}_N := \Big\{ Z_N \in \mathbf{R}^{2dN} \,/\, \forall i \neq j,\, |x_i - x_j| > \varepsilon \Big\}$$

and bounces off the boundary $\partial \mathcal{D}_N$ according to the laws of elastic reflection: if $|x_i - x_j| = \varepsilon$

$$\begin{aligned} v_i^{in} &= v_i^{out} - \nu^{i,j} \cdot (v_i^{out} - v_j^{out})\, \nu^{i,j} \\ v_j^{in} &= v_j^{out} + \nu^{i,j} \cdot (v_i^{out} - v_j^{out})\, \nu^{i,j} \end{aligned} \tag{4.1.2}$$

where $\nu^{i,j} := (x_i - x_j)/|x_i - x_j|$, and in the case when $\nu^{i,j} \cdot (v_i^{in} - v_j^{in}) < 0$ (meaning that the ingoing velocities are precollisional).

Contrary to the potential case studied in Part III, it is not obvious to check that (4.1.1) defines a global dynamics, at least for almost all initial data. Note indeed that this is not a simple consequence of the Cauchy-Lipschitz theorem since the boundary condition is not smooth, and even not defined for all configurations. We call *pathological* a trajectory such that

- either there exists a collision involving more than two particles, or the collision is grazing (meaning that $\nu^{i,j} \cdot (v_i^{in} - v_j^{in}) = 0$) hence the boundary condition is not well defined;

- or there are an infinite number of collisions in finite time so the dynamics cannot be globally defined.

In [**2**, Proposition 4.3], it is stated that outside a negligible set of initial data there are no pathological trajectories; the complete proof is provided in [**1**]. Actually the setting of [**1**] is more complicated than ours since an infinite number of particles is considered. The arguments of [**1**] can however be easily adapted to our case to yield the following result, whose proof we detail for the convenience of the reader.

***Proposition 4.1.1***. — *Let $N, \varepsilon$ be fixed. The set of initial configurations leading to a pathological trajectory is of measure zero in $\mathbf{R}^{2dN}$.*

We first prove the following elementary lemma, in which we have used the following notation: for any $s \in \mathbf{N}^*$ and $R > 0$, we denote $B_R^s := \{V_s \in \mathbf{R}^{ds},\, |V_s| \leq R\}$ where $|\cdot|$ is the euclidean norm; we often write $B_R := B_R^1$.

***Lemma 4.1.1***. — *Let $\rho, R > 0$ be given, and $\delta < \varepsilon/2$. Define*

$$I := \Big\{ Z_N \in B_\rho^N \times B_R^N \,/\, \textit{one particle will collide with two others on the time interval } [0,\delta] \Big\} .$$

*Then* $|I| \leq C(N,\varepsilon,R)\, \rho^{d(N-2)} \delta^2$ .

*Proof*. — We notice that $I$ is a subset of

$$\Big\{ Z_N \in B_\rho^N \times B_R^N \,/\, \exists \{i,j,k\} \,\text{distinct}\,, \quad |x_i - x_j| \in [\varepsilon, \varepsilon + 2R\delta] \quad \text{and} \quad |x_i - x_k| \in [\varepsilon, \varepsilon + 2R\delta] \Big\} ,$$

and the lemma follows directly. □

*Proof of Proposition 4.1.1*. — Let $R > 0$ be given and fix some time $t > 0$. Let $\delta < \varepsilon/2$ be a parameter such that $t/\delta$ is an integer.

Lemma 4.1.1 implies that there is a subset $I_0(\delta,R)$ of $B_R^N \times B_R^N$ of measure at most $C(N,\varepsilon,R)R^{d(N-2)}\delta^2$ such that any initial configuration belonging to $(B_R^N \times B_R^N) \setminus I_0(\delta,R)$ generates a solution on $[0,\delta]$ such that each particle encounters at most one other particle on $[0,\delta]$. Moreover up to removing a measure zero set of initial data each collision is non-grazing.

Now let us start again at time $\delta$. We recall that in the velocity variables, the ball of radius $R$ in $\mathbf{R}^{dN}$ is stable by the flow, whereas the positions at time $\delta$ lie in the ball $B_{R+R\delta}^N$. Let us apply Lemma 4.1.1 again to that new initial configuration space. Since the measure is invariant by the flow, we can construct a subset $I_1(\delta,R)$ of the initial positions $B_R^N \times B_R^N$, of size $C(N,\varepsilon,R)R^{d(N-2)}(1+\delta)^{d(N-2)}\delta^2$ such that outside $I_0 \cup I_1(\delta,R)$, the flow starting from any initial point in $B_R^N \times B_R^N$ is such that each particle encounters at most one other particle on $[0,\delta]$, and then at most one other particle on $[\delta, 2\delta]$, again in a non-grazing collision. We repeat the procedure $t/\delta$ times: we construct a subset

$$I_\delta(t,R) := \bigcup_{j=0}^{t/\delta - 1} I_j(\delta,R)$$

of $B_R^N \times B_R^N$, of measure

$$\begin{aligned} |I_\delta(t,R)| &\leq C(N,\varepsilon,R)R^{d(N-2)}\delta^2 \sum_{j=0}^{t/\delta-1} (1+j\delta)^{d(N-2)} \\ &\leq C(N,R,t,\varepsilon)\delta\,, \end{aligned}$$

such that for any initial configuration in $B_R^N \times B_R^N$ outside that set, the flow is well-defined up to time $t$. The intersection $I(t,R) := \bigcap_{\delta>0} I_\delta(t,R)$ is of measure zero, and any initial configuration in $B_R^N \times B_R^N$ outside $I(t,R)$ generates a well-defined flow until time $t$. Finally we consider the countable union of those zero measure sets $I := \bigcup_n I(t_n, R_n)$ where $t_n$ and $R_n$ go to infinity, and any initial configuration in $\mathbf{R}^{2dN}$ outside $I$ generates a globally defined flow. The proposition is proved. □

## 4.2. The Liouville equation and the BBGKY hierarchy

According to Part I, Paragraph 1.1, the Liouville equation relative to the particle system (4.1.1) is

$$\partial_t f_N + \sum_{i=1}^N v_i \cdot \nabla_{x_i} f_N = 0 \quad \text{on} \quad \mathcal{D}_N \tag{4.2.1}$$

with the boundary condition $f_N(t, Z_N^{in}) = f_N(t, Z_N^{out})$. We recall the assumption that $f_N$ is invariant by permutation in the sense of (1.1.1), meaning that the particles are indistinguishable.

The classical strategy to obtain asymptotically a kinetic equation such as (2.1.1) is to write the evolution equation for the first marginal of the distribution function $f_N$, namely

$$f_N^{(1)}(t, z_1) := \int_{\mathbf{R}^{2d(N-1)}} f_N(t, z_1, z_2, \dots, z_N) \mathbb{1}_{Z_N \in \mathcal{D}_N} \, dz_2 \dots dz_N \, .$$

The point to be noted is that the evolution of $f_N^{(1)}$ depends actually on $f_N^{(2)}$ because of the quadratic interaction imposed by the boundary condition. And in the same way, the equation on $f_N^{(2)}$ depends on $f_N^{(3)}$. Instead of a kinetic equation, we therefore obtain a hierarchy of equations involving all the marginals of $f_N$

$$f_N^{(s)}(t, Z_s) := \int_{\mathbf{R}^{2d(N-s)}} f_N(t, Z_s, z_{s+1}, \dots, z_N) \mathbb{1}_{Z_N \in \mathcal{D}_N} \, dz_{s+1} \cdots dz_N \, . \tag{4.2.2}$$

Notice that $f_N^{(s)}(t, Z_s)$ is defined on $\mathcal{D}_s$ only, and that

$$f_N^{(s)}(t, Z_s) = \int_{\mathbf{R}^{2d}} f_N^{(s+1)}(t, Z_s, z_{s+1}) \, dz_{s+1} \, . \tag{4.2.3}$$

Finally by integration of the boundary condition on $f_N$ we find that $f_N^{(s)}(t, Z_s^{in}) = f_N^{(s)}(t, Z_s^{out})$. An equation for the marginals is derived in weak form in Section 4.3, and from that equation we derive formally the Boltzmann hierarchy in the Boltzmann-Grad limit (see Section 4.4).

## 4.3. Weak formulation of Liouville's equation

Our goal in this section is to find the weak formulation of the system of equations satisfied by the family of marginals $\big(f_N^{(s)}\big)_{1 \leq s \leq N}$ defined above in (4.2.2). From now on we assume that $f_N$ decays at infinity in the velocity variable (the functional setting will be made precise in Chapter 5).

Given a smooth, compactly supported function $\phi$ defined on $\mathbf{R}_+ \times \mathcal{D}_s$ and satisfying the symmetry assumption (1.1.1) as well as the boundary condition $\phi(t, Z_s^{in}) = \phi(t, Z_s^{out})$, we have

$$\int_{\mathbf{R}_+ \times \mathbf{R}^{2dN}} \big(\partial_t f_N + \sum_{i=1}^N v_i \cdot \nabla_{x_i} f_N\big) \phi(t, Z_s) \mathbb{1}_{Z_N \in \mathcal{D}_N} \, dZ_N dt = 0 \, . \tag{4.3.1}$$

We now use integrations by parts to derive from (4.3.1) the weak form of the equation in the marginals $f_N^{(s)}$. On the one hand an integration by parts in the time variable gives

$$\begin{aligned}\int_{\mathbf{R}_+\times\mathbf{R}^{2dN}} \partial_t f_N(t,Z_N)\phi(t,Z_s)1\!\!1_{Z_N\in\mathcal{D}_N}\,dZ_N dt = -\int_{\mathbf{R}^{2dN}} f_N(0,Z_N)\phi(0,Z_s)1\!\!1_{Z_N\in\mathcal{D}_N}\,dZ_N\\ -\int_{\mathbf{R}_+\times\mathbf{R}^{2dN}} f_N(t,Z_N)\partial_t\phi(t,Z_s)1\!\!1_{Z_N\in\mathcal{D}_N}\,dZ_N dt\,,\end{aligned}$$

hence, by definition of $f_N^{(s)}$ in (4.2.2),

$$\begin{aligned}\int_{\mathbf{R}_+\times\mathbf{R}^{2dN}} \partial_t f_N(t,Z_N)\phi(t,Z_s)1\!\!1_{Z_N\in\mathcal{D}_N}\,dZ_N dt = -\int_{\mathbf{R}^{2ds}} f_N^{(s)}(0,Z_s)\phi(0,Z_s)\,dZ_s\\ -\int_{\mathbf{R}_+\times\mathbf{R}^{2ds}} f_N^{(s)}(t,Z_s)\partial_t\phi(t,Z_s)\,dZ_s dt\,.\end{aligned}$$

Now let us compute

$$\sum_{i=1}^N\int_{\mathbf{R}^{2dN}} v_i\cdot\nabla_{x_i} f_N(t,Z_N)\phi(t,Z_s)1\!\!1_{Z_N\in\mathcal{D}_N}\,dZ_N = \int_{\mathbf{R}^{2dN}} \mathrm{div}_{X_N}\big(V_N\,f_N(t,Z_N)\big)\phi(t,Z_s)1\!\!1_{Z_N\in\mathcal{D}_N}\,dZ_N$$

using Green's formula. The boundary terms involve configurations with at least one pair $(i,j)$ satisfying $|x_i-x_j|=\varepsilon$. According to Paragraph 4.1 we may neglect configurations where more than two particles collide at the same time, so the boundary condition is well defined. For any $i$ and $j$ in $\{1,\dots,N\}$ we denote

$$\Sigma_N(i,j) := \Big\{X_N\in\mathbf{R}^{2dN},\ |x_i-x_j|=\varepsilon\Big\}\,,$$

and $n^{i,j}$ is the outward normal to $\Sigma_N(i,j)$ in $\mathbf{R}^{dN}$. We obtain by Green's formula:

$$\begin{aligned}&\sum_{i=1}^N\int_{\mathbf{R}_+\times\mathbf{R}^{2dN}} v_i\cdot\nabla_{x_i} f_N(t,Z_N)\phi(t,Z_s)1\!\!1_{Z_N\in\mathcal{D}_N}\,dZ_N\,dt\\ &\quad= -\sum_{i=1}^s\int_{\mathbf{R}_+\times\mathbf{R}^{2dN}} f_N(t,Z_N)v_i\cdot\nabla_{x_i}\phi(t,Z_s)1\!\!1_{Z_N\in\mathcal{D}_N}\,dZ_N dt\\ &\qquad+\sum_{1\le i\ne j\le N}\int_{\mathbf{R}_+\times\mathbf{R}^{dN}\times\Sigma_N(i,j)} n^{i,j}\cdot V_N\,f_N(t,Z_N)\phi(t,Z_s)\,d\sigma_N^{i,j}dV_N dt\,,\end{aligned}$$

with $d\sigma_N^{i,j}$ the surface measure on $\Sigma_N(i,j)$, induced by the Lebesgue measure. Now we split the last term into four parts:

$$\begin{aligned}&\sum_{1\le i\ne j\le N}\int_{\mathbf{R}_+\times\mathbf{R}^{dN}\times\Sigma_N(i,j)} n^{i,j}\cdot V_N\,f_N(t,Z_N)\phi(t,Z_s)\,d\sigma_N^{i,j}dV_N dt\\ &\quad=\sum_{i=1}^s\sum_{j=s+1}^N\int_{\mathbf{R}_+\times\mathbf{R}^{dN}\times\Sigma_N(i,j)} n^{i,j}\cdot V_N\,f_N(t,Z_N)\phi(t,Z_s)\,d\sigma_N^{i,j}dV_N dt\\ &\qquad+\sum_{i=s+1}^N\sum_{j=1}^s\int_{\mathbf{R}_+\times\mathbf{R}^{dN}\times\Sigma_N(i,j)} n^{i,j}\cdot V_N\,f_N(t,Z_N)\phi(t,Z_s)\,d\sigma_N^{i,j}dV_N dt\\ &\qquad+\sum_{1\le i\ne j\le s}\int_{\mathbf{R}_+\times\mathbf{R}^{dN}\times\Sigma_N(i,j)} n^{i,j}\cdot V_N\,f_N(t,Z_N)\phi(t,Z_s)\,d\sigma_N^{i,j}dV_N dt\\ &\qquad+\sum_{s+1\le i\ne j\le N}\int_{\mathbf{R}_+\times\mathbf{R}^{dN}\times\Sigma_N(i,j)} n^{i,j}\cdot V_N\,f_N(t,Z_N)\phi(t,Z_s)\,d\sigma_N^{i,j}dV_N dt\,.\end{aligned}$$

The boundary condition on $f_N$ and $\phi$ imply that the two last terms of on the right-hand side are zero. By symmetry (1.1.1) and by definition of $f_N^{(s)}$, we can write

$$
\begin{aligned}
&\sum_{i=1}^{s}\sum_{j=s+1}^{N}\int_{\mathbf{R}_+\times\mathbf{R}^{dN}\times\Sigma_N(i,j)} n^{i,j}\cdot V_N\, f_N(t,Z_N)\phi(t,Z_s)\,d\sigma_N^{i,j}dV_N dt\\
&\qquad+\sum_{i=s+1}^{N}\sum_{j=1}^{s}\int_{\mathbf{R}_+\times\mathbf{R}^{dN}\times\Sigma_N(i,j)} n^{i,j}\cdot V_N\, f_N(t,Z_N)\phi(t,Z_s)\,d\sigma_N^{i,j}dV_N dt\\
&=-(N-s)\sum_{i=1}^{s}\int_{\mathbf{R}_+\times\mathbf{S}_\varepsilon^{d-1}\times\mathbf{R}^d\times\mathbf{R}^{2ds}}\frac{(x_{s+1}-x_i)}{|x_{s+1}-x_i|}\cdot(v_{s+1}-v_i)\,f_N^{(s+1)}(t,Z_s,x_{s+1},v_{s+1})\phi(t,Z_s)\,dZ_s d\sigma(x_{s+1})dv_{s+1}dt\\
&=-(N-s)\varepsilon^{d-1}\sum_{i=1}^{s}\int_{\mathbf{R}_+\times\mathbf{S}_1^{d-1}\times\mathbf{R}^d\times\mathbf{R}^{2ds}}\omega\cdot(v_{s+1}-v_i)\,f_N^{(s+1)}(t,Z_s,x_i+\varepsilon\omega,v_{s+1})\phi(t,Z_s)\,dZ_s d\omega dv_{s+1}dt\,.
\end{aligned}
$$

Finally we obtain

$$
\begin{aligned}
&\sum_{i=1}^{N}\int_{\mathbf{R}_+\times\mathbf{R}^{2dN}} v_i\cdot\nabla_{x_i}f_N(t,Z_N)\phi(t,Z_s)\mathbb{1}_{Z_N\in\mathcal{D}_N}\,dZ_N\,dt\\
&=-\sum_{i=1}^{s}\int_{\mathbf{R}_+\times\mathbf{R}^{2ds}} f_N^{(s)}(t,Z_s)v_i\cdot\nabla_{x_i}\phi(t,Z_s)\,dZ_s dt\\
&-(N-s)\varepsilon^{d-1}\sum_{i=1}^{s}\int_{\mathbf{R}_+\times\mathbf{S}_1^{d-1}\times\mathbf{R}^d\times\mathbf{R}^{2ds}}\omega\cdot(v_{s+1}-v_i)\,f_N^{(s+1)}(t,Z_s,x_i+\varepsilon\omega,v_{s+1})\phi(t,Z_s)\,dZ_s d\omega dv_{s+1}dt\,.
\end{aligned}
$$

It remains to define the *collision operator*

$$
\begin{aligned}
\big(\mathcal{C}_{s,s+1}f_N^{(s+1)}\big)(t,Z_s):=(N-s)\varepsilon^{d-1}\sum_{i=1}^{s}\int_{\mathbf{S}_1^{d-1}\times\mathbf{R}^d}&\omega\cdot(v_{s+1}-v_i)\\
&\times f_N^{(s+1)}(t,Z_s,x_i+\varepsilon\omega,v_{s+1})d\omega dv_{s+1}\,,
\end{aligned}
\tag{4.3.2}
$$

where recall that $\mathbf{S}_1^{d-1}$ is the unit sphere of $\mathbf{R}^d$, and in the end we obtain the weak formulation of the BBGKY hierarchy

$$
\partial_t f_N^{(s)}+\sum_{1\le i\le s} v_i\cdot\nabla_{x_i}f_N^{(s)}=\mathcal{C}_{s,s+1}f_N^{(s+1)}\quad\text{in}\quad\mathbf{R}_+\times\mathcal{D}_s\,,
\tag{4.3.3}
$$

with the boundary conditions $f_N^{(s)}(t,Z_s^{in})=f_N^{(s)}(t,Z_s^{out})$.

In the integrand of the collision operators $\mathcal{C}_{s,s+1}$ defined in (4.3.2), we now distinguish between pre- and post-collisional configurations, as we decompose

$$
\mathcal{C}_{s,s+1}=\mathcal{C}^+_{s,s+1}-\mathcal{C}^-_{s,s+1}
$$

where

$$
\mathcal{C}^\pm_{s,s+1}f^{(s+1)}=\sum_{i=1}^{s}\mathcal{C}^{\pm,i}_{s,s+1}f^{(s+1)}
\tag{4.3.4}
$$

the index $i$ referring to the index of the interaction particle among the $s$ "fixed" particles, with the notation

$$
\big(\mathcal{C}^{\pm,i}_{s,s+1}f^{(s+1)}\big)(Z_s):=(N-s)\varepsilon^{d-1}\int_{\mathbf{S}_1^{d-1}\times\mathbf{R}^d}(\omega\cdot(v_{s+1}-v_i))_\pm f^{(s+1)}(Z_s,x_i+\varepsilon\omega,v_{s+1})\,d\omega dv_{s+1}\,,
$$

the index + corresponding to post-collisional configurations and the index − to pre-collisional configurations.

Denote by $\mathbf{\Psi}_s(t)$ the $s$-particle flow associated with the hard-spheres system, and by $\mathbf{T}_s$ the associated solution operator:

$$\mathbf{T}_s(t) : \qquad f \in L^\infty(\mathcal{D}_s;\mathbf{R}) \mapsto f(\mathbf{\Psi}_s(-t,\cdot)) \in L^\infty(\mathcal{D}_s;\mathbf{R})\,. \tag{4.3.5}$$

The time-integrated form of equation (4.3.3) is

$$f_N^{(s)}(t,Z_s) = \mathbf{T}_s(t) f_N^{(s)}(0,Z_s) + \int_0^t \mathbf{T}_s(t-\tau)\mathcal{C}_{s,s+1} f_N^{(s+1)}(\tau,Z_s)\,d\tau\,. \tag{4.3.6}$$

The *total flow* and *total collision* operators $\mathbf{T}$ and $\mathbf{C}_N$ are defined on finite sequences $G_N = (g_s)_{1\leq s\leq N}$ as follows:

$$\begin{cases} \forall s \leq N\,,\ \left(\mathbf{T}(t)G_N\right)_s := \mathbf{T}_s(t) g_s\,, \\ \forall\, s \leq N-1\,,\ \left(\mathbf{C}_N G_N\right)_s := \mathcal{C}_{s,s+1} g_{s+1}\,, \quad \big(\mathbf{C}_N G_N\big)_N := 0\,. \end{cases} \tag{4.3.7}$$

We finally define *mild solutions* to the BBGKY hierarchy (4.3.6) to be solutions of

$$F_N(t) = \mathbf{T}(t)F_N(0) + \int_0^t \mathbf{T}(t-\tau)\mathbf{C}_N F_N(\tau)\,d\tau\,, \qquad F_N = (f_N^{(s)})_{1\leq s\leq N}\,. \tag{4.3.8}$$

## 4.4. The Boltzmann hierarchy and the Boltzmann equation

Starting from (4.3.8) we now consider the limit $N\to\infty$ under the Boltzmann-Grad scaling $N\varepsilon^{d-1}\equiv 1$, in order to derive formally the expected form of the Boltzmann hierarchy.

Because of the scaling assumption $N\varepsilon^{d-1}\equiv 1$, the collision term $\mathcal{C}_{s,s+1}f^{(s+1)}(Z_s)$ is approximately equal to

$$\sum_{i=1}^s \int_{\mathbf{S}_1^{d-1}\times\mathbf{R}^d} \omega\cdot(v_{s+1}-v_i) f_N^{(s+1)}(Z_s, x_i+\varepsilon\omega, v_{s+1})\,d\omega dv_{s+1}$$

which we may split into two terms, depending on the sign of $\omega\cdot(v_{s+1}-v_i)$, as in (4.3.4):

$$\begin{aligned} &\sum_{i=1}^s \int_{\mathbf{S}_1^{d-1}\times\mathbf{R}^d} \Big(\omega\cdot(v_{s+1}-v_i)\Big)_+ f_N^{(s+1)}(Z_s, x_i+\varepsilon\omega, v_{s+1})\,d\omega dv_{s+1} \\ &\quad - \sum_{i=1}^s \int_{\mathbf{S}_1^{d-1}\times\mathbf{R}^d} \Big(\omega\cdot(v_{s+1}-v_i)\Big)_- f_N^{(s+1)}(Z_s, x_i+\varepsilon\omega, v_{s+1})\,d\omega dv_{s+1}\,. \end{aligned}$$

Changing $\omega$ in $-\omega$ in the second term, we get

$$\begin{aligned} &\sum_{i=1}^s \int_{\mathbf{S}_1^{d-1}\times\mathbf{R}^d} \Big(\omega\cdot(v_{s+1}-v_i)\Big)_+ f_N^{(s+1)}(Z_s, x_i+\varepsilon\omega, v_{s+1})\,d\omega dv_{s+1} \\ &\quad - \sum_{i=1}^s \int_{\mathbf{S}_1^{d-1}\times\mathbf{R}^d} \Big(\omega\cdot(v_{s+1}-v_i)\Big)_+ f_N^{(s+1)}(Z_s, x_i-\varepsilon\omega, v_{s+1})\,d\omega dv_{s+1}\,. \end{aligned}$$

Recall that pre-collisional particles are particles $(x_i,v_i)$ and $(x_{s+1},v_{s+1})$ whose distance is decreasing up to collision time, meaning that for which

$$(x_{s+1}-x_i)\cdot(v_{s+1}-v_i) < 0\,.$$

With the above notation this means that

$$\omega\cdot(v_{s+1}-v_i) < 0\,.$$

On the contrary the case when $\omega\cdot(v_{s+1}-v_i) > 0$ is called the post-collisional case; we recall that grazing collisions, satisyfing $\omega\cdot(v_{s+1}-v_i) = 0$ can be neglected (see Paragraph 4.1 above).

Consider a set of particles $Z_{s+1} = (Z_s, x_i + \varepsilon\omega, v_{s+1})$ such that $(x_i, v_i)$ and $(x_i + \varepsilon\omega, v_{s+1})$ are post-collisional. We recall the boundary condition

$$f_N^{(s+1)}(t, Z_s, x_i + \varepsilon\omega, v_{s+1}) = f_N^{(s+1)}(t, Z_s^*, x_i + \varepsilon\omega, v_{s+1}^*)$$

where $Z_s^* = (z_1, \dots, z_i^*, \dots z_s)$ and $(v_i^*, v_{s+1}^*)$ is the pre-image of $(v_i, v_{s+1})$ by (4.1.1):

$$\begin{aligned} v_i^* &:= v_i - \omega \cdot (v_i - v_{s+1})\, \omega \\ v_{s+1}^* &:= v_{s+1} + \ \omega \cdot (v_i - v_{s+1})\, \omega\,, \end{aligned} \tag{4.4.1}$$

while $x_i^* := x_i$. In the following writing also $x_{s+1}^* := x_{s+1}$ we shall use the notation

$$\sigma(z_i^*, z_{s+1}^*) := (z_i, z_{s+1})\,. \tag{4.4.2}$$

Then neglecting the small spatial translations in the arguments of $f_N^{(s+1)}$ and using the fact that $f_N^{(s+1)}$ is left-continuous in time for all $s$ we obtain the following asymptotic expression for the collision operator at the limit:

$$\begin{aligned} \mathcal{C}^0_{s,s+1} f^{(s+1)}(t, Z_s) &:= \sum_{i=1}^s \int \big(\omega \cdot (v_{s+1} - v_i)\big)_+ \\ &\times \Big( f^{(s+1)}(t, x_1, v_1, \dots, x_i, v_i^*, \dots, x_s, v_s, x_i, v_{s+1}^*) - f^{(s+1)}(t, Z_s, x_i, v_{s+1}) \Big) d\omega dv_{s+1}\,. \end{aligned} \tag{4.4.3}$$

The asymptotic dynamics are therefore governed by the following integral form of the Boltzmann hierarchy:

$$f^{(s)}(t) = \mathbf{S}_s(t) f_0^{(s)} + \int_0^t \mathbf{S}_s(t-\tau) \mathcal{C}^0_{s,s+1} f^{(s+1)}(\tau)\, d\tau\,, \tag{4.4.4}$$

where $\mathbf{S}_s(t)$ denotes the $s$-particle free-flow.

Similarly to (4.3.7), we can define the total Boltzmann flow and collision operators $\mathbf{S}$ and $\mathbf{C}$ as follows:

$$\begin{cases} \forall s \geq 1\,,\ \big(\mathbf{S}(t) G\big)_s := \mathbf{S}_s(t) g_s\,, \\ \forall\, s \geq 1\,,\ \big(\mathbf{C^0} G\big)_s := \mathcal{C}^0_{s,s+1} g_{s+1}\,, \end{cases} \tag{4.4.5}$$

so that *mild solutions* to the Boltzmann hierarchy (4.4.4) are solutions of

$$F(t) = \mathbf{S}(t) F(0) + \int_0^t \mathbf{S}(t-\tau) \mathbf{C^0} F(\tau)\, d\tau\,, \qquad F = (f^{(s)})_{s \geq 1}\,. \tag{4.4.6}$$

Note that if $f^{(s)}(t, Z_s) = \displaystyle\prod_{i=1}^s f(t, z_i)$ (meaning $f^{(s)}(t)$ is *tensorized*) then $f$ satisfies the Boltzmann equation (2.1.1)-(2.1.2), where the cross-section is $b(w, \omega) := \big(\omega \cdot w\big)_+$.

# CHAPTER 5

# UNIFORM A PRIORI ESTIMATES FOR THE BBGKY AND BOLTZMANN HIERARCHIES

*This is a revised version of Chapter 5: in the original version, there were inconsistencies in the way the function spaces were introduced, and the present Paragraph 5.1 has been added to this Chapter in order to settle the functional framework.*

*The first two authors wish to thank Thierry Bodineau for his help in the writing of this new version.*

This chapter is devoted to the statement and proof of uniform a priori estimates for mild solutions to the BBGKY hierarchy, defined formally in (4.3.8), which we reproduce here:

$$F_N(t) = \mathbf{T}(t)F_N(0) + \int_0^t \mathbf{T}(t-\tau)\mathbf{C}_N F_N(\tau)\, d\tau\,, \qquad F_N = (f_N^{(s)})_{1\leq s\leq N}\,, \tag{5.0.1}$$

as well as for the limit Boltzmann hierarchy defined in (4.4.6)

$$F(t) = \mathbf{S}(t)F(0) + \int_0^t \mathbf{S}(t-\tau)\mathbf{C^0} F(\tau)\, d\tau\,, \qquad F = (f^{(s)})_{s\geq 1}\,. \tag{5.0.2}$$

Those results are obtained in Paragraphs 5.3 and 5.4 by use of a Cauchy-Kowalevskaya type argument. Before that we need to make sense of the formulation (5.0.1), which is not an obvious fact since characteristics of the transport are defined only almost everywhere (see Chapter 4) while the collision operators are defined by integrals on manifolds of codimension 1 [(1)]. In Paragraph 5.1 we show that the collision integrals make sense in $L^\infty$ outside some measure zero sets, provided that they are combined with the transport operator. Then Paragraph 5.2 is devoted to the definition of adequate function spaces in which the equations will be shown to be wellposed, and to the statements of the wellposedness results.

## 5.1. Rigorous formulation of the BBGKY hierarchy

In this paragraph we show how to make sense of the collision operators in (5.0.1). To this end, we define a new hierarchy by filtering of the transport operator:

$$G_N(t) = F_N(0) + \int_0^t \mathbf{T}(-\tau)\mathbf{C}_N\mathbf{T}(\tau)G_N(\tau)\, d\tau\,. \tag{5.1.1}$$

1. The question of correctly defining the hierarchy is also addressed in the work by S. Simonella, *Evolution of correlation functions in the hard sphere dynamics*, J. Stat. Phys. 155 (2014), no. 6, 1191-1221.

Notice that although $G_N$ and $F_N$ are related by the simple fact that

$$G_N(t) = (\mathbf{T}_s(-t)f_N^{(s)}(t))_{1\leq s\leq N}\,,$$

the hierarchy $G_N$ has much better regularity properties. In particular one can see (see the discussion in Remark 5.4.4 at the end of this chapter) that writing $G_N = (g_{n,s})_{1\leq s\leq N}$ then $g_{n,s}$ is a continuous function of time, with values in $L^\infty(\mathcal{D}_s)$, which is not the case of $f_N^{(s)}$. Moreover the idea of combining the collision integral $\mathcal{C}_{s,s+1}$ with the transport operator $\mathbf{T}_s(\tau)$ comes from the fact that time can be viewed as the missing coordinate on $\partial\mathcal{D}_{s+1}$ in the direction orthogonal to the boundary. We then expect to define the collision integral in $L^\infty$ by using Fubini's theorem.

**5.1.1. A local system of coordinates near the boundary.** — From now on we fix two integers $1\leq i\leq s$ and we note that for all $\delta>0$, the change of variables

$$\begin{aligned}\iota_s := \mathcal{D}_s\times[0,\delta]\times\mathbf{S}_1^{d-1}\times\mathbf{R}^d &\to \mathbf{R}^{2d(s+1)}\\ (Z_s,,t,\omega,v_{s+1}) &\mapsto Z_{s+1} = (X_s - tV_s, V_s, x_i+\varepsilon\omega - tv_{s+1}, v_{s+1})\end{aligned} \tag{5.1.2}$$

maps the measure $d\mu_i^- := \varepsilon^{d-1}\big((v_{s+1}-v_i)\cdot\omega\big)_- dZ_s dt d\omega dv_{s+1}$ on the Lebesgue measure $dZ_{s+1}$. Of course $Z_{s+1}$ defined in (5.1.2) is simply the mapping of $\tilde Z_{s+1} := (Z_s, x_i+\varepsilon\omega, v_{s+1})$ by the free transport operator. Similarly one can consider a post-collisional situation and notice that as the scattering preserves the measure, we have that for any $i\leq s$, with notation (4.4.1),

$$\iota_s^* := (Z_s,t,\omega,v_{s+1})\in\mathcal{D}_s\times[0,\delta]\times\mathbf{S}_1^{d-1}\times\mathbf{R}^d \mapsto Z_{s+1} = (X_s - tV_s^*, V_s^*, x_i+\varepsilon\omega - tv_{s+1}^*, v_{s+1}^*) \tag{5.1.3}$$

maps the measure $d\mu_i^+ := \varepsilon^{d-1}\big((v_{s+1}-v_i)\cdot\omega\big)_+ dZ_s dt d\omega dv_{s+1}$ on the Lebesgue measure $dZ_{s+1}$. In the following we write $\iota_s^-$ and $\iota_s^{-*}$ the above mappings where $t$ is replaced by $-t$.

Our aim is to extend this to the case when the free transport in the mappings $\iota_s, \iota_s^*$ is replaced by the transport $\mathbf{\Psi}_{s+1}$ with exclusion

$$Z_{s+1} = \mathbf{\Psi}_{s+1}(-t)\tilde Z_{s+1}\,,\quad \tilde Z_{s+1} := (Z_s, x_i+\varepsilon\omega, v_{s+1})$$

so that the image belongs to $\mathcal{D}_{s+1}$.

To do so, we are going to consider trajectories away from pathological configurations. From now on we fix $R_1, R>0$ (which will go to infinity at the very end), as well as the set

$$B_{R_1,R}^{2(s+1)} := \Big\{Z_{s+1}\in\mathbf{R}^{2d(s+1)}\,/\,|X_{s+1}|\leq R_1 \quad\text{and}\quad |V_{s+1}|\leq R\Big\}$$

and we define for all $\delta>0$, the sets

$$\begin{aligned}\partial\mathcal{D}_\delta^{i,s+1,\pm} := \Big\{Z_{s+1}\in B_{R_1,R}^{2(s+1)}\,/\,|x_i-x_{s+1}|=\varepsilon\,,\quad \pm(v_i-v_{s+1})\cdot(x_i-x_{s+1})>0\\ \text{and}\quad \forall(k,\ell)\in[1,s+1]^2\setminus\{(i,s+1)\}\,,\quad |x_k-x_\ell|>\varepsilon+R\delta\Big\}\,,\end{aligned}$$

and $\partial\mathcal{D}_\delta^{i,s+1} := \partial\mathcal{D}_\delta^{i,s+1,+}\cup\partial\mathcal{D}_\delta^{i,s+1,-}$. When $\delta=0$ we write $\partial\mathcal{D}^{i,s+1,\pm} := \partial\mathcal{D}_0^{i,s+1,\pm}$.

Note that $\Big(\partial\mathcal{D}_\delta^{i,s+1,+}\Big)_{\delta>0}$ are decreasing families.

### 5.1.2. Definition of the truncated collision integral. —

The collision operator is obtained by integration on each component of the boundary $\partial\mathcal{D}^{i,s+1,\pm}$ with respect to a partial set of variables, namely $\omega, v_{s+1}$, with the measure $d\mu_i^\pm$. For functions in $L^\infty$ (which are defined almost everywhere), such integrals are defined by Fubini's theorem.

More precisely, let us define truncated collision operators as follows: for any $\delta > 0$ and any continuous function $\varphi_{s+1}$ defined on $\mathcal{D}_{s+1}$,

$$\begin{aligned}\big(\mathcal{C}^{\pm,\delta}_{s,s+1}\varphi_{s+1}\big)(Z_s) &:= \sum_{i=1}^s \big(\mathcal{C}^{\pm,i,\delta}_{s,s+1}\varphi_{s+1}\big)(Z_s)\\ &:= (N-s)\varepsilon^{d-1}\sum_{i=1}^s\int_{\mathbf{S}_1^{d-1}\times\mathbf{R}^d}\big(\omega\cdot(v_{s+1}-v_i)\big)_\pm\\ &\quad\times\varphi_{s+1}(Z_s,x_i+\varepsilon\omega,v_{s+1})\Big(\prod_{(k,\ell)\in[1,s+1]^2\setminus\{(i,s+1)\}}1\!\!1_{|x_k-x_\ell|>\varepsilon+\delta R}\Big)\,d\omega dv_{s+1}\,.\end{aligned}$$

In the above integral to simplify notation we have written $x_{s+1} = x_i + \varepsilon\omega$ in the exclusion function $\displaystyle\prod_{(k,\ell)\in[1,s+1]^2\setminus\{(i,s+1)\}}1\!\!1_{|x_k-x_\ell|>\varepsilon+\delta R}$.

Now let us fix $T > 0$ and let us make sense of the functions $\mathcal{C}^{\pm,\delta}_{s,s+1}\mathbf{T}_{s+1}(t)\varphi_{s+1}$ in $L^\infty$, for $\varphi_{s+1}$ belonging to $L^\infty(\mathcal{D}_{s+1})$ and $t\in[0,T]$.

- We start by proving that those functions are locally integrable on $\mathcal{D}_s\times[0,T]$ (equipped with the Lebesgue measure $dZ_sdt$).

In the case when $t\in[0,\delta]$ then writing

$$\mathcal{C}^{\pm,\delta}_{s,s+1}(\mathbf{T}_{s+1}(t)\varphi_{s+1}) = \mathcal{C}^{\pm,\delta}_{s,s+1}(\varphi_{s+1}(\tilde Z_{s+1}))$$

then by definition there is no recollision since $\tilde Z_{s+1}$ belongs to $\partial\mathcal{D}^{i,s+1,\pm}_\delta$. Using the change of variables (5.1.2) in the pre-collisional case, and (5.1.3) in the post-collisional one, one finds that for any function $\varphi_{s+1}$ belonging to $L^\infty(\mathcal{D}_{s+1})\subset L^1_{loc}(\mathcal{D}_{s+1})$, the volumic integral is well defined: the domain of integration is indeed included in $\iota_s^-(B^s_{R_1}\times[0,\delta]\times\mathbf{S}_1^{d-1}\times B^1_R)\cup\iota_s^{-*}(B^s_{R_1}\times[0,\delta]\times\mathbf{S}_1^{d-1}\times B^1_R)$, or in other words in

$$\begin{aligned}&\{Z_{s+1}\in B^{2(s+1)}_{R_1,R}\,/\,\exists t\in[0,\delta],\quad |x_i-x_{s+1}+t(v_i-v_{s+1})|=\varepsilon\}\\ &\cup\{Z_{s+1}\in B^{2(s+1)}_{R_1,R}\,/\,\exists t\in[0,\delta],\quad |x_i-x_{s+1}+t(v_i^*-v_{s+1}^*)|=\varepsilon\}\end{aligned}$$

the volume of which is $O(R\delta\varepsilon^{d-1}R^{d(s+1)}R_1^{ds})$. Then,

$$\Big|\int_0^\delta\int_{\mathcal{D}_s}\Big(\mathcal{C}^{\pm,i,\delta}_{s,s+1}\mathbf{T}_{s+1}(t)\varphi_{s+1}\Big)\,dZ_sdt\Big|\le C_d\delta\varepsilon^{d-1}R_1^{ds}R^{d(s+1)+1}\|\varphi_{s+1}\|_{L^\infty(\mathcal{D}_{s+1})}\,.$$

Next we cover $[0,T]$ by $T/\delta$ intervals $[n\delta,(n+1)\delta]$

$$\int_{n\delta}^{(n+1)\delta}\int_{\mathcal{D}_s}\Big(\mathcal{C}^{\pm,i,\delta}_{s,s+1}\mathbf{T}_{s+1}(t)\varphi_{s+1}\Big)\,dZ_sdt = \int_0^\delta\int_{\mathcal{D}_s}\Big(\mathcal{C}^{\pm,i,\delta}_{s,s+1}\mathbf{T}_{s+1}(\tau)\mathbf{T}_{s+1}(n\delta)\varphi_{s+1}\Big)\,dZ_sd\tau$$

and we know that thanks to Alexander [**2**] (see also Paragraph 4.1),

$$\big|(\mathbf{T}_{s+1}(n\delta)\varphi_{s+1})(Z_{s+1})\big|\le\|\varphi_{s+1}\|_{L^\infty(\mathcal{D}_{s+1})}\,.$$

As above one infers after changing variables that

$$\int_{n\delta}^{(n+1)\delta}\int_{\mathcal{D}_s}\Big|\Big(\mathcal{C}^{\pm,i,\delta}_{s,s+1}\mathbf{T}_{s+1}(t)\varphi_{s+1}\Big)\Big|\,dZ_sdt \leq C_d\delta\varepsilon^{d-1}R_1^{ds}R^{d(s+1)+1}\|\varphi_{s+1}\|_{L^\infty(\mathcal{D}_{s+1})}$$

and therefore finally

$$\int_0^T\int_{\mathcal{D}_s}\Big|\Big(\mathcal{C}^{\pm,i,\delta}_{s,s+1}\mathbf{T}_{s+1}(t)\varphi_{s+1}\Big)\Big|\,dZ_sdt \leq C_dT\varepsilon^{d-1}R_1^{ds}R^{d(s+1)+1}\|\varphi_{s+1}\|_{L^\infty(\mathcal{D}_{s+1})}\,.$$

Then, by Fubini's theorem, we conclude that $\mathcal{C}^{\pm,i,\delta}_{s,s+1}\mathbf{T}_{s+1}(t)\varphi_{s+1}\in L^1([0,T]\times\mathcal{D}_s)$, in particular they are measurable functions.

• Returning to the control of the $L^\infty$ norm, we find from the above analysis that for any subset $A$ of $[0,\delta]\times\mathcal{D}_s$,

$$\int_A\Big|\Big(\mathcal{C}^{\pm,i,\delta}_{s,s+1}\mathbf{T}_{s+1}(t)\varphi_{s+1}\Big)\Big|\,dZ_sdt \leq C_d|A|R^{d+1}\varepsilon^{d-1}\|\varphi_{s+1}\|_{L^\infty(\mathcal{D}_{s+1})}\,,$$

since the domain of integration is included in $\iota_s^-(A\times\mathbf{S}_1^{d-1}\times B_R^1)\cup\iota_s^{-*}(A\times\mathbf{S}_1^{d-1}\times B_R^1)$. It is then easy to conclude that

$$\Big|\Big(\mathcal{C}^{\pm,i,\delta}_{s,s+1}\mathbf{T}_{s+1}(t)\varphi_{s+1}\Big)(Z_s)\Big| \leq C_dR^{d+1}\varepsilon^{d-1}\|\varphi_{s+1}\|_{L^\infty(\mathcal{D}_{s+1})}$$

almost everywhere in $[0,\delta]\times\mathcal{D}_s$ (since the set where these inequalities are not satisfied is of measure 0). We then extend the reasoning to any set of the type $[n\delta,(n+1)\delta]\times\mathcal{D}_s$ as in the previous paragraph: for any subset $A_n$ of $[n\delta,(n+1)\delta]\times\mathcal{D}_s$, we have

$$\begin{aligned}\int_{A_n}\Big(\mathcal{C}^{\pm,i,\delta}_{s,s+1}\mathbf{T}_{s+1}(t)\varphi_{s+1}\Big)(Z_s)\,dZ_sdt &= \int_{A_n}\Big(\mathcal{C}^{\pm,i,\delta}_{s,s+1}\mathbf{T}_{s+1}(t-n\delta)\mathbf{T}_{s+1}(n\delta)\varphi_{s+1}\Big)(Z_s)\,dZ_sdt\\ &= \int_{A_n^\delta}\Big(\mathcal{C}^{\pm,i,\delta}_{s,s+1}\mathbf{T}_{s+1}(\tau)\mathbf{T}_{s+1}(n\delta)\varphi_{s+1}\Big)(Z_s)\,dZ_sd\tau\end{aligned}$$

where $A_n^\delta := \{(\tau,Z_s)\,/\,(\tau+n\delta,Z_s)\in A_n\}$. Since $|A_n^\delta|=|A_n|$ we find that

$$\int_{A_n}\Big|\Big(\mathcal{C}^{\pm,i,\delta}_{s,s+1}\mathbf{T}_{s+1}(t)\varphi_{s+1}\Big)(Z_s)\Big|\,dZ_sdt \leq C_d|A_n|R^{d+1}\varepsilon^{d-1}\|\varphi_{s+1}\|_{L^\infty(\mathcal{D}_{s+1})}\,,$$

so

$$\Big|\Big(\mathcal{C}^{\pm,i,\delta}_{s,s+1}\mathbf{T}_{s+1}(t)\varphi_{s+1}\Big)(Z_s)\Big| \leq C_dR^{d+1}\varepsilon^{d-1}\|\varphi_{s+1}\|_{L^\infty(\mathcal{D}_{s+1})}$$

almost everywhere in $[n\delta,(n+1)\delta]\times\mathcal{D}_s$. Finally this implies that

$$\Big|\Big(\mathcal{C}^{\pm,i,\delta}_{s,s+1}\mathbf{T}_{s+1}(t)\varphi_{s+1}\Big)(Z_s)\Big| \leq C_dR^{d+1}\varepsilon^{d-1}\|\varphi_{s+1}\|_{L^\infty(\mathcal{D}_{s+1})}$$

almost everywhere in $[0,T]\times\mathcal{D}_s$.

We have thus defined truncated collision integrals far from the singular points of the boundary of $\mathcal{D}_{s+1}$. It remains then to check that the sequence of operators thus constructed is a Cauchy sequence with respect to the truncation parameter in $L^\infty$, outside a set of measure going to zero with the truncation parameter.

### 5.1.3. Removing the truncation. —

Let $0 < \delta' < \delta$ be given and consider the truncated operators

$$\mathcal{C}^{\pm,i,\delta',\delta}_{s,s+1} := \mathcal{C}^{\pm,i,\delta'}_{s,s+1} - \mathcal{C}^{\pm,i,\delta}_{s,s+1} \,.$$

We shall prove that the partial integral $\mathcal{C}^{\pm,i,\delta',\delta}_{s,s+1}\mathbf{T}_{s+1}(t)\varphi_{s+1}$ is small (of the order $\sqrt{\delta}$) outside a small subset of $\mathcal{D}_s \times [0,T]$, of measure going to zero with $\delta$. Indeed we have

$$\int_0^{\delta'}\int_{\mathcal{D}_s}\Big|\Big(\mathcal{C}^{\pm,i,\delta',\delta}_{s,s+1}\mathbf{T}_{s+1}(t)\varphi_{s+1}\Big)\Big|\,dZ_s dt = \int_{V_{\delta,\delta'}} |\varphi_{s+1}(Z_{s+1})|\,dZ_{s+1}\,,$$

where $V_{\delta,\delta'}$ is a subdomain of

$$\begin{aligned}
&\Big\{Z_{s+1} \in B^{2(s+1)}_{R_1,R} \,/\, \exists t \in [0,\delta']\,, (j,j') \neq (i,s+1)\,, \quad |x_i - x_{s+1} + t(v_i - v_{s+1})| = \varepsilon \\
&\qquad\qquad \text{and } \varepsilon \leq |x_j - x_{j'} + t(v_j - v_{j'})| \leq \varepsilon + R\delta\Big\} \\
&\cup \Big\{Z_{s+1} \in B^{2(s+1)}_{R_1,R} \,/\, \exists t \in [0,\delta']\,, (j,j') \neq (i,s+1)\,, \ell \neq i, s+1\,, \quad |x_i - x_{s+1} + t(v^*_i - v^*_{s+1})| = \varepsilon \\
&\qquad \text{and } \begin{cases} \text{either } \varepsilon \leq |x_i - x_\ell + t(v^*_i - v_\ell)| \leq \varepsilon + R\delta \\ \text{or } \varepsilon \leq |x_{s+1} - x_\ell + t(v^*_{s+1} - v_\ell)| \leq \varepsilon + R\delta \quad \text{ or } \varepsilon \leq |x_j - x_{j'} + t(v_j - v_{j'})| \leq \varepsilon + R\delta \end{cases}\Big\}\,.
\end{aligned}$$

In particular, $|V_{\delta,\delta'}| \leq C(R,\varepsilon)\delta\delta'$. Arguing as in the previous section we deduce the estimate on $[0,T]$

$$\int_0^{T}\int_{\mathcal{D}_s}\Big|\Big(\mathcal{C}^{\pm,i,\delta',\delta}_{s,s+1}\mathbf{T}_{s+1}(t)\varphi_{s+1}\Big)\Big|\,dZ_s dt \leq C(R,T)\delta\|\varphi_{s+1}\|_{L^\infty(\mathcal{D}_{s+1})}\,, \tag{5.1.4}$$

uniformly in $\delta'$. Finally we introduce the set

$$I_{\delta,\delta',i,\pm} = \Big\{(t,Z_s) \in [0,T]\times\mathcal{D}_s \,\Big|\, \Big|\Big(\mathcal{C}^{\pm,i,\delta',\delta}_{s,s+1}\mathbf{T}_{s+1}(t)\varphi_{s+1}\Big)(Z_s)\Big| \geq \sqrt{\delta}\Big\}\,.$$

Thanks to the Bienaymé-Tchebichev inequality and to (5.1.4), we have uniformly in $\delta'$

$$|I_{\delta,\delta',i,\pm}| = O(\sqrt{\delta})\,.$$

Note furthermore that $I_{\delta,\delta',i,\pm}$ is a decreasing function of $\delta$. On the complement of $I_{\delta,\delta',i,\pm}$, for any function $\varphi_{s+1} \in L^\infty(\mathcal{D}_{s+1})$

$$\|\mathcal{C}^{\pm,i,\delta',\delta}_{s,s+1}\mathbf{T}_{s+1}(t)\varphi_{s+1}\|_{L^\infty} \leq C(R)\|\varphi_{s+1}\|_{L^\infty}\sqrt{\delta}\,.$$

This tells us exactly that the sequence $\mathcal{C}^{\pm,i,\delta}_{s,s+1}\mathbf{T}_{s+1}(t)\varphi_{s+1}$ is a Cauchy sequence and converges weakly-$*$ in $L^\infty([0,T]\times\mathcal{D}_s)$ as $\delta \to 0$.

### 5.1.4. Dependence with respect to time and conclusion. —

Finally to define $\mathcal{C}_{s,s+1}\mathbf{T}_{s+1}(t)$ on time-dependent functions belonging to $C([0,T];L^\infty(\mathcal{D}_{s+1}))$ supported in $[0,T]\times B^{2(s+1)}_{R_1,R}$, we notice that the above arguments are very easily adapted to the case of piecewise constant functions in time, denoted $PC([0,T];L^\infty(\mathcal{D}_{s+1}))$. Then we conclude by density of $PC([0,T];L^\infty(\mathcal{D}_{s+1}))$ in $C([0,T];L^\infty(\mathcal{D}_{s+1}))$. Indeed if $\varphi_{s+1}$ is a function in $C([0,T];L^\infty(\mathcal{D}_{s+1}))$ supported in $[0,T]\times B^{2(s+1)}_{R_1,R}$ and if $(\varphi^n_{s+1})_{n\in\mathbf{N}}$ is a sequence of approximations of $\varphi_{s+1}$, we have the following estimate

$$\big\|\mathcal{C}^{\pm}_{s,s+1}\mathbf{T}_{s+1}(t)\big(\varphi^n_{s+1}(t) - \varphi^m_{s+1}(t)\big)\big\|_{L^\infty} \leq C(R)\|\varphi^n_{s+1}(t) - \varphi^m_{s+1}(t)\|_{L^\infty}\,,$$

which tends to 0 as $n,m \to \infty$, uniformly in $t \in [0,T]$.

Letting $R_1$ and $R$ go to infinity, we conclude that the operator $\mathcal{C}_{s,s+1}\mathbf{T}_{s+1}(t)$ is well defined on functions of $C([0,T];L^\infty(\mathcal{D}_{s+1}))$ with bounded support in $V_{s+1}$ (or decaying sufficiently fast at infinity). A quantitative estimate of this decay will be given by introducing appropriate weighted spaces in the next section.

Notice that for the Boltzmann hierarchy (5.0.2), the collision operators are defined by integrals on manifolds of codimension $d$ but since free transport preserves continuity one can require that all functions under study are continuous.

## 5.2. Functional spaces and statement of the results

In order to obtain uniform a priori bounds for mild solutions to the (filtered) BBGKY hierarchy, we need to introduce some norms on the space of sequences $(g_s)_{s\geq 1}$. Given $\varepsilon > 0$, $\beta > 0$, an integer $s \geq 1$, and a measurable function $g_s : \mathcal{D}_s \to \mathbf{R}$, we let

$$|g_s|_{\varepsilon,s,\beta} := \operatorname{supess}_{Z_s\in\mathcal{D}_s}\big(|g_s(Z_s)|\exp\big(\beta E_0(Z_s)\big)\big) \tag{5.2.1}$$

where $E_0$ is the free Hamiltonian:

$$E_0(Z_s) := \sum_{1\leq i\leq s}\frac{|v_i|^2}{2}\,. \tag{5.2.2}$$

Note that the dependence on $\varepsilon$ of the norm is through the constraint $Z_s \in \mathcal{D}_s$.

We also define, for a continuous function $g_s : \mathbf{R}^{2ds} \to \mathbf{R}$,

$$|g_s|_{0,s,\beta} := \sup_{Z_s\in\mathbf{R}^{2ds}}\big(|g_s(Z_s)|\exp\big(\beta E_0(Z_s)\big)\big)\,. \tag{5.2.3}$$

***Definition 5.2.1***. — *For $\varepsilon > 0$ and $\beta > 0$, we denote $X_{\varepsilon,s,\beta}$ the Banach space of measurable functions $\mathcal{D}_s \to \mathbf{R}$ with finite $|\cdot|_{\varepsilon,s,\beta}$ norm, and similarly $X_{0,s,\beta}$ is the Banach space of continuous functions $\mathbf{R}^{2ds} \to \mathbf{R}$ with finite $|\cdot|_{0,s,\beta}$ norm.*

For sequences of measurable functions $G = (g_s)_{s\geq 1}$, with $g_s : \mathcal{D}_s \to \mathbf{R}$, we let for $\varepsilon > 0$, $\beta > 0$, and $\mu \in \mathbf{R}$,

$$\|G\|_{\varepsilon,\beta,\mu} := \sup_{s\geq 1}\Big(|g_s|_{\varepsilon,s,\beta}\exp(\mu s)\Big)\,.$$

We define similarly for $G = (g_s)_{s\geq 1}$, with $g_s : \mathbf{R}^{2ds} \to \mathbf{R}$ continuous,

$$\|G\|_{0,\beta,\mu} := \sup_{s\geq 1}\Big(|g_s|_{0,s,\beta}\exp(\mu s)\Big)\,.$$

***Definition 5.2.2***. — *For $\varepsilon \geq 0$, $\beta > 0$, and $\mu \in \mathbf{R}$, we denote $\mathbf{X}_{\varepsilon,\beta,\mu}$ the Banach space of sequences of functions $G = (g_s)_{1\leq s\leq N}$, with $g_s \in X_{\varepsilon,s,\beta}$ and $\|G\|_{\varepsilon,\beta,\mu} < \infty$, and similarly $\mathbf{X}_{0,\beta,\mu}$ the Banach space of sequences of continuous functions $G = (g_s)_{s\geq 1}$, with $g_s \in X_{0,s,\beta}$ and $\|G\|_{0,\beta,\mu} < \infty$.*

The following inclusions hold:

$$\text{if } \beta' \leq \beta \text{ and } \mu' \leq \mu, \text{ then } \quad X_{\varepsilon,s,\beta'} \subset X_{\varepsilon,s,\beta}\,, \quad \mathbf{X}_{\varepsilon,\beta',\mu'} \subset \mathbf{X}_{\varepsilon,\beta,\mu}\,. \tag{5.2.4}$$

***Remark 5.2.3***. — *These norms are rather classical in statistical physics (up to replacing the $L^\infty$ norm by an $L^1$ norm), where probability measures are called "ensembles".*

*At the canonical level, the ensemble $1\!\!1_{Z_s \in \mathcal{D}_s} e^{-\beta E_0(Z_s)} dZ_s$ is a normalization of the Lebesgue measure, where $\beta \sim \theta^{-1}$ (and $\theta$ is the absolute temperature) specifies fluctuations of energy. The Boltzmann-Gibbs principle states that the average value of any quantity in the canonical ensemble is its equilibrium value at temperature $\theta$.*

*The micro-canonical level consists in restrictions of the ensemble to energy surfaces.*

*At the grand-canonical level the number of particles may vary, with variations indexed by chemical potential $\mu \in \mathbf{R}$.*

Existence and uniqueness for (5.0.1) comes from the theory of linear transport equations which provides a unique, global solution to the Liouville equation (4.2.1) by the method of characteristics. Nevertheless, in order to obtain a similar result for the limiting hierarchy (5.0.2), we need to obtain uniform a priori estimates with respect to $N$, on the marginals $f_N^{(s)}$ for any fixed $s$. We shall thus deal with both systems (5.0.1) and (5.0.2) simultaneously, using analytical-type techniques which will provide short-time existence (with uniform bounds) in the spaces of $\mathbf{X}_{\varepsilon,\beta,\mu}$-valued functions of time (resp. $\mathbf{X}_{0,\beta,\mu}$). Actually the parameters $\beta$ and $\mu$ will themselves depend on time: in the sequel we choose for simplicity a linear dependence in time, though other, decreasing functions of time could be chosen just as well. Such a time dependence on the parameters of the function spaces is a situation which occurs whenever continuity estimates involve a loss, which is the case here since the continuity estimates on the collision operators lead to a deterioration in the parameters $\beta$ and $\mu$.

***Definition 5.2.4***. — *Given $T > 0$, a positive function $\boldsymbol{\beta}$ and a real valued function $\boldsymbol{\mu}$ both defined on $[0, T]$, we denote by $\mathbf{X}_{\varepsilon,\boldsymbol{\beta},\boldsymbol{\mu}}$ the space of time continuous functions*

$$G : t \in [0, T] \mapsto G(t) = (g_s(t))_{s\geq 1} \in \mathbf{X}_{\varepsilon,\beta(t),\mu(t)} ,$$

*such that*

$$\begin{aligned} &|||G|||_{\varepsilon,\boldsymbol{\beta},\boldsymbol{\mu}} := \sup_{0\leq t\leq T} \|G(t)\|_{\varepsilon,\beta(t),\mu(t)} < \infty , \\ &\lim_{s\to t^-} \|G(t) - G(s)\|_{\mathbf{X}_{\varepsilon,\beta(t),\mu(t)}} = 0 . \end{aligned} \tag{5.2.5}$$

*We define similarly*

$$|||G|||_{0,\boldsymbol{\beta},\boldsymbol{\mu}} := \sup_{0\leq t\leq T} \|G(t)\|_{0,\beta(t),\mu(t)} .$$

We shall prove the following uniform bounds for the BBGKY hierarchy.

***Theorem 6* (Uniform estimates for the BBGKY hierarchy)**. — *Let $\beta_0 > 0$ and $\mu_0 \in \mathbf{R}$ be given. There is a time $T > 0$ as well as two nonincreasing functions $\boldsymbol{\beta} > 0$ and $\boldsymbol{\mu}$ defined on $[0, T]$, satisfying $\boldsymbol{\beta}(0) = \beta_0$ and $\boldsymbol{\mu}(0) = \mu_0$, such that in the Boltzmann-Grad scaling $N\varepsilon^{d-1} \equiv 1$, any family of initial marginals $F_N(0) = \big(f_N^{(s)}(0)\big)_{1\leq s\leq N}$ in $\mathbf{X}_{\varepsilon,\beta_0,\mu_0}$ gives rise to a unique solution $G_N(t) = (\mathbf{T}_s(-t) f_N^{(s)}(t))_{1\leq s\leq N}$ in $\mathbf{X}_{\varepsilon,\boldsymbol{\beta},\boldsymbol{\mu}}$ to the BBGKY hierarchy (5.0.1) satisfying the following bound:*

$$|||G_N|||_{\varepsilon,\boldsymbol{\beta},\boldsymbol{\mu}} \leq 2\|F_N(0)\|_{\varepsilon,\beta_0,\mu_0} .$$

***Remark 5.2.5***. — *The proof of Theorem 6 provides a lower bound for the time $T$ on which one has a uniform bound, in terms of the initial parameters $\beta_0$, $\mu_0$ and the dimension $d$: one finds*

$$T \geq C_d e^{\mu_0} (1 + \beta_0^{\frac{1}{2}})^{-1} \max_{\beta \in [0,\beta_0]} \beta e^{-\beta} (\beta_0 - \beta)^{\frac{d+1}{2}} \,, \tag{5.2.6}$$

*where $C_d$ is a constant depending only on $d$.*

*In particular if $d \ll \beta_0$, there holds $\max_{\beta \in [0,\beta_0]} \beta e^{-\beta} (\beta_0 - \beta)^{\frac{d+1}{2}} = \beta_0^{\frac{d+1}{2}} \big(1 + o(1)\big)$, hence an existence time of the order of $e^{\mu_0} \beta_0^{d/2}$.*

The proof of Theorem 6 uses neither the fact that the BBGKY hierarchy is closed by the transport equation satisfied by $f_N$, nor possible cancellations of the collision operators. It only relies on crude estimates and in particular the limiting hierarchy satisfies the same result, proved similarly. Note that the functional setting is simpler in the case of the Boltzmann hierarchy as all functions are continuous with respect to all parameters.

***Theorem 7* (Existence for the Boltzmann hierarchy)**. — *Let $\beta_0 > 0$ and $\mu_0 \in \mathbf{R}$ be given. There is a time $T > 0$ as well as two nonincreasing functions $\boldsymbol{\beta} > 0$ and $\boldsymbol{\mu}$ defined on $[0,T]$, satisfying $\boldsymbol{\beta}(0) = \beta_0$ and $\boldsymbol{\mu}(0) = \mu_0$, such that any family of initial marginals $F(0) = \big(f^{(s)}(0)\big)_{s\geq 1}$ in $\mathbf{X}_{0,\beta_0,\mu_0}$ gives rise to a unique solution $G(t) = (\mathbf{S}_s(-t) f^{(s)}(t))_{s\geq 1}$ in $\mathbf{X}_{0,\boldsymbol{\beta},\boldsymbol{\mu}}$ to the Boltzmann hierarchy (5.0.2), satisfying the following bound:*

$$|||G|||_{0,\boldsymbol{\beta},\boldsymbol{\mu}} \leq 2 \|F(0)\|_{0,\beta_0,\mu_0} \,.$$

## 5.3. Main steps of the proofs

The proofs of Theorems 6 and 7 are typical of analytical-type results, such as the classical Cauchy-Kowalevskaya theorem. We follow here Ukai's approach [**45**], which turns out to be remarkably short and self-contained.

Let us give the main steps of the proof: we start by noting that the conservation of energy for the $s$-particle flow is reflected in identities

$$|\mathbf{T}_s(t) g_s|_{\varepsilon,s,\beta} = |g_s|_{\varepsilon,s,\beta} \quad \text{and} \quad \|\mathbf{T}(t) G_N\|_{\varepsilon,\beta,\mu} = \|G_N\|_{\varepsilon,\beta,\mu} \,, \tag{5.3.1}$$

for all parameters $\beta > 0$, $\mu \in \mathbf{R}$, and for all $g_s \in X_{\varepsilon,s,\beta}$, $G_N = (g_s)_{1\leq s\leq N} \in \mathbf{X}_{\varepsilon,\beta,\mu}$, and all $t \geq 0$. Similarly,

$$|\mathbf{S}_s(t) g_s|_{0,s,\beta} = |g_s|_{s,\beta} \quad \text{and} \quad \|\mathbf{S}(t) G\|_{0,\beta,\mu} = \|G\|_{0,\beta,\mu} \,, \tag{5.3.2}$$

for all parameters $\beta > 0$, $\mu \in \mathbf{R}$, and for all $g_s \in X_{0,s,\beta}$, $G = (g_s)_{s\geq 1} \in \mathbf{X}_{0,\beta,\mu}$, and all $t \geq 0$.

Next assume that in the Boltzmann-Grad scaling $N\varepsilon^{d-1} \equiv 1$, there holds the bound

$$\forall\, 0 < \varepsilon \leq \varepsilon_0 \,, \quad \Big|\Big|\Big| \int_0^t \mathbf{T}(-\tau) \mathbf{C}_N \mathbf{T}(\tau) G_N(\tau)\, d\tau \Big|\Big|\Big|_{\varepsilon,\boldsymbol{\beta},\boldsymbol{\mu}} \leq \frac{1}{2} |||G_N|||_{\varepsilon,\boldsymbol{\beta},\boldsymbol{\mu}} \,, \tag{5.3.3}$$

for some functions $\boldsymbol{\beta}$ and $\boldsymbol{\mu}$ as in the statement of Theorem 6. Under (5.3.3), the linear operator

$$\mathfrak{L} : \quad G_N \in \mathbf{X}_{\varepsilon,\boldsymbol{\beta},\boldsymbol{\mu}} \mapsto \Big( t \mapsto \int_0^t \mathbf{T}(-\tau) \mathbf{C}_N \mathbf{T}(\tau) G_N(\tau)\, d\tau \Big) \in \mathbf{X}_{\varepsilon,\boldsymbol{\beta},\boldsymbol{\mu}}$$

is linear continuous from $\mathbf{X}_{\varepsilon,\boldsymbol{\beta},\boldsymbol{\mu}}$ to itself with norm strictly smaller than one. In particular, the operator $\mathrm{Id}-\mathfrak{L}$ is invertible in the Banach algebra $\mathcal{L}(\mathbf{X}_{\varepsilon,\boldsymbol{\beta},\boldsymbol{\mu}})$. Hence, there exists a unique solution $G_N$ in $\mathbf{X}_{\varepsilon,\boldsymbol{\beta},\boldsymbol{\mu}}$ to $(\mathrm{Id}-\mathfrak{L})G_N = F_N(0)$, an equation which is equivalent to (5.0.1).

The reasoning is identical for Theorem 7, replacing (5.3.3) by

$$\left\vert\kern-0.25ex\left\vert\kern-0.25ex\left\vert \int_0^t \mathbf{S}(-\tau)\mathbf{C}^0\mathbf{S}(\tau)G(\tau)\,d\tau \right\vert\kern-0.25ex\right\vert\kern-0.25ex\right\vert_{0,\boldsymbol{\beta},\boldsymbol{\mu}} \leq \frac{1}{2}\,|\!|\!| G |\!|\!|_{0,\boldsymbol{\beta},\boldsymbol{\mu}}\,. \tag{5.3.4}$$

The next section is devoted to the proofs of (5.3.3) and (5.3.4).

## 5.4. Continuity estimates

In order to prove (5.3.3) and (5.3.4), we first establish bounds, in the above defined functional spaces, for the collision operators defined in (4.3.2) and (4.4.3), and for the total collision operators. In $\mathcal{C}_{s,s+1}$, the sum in $i$ over $[1,s]$ will imply a loss in $\mu$, while the linear velocity factor will imply a loss in $\beta$.

The next statement concerns the BBGKY collision operator.

***Proposition 5.4.1.*** — *Given $\beta > 0$ and $\mu \in \mathbf{R}$, for $1 \leq s \leq N-1$, the collision operator $\mathcal{C}_{s,s+1}$ satisfies the bound, for all $G_N = (g_s)_{1\leq s\leq N} \in \mathbf{X}_{\varepsilon,\beta,\mu}$ in the Boltzmann-Grad scaling $N\varepsilon^{d-1} \equiv 1$, and for almost all $t$ and $Z_s$,*

$$\left|\big(\mathcal{C}_{s,s+1}\mathbf{T}_{s+1}(t)g_{s+1}\big)(Z_s)\right| \leq C_d\,\beta^{-\frac{d}{2}}\Big(s\beta^{-\frac{1}{2}} + \sum_{1\leq i\leq s}|v_i|\Big)e^{-\beta E_0(Z_s)}|g_{s+1}|_{\varepsilon,s+1,\beta}\,, \tag{5.4.1}$$

*for some $C_d > 0$ depending only on $d$.*

*Proof.* — Recall that as in (4.3.2),

$$\begin{aligned}\big(\mathcal{C}_{s,s+1}\mathbf{T}_{s+1}(t)g^{(s+1)}\big)(t,Z_s) :=\,& (N-s)\varepsilon^{d-1}\\ &\times\sum_{i=1}^{s}\int_{\mathbf{S}_1^{d-1}\times\mathbf{R}^d}\omega\cdot(v_{s+1}-v_i)\,\mathbf{T}_{s+1}(t)g^{(s+1)}(t,Z_s,x_i+\varepsilon\omega,v_{s+1})d\omega dv_{s+1}\,.\end{aligned}$$

Estimating each term in the sum separately, regardless of possible cancellations between "gain" and "loss" terms, it is obvious that

$$|\mathcal{C}_{s,s+1}\mathbf{T}_{s+1}(t)g_{s+1}(Z_s)| \leq \kappa_d\varepsilon^{d-1}(N-s)|g_{s+1}|_{\varepsilon,s+1,\beta}\sum_{1\leq i\leq s}I_i(V_s)\,,$$

where $\kappa_d$ is the volume of the unit ball of $\mathbf{R}^d$, and where

$$I_i(V_s) := \int_{\mathbf{R}^d}\big(|v_{s+1}|+|v_i|\big)\exp\Big(-\frac{\beta}{2}\sum_{j=1}^{s+1}|v_j|^2\Big)dv_{s+1}\,.$$

Since a direct calculation gives

$$I_i(V_s) \leq C_d\,\beta^{-\frac{d}{2}}\big(\beta^{-\frac{1}{2}}+|v_i|\big)\exp\Big(-\frac{\beta}{2}\sum_{1\leq j\leq s}|v_j|^2\Big)\,,$$

the result (5.4.1) is deduced directly in the Boltzmann-Grad scaling $N\varepsilon^{d-1}\equiv 1$. Proposition 5.4.1 is proved. □

A similar result holds for the limiting collision operator.

***Proposition 5.4.2.*** — *Given $\beta > 0$, $\mu \in \mathbf{R}$, the collision operator $\mathcal{C}^0_{s,s+1}$ satisfies the following bound, for all $g_{s+1} \in X_{0,s+1,\beta}$ :*

$$\big|(\mathcal{C}^0_{s,s+1} g_{s+1})(Z_s)\big| \leq C_d \beta^{-\frac{d}{2}} \Big( s\beta^{-\frac12} + \sum_{1\leq i\leq s} |v_i| \Big) e^{-\beta E_0(Z_s)} |g_{s+1}|_{0,s+1,\beta} \, , \tag{5.4.2}$$

*for some $C_d > 0$ depending only on $d$.*

*Proof.* — There holds

$$\big|(\mathcal{C}^0_{s,s+1} g_{s+1})(Z_s)\big| \leq \sum_{1\leq i\leq s} \int_{\mathbf{S}^{d-1}\times\mathbf{R}^d} \big(|v_{s+1}| + |v_i|\big)\big(|g_{s+1}(v_i^*, v_{s+1}^*)| + |g_{s+1}(v_i, v_{s+1})|\big) d\omega dv_{s+1},$$

omitting most of the arguments of $g_{s+1}$ in the integrand. By definition of $|\cdot|_{0,s,\beta}$ norms and conservation of energy (5.3.1), there holds

$$\begin{aligned} |g_{s+1}(v_i^*, v_{s+1}^*)| + |g_{s+1}(v_i, v_{s+1})| &\leq \big(e^{-\beta E_0(Z_s^*)} + e^{-\beta E_0(Z_s)}\big) |g_{s+1}|_{0,\beta} \\ &= 2e^{-\beta E_0(Z_s)} |g_{s+1}|_{0,s+1,\beta}, \end{aligned}$$

where $Z_s^*$ is identical to $Z_s$ except for $v_i$ and $v_{s+1}$ changed to $v_i^*$ and $v_{s+1}^*$. This gives

$$\big|(\mathcal{C}^0_{s,s+1} g_{s+1})(Z_s)\big| \leq C_d |g_{s+1}|_{0,s+1,\beta} e^{-\beta E_0(Z_s)} \sum_{1\leq i\leq s} I_i(V_s) \, ,$$

borrowing notation from the proof of Proposition 5.4.1, and we conclude as above. □

Propositions 5.4.1 and 5.4.2 are the key to the proof of (5.3.3) and (5.3.4). Let us first prove a continuity estimate based on Proposition 5.4.1, which implies directly (5.3.3).

***Lemma 5.4.3.*** — *Let $\beta_0 > 0$ and $\mu_0 \in \mathbf{R}$ be given. For all $\lambda > 0$ and $t > 0$ such that $\lambda t < \beta_0$, there holds the bound*

$$e^{s(\mu_0 - \lambda t)} \Big| \int_0^t \mathbf{T}_s(-\tau) \mathcal{C}_{s,s+1} \mathbf{T}_{s+1}(\tau) g_{s+1}(\tau) \, d\tau \Big|_{\varepsilon,s,\beta_0-\lambda t} \leq \bar c(\beta_0, \mu_0, \lambda, T) |\!|\!| G_N |\!|\!|_{\varepsilon,\boldsymbol{\beta},\boldsymbol{\mu}} \, , \tag{5.4.3}$$

*for all $G_N = (g_s)_{1\leq s\leq N} \in \mathbf{X}_{\varepsilon,\boldsymbol{\beta},\boldsymbol{\mu}}$, with $\bar c(\beta_0, \mu_0, \lambda, T)$ computed explicitly in (5.4.9) below. In particular there is $T > 0$ depending only on $\beta_0$ and $\mu_0$ such that for an appropriate choice of $\lambda$ in $(0, \beta_0/T)$, there holds for all $t \in [0, T]$*

$$e^{s(\mu_0 - \lambda t)} \Big| \int_0^t \mathbf{T}_s(-\tau) \mathcal{C}_{s,s+1} \mathbf{T}_{s+1}(\tau) g_{s+1}(\tau) \, d\tau \Big|_{\varepsilon,s,\beta_0-\lambda t} \leq \frac12 \, |\!|\!| G_N |\!|\!|_{\varepsilon,\boldsymbol{\beta},\boldsymbol{\mu}} \, . \tag{5.4.4}$$

*Proof.* — Let us define, for all $\lambda > 0$ and $t > 0$ such that $\lambda t < \beta_0$, the functions

$$\beta_0^\lambda(t) := \beta_0 - \lambda t \quad \text{and} \quad \mu_0^\lambda(t) := \mu_0 - \lambda t \, . \tag{5.4.5}$$

By conservation of energy (5.3.1), there holds the bound

$$\Big| \int_0^t \mathbf{T}_s(-\tau) \mathcal{C}_{s,s+1} \mathbf{T}_{s+1}(\tau) g_{s+1}(\tau) \, d\tau \Big|_{\varepsilon,s,\beta_0^\lambda(t)} \leq \sup_{Z_s\in\mathbf{R}^{2ds}} \int_0^t e^{\beta_0^\lambda(t) E_0(Z_s)} \big| \mathcal{C}_{s,s+1} \mathbf{T}_{s+1}(\tau) g_{s+1}(\tau, Z_s) \big| \, d\tau \, .$$

Estimate (5.4.1) from Proposition 5.4.1 gives

$$\begin{aligned} &e^{\beta_0^\lambda(t) E_0(Z_s)} \big| \mathcal{C}_{s,s+1} \mathbf{T}_{s+1}(\tau) g_{s+1}(\tau, Z_s) \big| \\ &\qquad \leq C_d \left(\beta_0^\lambda(\tau)\right)^{-\frac{d}{2}} |g_{s+1}(\tau)|_{\varepsilon,s+1,\beta_0^\lambda(\tau)} \Big( s(\beta_0^\lambda(\tau))^{-\frac12} + \sum_{1\leq i\leq s} |v_i| \Big) e^{\lambda(\tau - t) E_0(Z_s)} \, . \end{aligned}$$

By definition of norms $\|\cdot\|_{\varepsilon,\beta,\mu}$ and $\|\!|\cdot|\!\|_{\varepsilon,\boldsymbol{\beta},\boldsymbol{\mu}}$ we have

$$
\begin{aligned}
|g_{s+1}(\tau)|_{\varepsilon,s+1,\beta_0^\lambda(\tau)} &\leq e^{-(s+1)\mu_0^\lambda(\tau)}\|G_N(\tau)\|_{\varepsilon,\beta_0^\lambda(\tau),\mu_0^\lambda(\tau)} \\
&\leq e^{-(s+1)\mu_0^\lambda(\tau)}\|\!|G_N|\!\|_{\varepsilon,\boldsymbol{\beta},\boldsymbol{\mu}}\,.
\end{aligned}
\tag{5.4.6}
$$

The above bounds yield, since $\beta_0^\lambda$ and $\mu_0^\lambda$ are nonincreasing,

$$
\begin{aligned}
e^{s\mu_0^\lambda(t)}\Big|\int_0^t \mathbf{T}_s(-\tau)\mathcal{C}_{s,s+1}\mathbf{T}_{s+1}(\tau)g_{s+1}(\tau)\,d\tau\Big|_{\varepsilon,s,\beta_0^\lambda(t)} \\
\leq C_d\|\!|G_N|\!\|_{\varepsilon,\boldsymbol{\beta},\boldsymbol{\mu}}e^{-\mu_0^\lambda(T)}\big(\beta_0^\lambda(T)\big)^{-\frac{d}{2}}\sup_{Z_s\in\mathbf{R}^{2ds}}\int_0^t\overline{C}(\tau,t,Z_s)\,d\tau\,,
\end{aligned}
$$

where, for $\tau\leq t$,

$$
\overline{C}(\tau,t,Z_s) := \Big(s(\beta_0^\lambda(\tau))^{-\frac12}+\sum_{1\leq i\leq s}|v_i|\Big)e^{\lambda(\tau-t)(s+E_0(Z_s))}\,.
\tag{5.4.7}
$$

Since

$$
\sup_{Z_s\in\mathbf{R}^{2ds}}\int_0^t\overline{C}(\tau,t,Z_s)\,d\tau \leq \frac{C_d}{\lambda}\Big(1+\big(\beta_0^\lambda(T)\big)^{-\frac12}\Big)\,,
\tag{5.4.8}
$$

there holds finally

$$
e^{s\mu_0^\lambda(t)}\Big|\int_0^t \mathbf{T}_s(-\tau)\mathcal{C}_{s,s+1}\mathbf{T}_{s+1}(\tau)g_{s+1}(\tau)\,d\tau\Big|_{\varepsilon,s,\beta_0^\lambda(t)} \leq \bar{c}(\beta_0,\mu_0,\lambda,T)\|\!|G_N|\!\|_{\varepsilon,\boldsymbol{\beta},\boldsymbol{\mu}}\,,
$$

where, with a possible change of the constant $C_d$,

$$
\bar{c}(\beta_0,\mu_0,\lambda,T) := C_d\,e^{-\mu_0^\lambda(T)}\lambda^{-1}\big(\beta_0^\lambda(T)\big)^{-\frac{d}{2}}\Big(1+\big(\beta_0^\lambda(T)\big)^{-\frac12}\Big)\,.
\tag{5.4.9}
$$

The result (5.4.3) follows. To deduce (5.4.4) we need to find $T>0$ and $\lambda>0$ such that $\lambda T<\beta_0$ and

$$
C_d(1+(\beta_0-\lambda T)^{-\frac12})e^{-\mu_0+\lambda T}(\beta_0-\lambda T)^{-\frac{d}{2}} = \frac{\lambda}{2}\,.
\tag{5.4.10}
$$

With $\beta:=\lambda T\in(0,\beta_0)$, condition (5.4.10) becomes

$$
\begin{aligned}
T &= C_de^{\mu_0}\beta e^{-\beta}\frac{(\beta_0-\beta)^{\frac{d+1}{2}}}{1+(\beta_0-\beta)^{\frac12}} \\
&\geq C_de^{\mu_0}(1+\beta_0^{\frac12})^{-1}\beta e^{-\beta}(\beta_0-\beta)^{\frac{d+1}{2}}\,,
\end{aligned}
$$

up to changing the constant $C_d$ and (5.4.4) follows. Notice that (5.2.6) is a consequence of this computation. □

The proof of the corresponding result (5.3.4) for the Boltzmann hierarchy is identical, since the estimates for $\mathcal{C}^0_{s,s+1}$ and $\mathcal{C}_{s,s+1}$ are essentially identical (compare estimate (5.4.1) from Proposition 5.4.1 with estimate (5.4.2) from Proposition 5.4.2).

***Remark 5.4.4***. — *The above arguments provide the global in time wellposedness of the BBGKY hierarchy for each fixed $N$ — though with no uniform bound on $N$. Indeed the exponential weight $\exp\big(-\mu_0N-\beta_0E_0(Z_N)\big)\mathbb{1}_{\mathcal{D}_N}$ is an invariant measure for the flow of the transport equation*

$$
\partial_tf_N+V_N\cdot\nabla_{X_N}f_N=0\,.
$$

*The maximum principle then implies that for all $t\geq0$*

$$
0\leq f_N(t,Z_N)\leq\exp\big(-\mu_0N-\beta_0E_0(Z_N)\big)\mathbb{1}_{\mathcal{D}_N}\,.
$$

*By integration we find*

$$0 \leq f_N^{(s)}(t, Z_s) \leq \exp\big(-\mu_0 N - \beta_0 E_0(Z_s)\big) \mathbb{1}_{\mathcal{D}_s}$$

*As the measure* $\exp\big(-\mu_0 N - \beta_0 E_0(Z_s)\big) \mathbb{1}_{\mathcal{D}_s}$ *is invariant by the flow* $\mathbf{T}_s$*, we get by filtering that with the notation introduced in Pargagraph* 5.1*,* $G_N = (g_{N,s})_{1 \leq s \leq N}$ *satisfies*

$$0 \leq g_{N,s}(t, Z_s) \leq \exp\big(-\mu_0 N - \beta_0 E_0(Z_s)\big) \mathbb{1}_{\mathcal{D}_s}$$

*hence a bound for which no parameters depend on* $t$ *(though the bound is very poor in* $N$*).*

*Then we can iterate the fixed point method used in the proof of Theorem* 6 *to prove that the marginals belong for all time to the space* $\mathbf{X}_{\varepsilon,\boldsymbol{\beta},\boldsymbol{\mu}}$ *and not only on a short time interval . However the size of the functions grows with* $N$ *so that fact cannot be used to obtain a convergence result.*

# CHAPTER 6

# STATEMENT OF THE CONVERGENCE RESULT

We state here our first main result, describing convergence of mild solutions to the BBGKY hierarchy (4.3.6) to mild solutions of the Boltzmann hierarchy (4.4.4). This result implies in particular Theorem 5 stated in the Introduction page 17.

The first part of this chapter is devoted to a precise description of Boltzmann initial data which are *admissible*, i.e., which can be obtained as the limit of BBGKY initial data satisfying the required uniform bounds. This involves discussing the notion of "quasi-independence" mentioned in the Introduction, via a conditioning of the initial data. Then we state the main convergence result (Theorem 8 page 51) and sketch the main steps of its proof.

## 6.1. Quasi-independence

In this paragraph we discuss the notion of "quasi-independent" initial data. We first define admissible Boltzmann initial data, meaning data which can be reached from BBGKY initial data (which are bounded families of marginals) by a limiting procedure, and then show how to "condition" the initial BBGKY initial data so as to converge towards admissible Boltzmann initial data. Finally we characterize admissible Boltzmann initial data.

### 6.1.1. Admissible Boltzmann data. — In the following we denote

$$\Omega_s := \{ Z_s \in \mathbf{R}^{2ds} \, , \; \forall i \neq j \, , \, x_i \neq x_j \} \, .$$

***Definition 6.1.1* (Admissible Boltzmann data).** — *Admissible Boltzmann data are defined as families $F_0 = (f_0^{(s)})_{s\geq 1}$, with each $f_0^{(s)}$ nonnegative, integrable and continuous over $\Omega_s$, such that*

$$\int_{\mathbf{R}^{2d}} f_0^{(s+1)}(Z_s, z_{s+1}) \, dz_{s+1} = f_0^{(s)}(Z_s) \, , \tag{6.1.1}$$

*and which are limits of BBGKY initial data $F_{0,N} = (f_{0,N}^{(s)})_{1\leq s\leq N} \in \mathbf{X}_{\varepsilon,\beta_0,\mu_0}$ in the following sense: for some $F_{0,N}$ satisfying*

$$\sup_{N\geq 1} \|F_{0,N}\|_{\varepsilon,\beta_0,\mu_0} < \infty \, , \quad \textit{for some } \beta_0 > 0 \, , \, \mu_0 \in \mathbf{R} \, , \, \textit{as } N\varepsilon^{d-1} \equiv 1 \, , \tag{6.1.2}$$

*for each given $s \in [1, N]$, the marginal of order $s$ defined by*

$$f_{0,N}^{(s)}(Z_s) = \int_{\mathbf{R}^{2d(N-s)}} 1\!\!1_{Z_N \in \mathcal{D}_N} f_{0,N}^{(N)}(Z_N)\, dz_{s+1} \dots dz_N\,, \quad 1 \leq s < N\,, \tag{6.1.3}$$

*converges in the Boltzmann-Grad limit:*

$$f_{0,N}^{(s)} \longrightarrow f_0^{(s)} \quad \textit{as } N \to \infty \textit{ with } N\varepsilon^{d-1} \equiv 1\,, \textit{ locally uniformly in } \Omega_s\,. \tag{6.1.4}$$

In this section we shall prove the following result.

***Proposition 6.1.1***. — *The set of admissible Boltzmann data, in the sense of Definition* 6.1.1*, is the set of families of marginals $F_0$ as in* (6.1.1) *satisfying a uniform bound $\|F_0\|_{0,\beta_0,\mu_0} < \infty$ for some $\beta_0 > 0$ and $\mu_0 \in \mathbf{R}$.*

**6.1.2. Conditioning.** — We first consider "chaotic" configurations, corresponding to tensorized initial measures, or initial densities which are products of one-particle distributions:

$$f_0^{\otimes s}(Z_s) = \prod_{1 \leq i \leq s} f_0(z_i)\,, \qquad 1 \leq s \leq N\,, \tag{6.1.5}$$

where $f_0$ is nonnegative, normalized, and belongs to some $X_{0,1,\beta_0}$ space (see Definition 5.2.1 page 38):

$$f_0 \geq 0\,, \quad \int_{\mathbf{R}^{2d}} f_0(z)dz = 1\,, \quad e^{\mu_0}|f_0|_{0,1,\beta_0} \leq 1 \quad \text{for some } \beta_0 > 0\,, \mu_0 \in \mathbf{R}\,. \tag{6.1.6}$$

Such initial data are particularly meaningful insofar as they will produce the Boltzmann equation (2.1.1), and we shall show in Proposition 6.1.2 that they are admissible.

We then consider the initial data with exclusion $1\!\!1_{Z_N \in \mathcal{D}_N} f_0^{\otimes N}(Z_N)$, and the property of normalization is preserved by introduction of the partition function

$$\mathcal{Z}_N := \int_{\mathbf{R}^{2dN}} 1\!\!1_{Z_N \in \mathcal{D}_N} f_0^{\otimes N}(Z_N)\, dZ_N\,. \tag{6.1.7}$$

*Conditioned datum built on $f_0$* is then defined as $\mathcal{Z}_N^{-1} 1\!\!1_{Z_N \in \mathcal{D}_N} f_0^{\otimes N}(Z_N)$. This operation is called *conditioning on energy surfaces*, and is a classical tool in statistical mechanics (see [**20, 32, 33**] for instance).

The partition function defined in (6.1.7) satisfies the next result, which will be useful in the following.

***Lemma 6.1.2***. — *Given $f_0$ satisfying* (6.1.6)*, there holds for $1 \leq s \leq N$ the bound*

$$1 \leq \mathcal{Z}_N^{-1} \mathcal{Z}_{N-s} \leq \big(1 - \varepsilon \kappa_d |f_0|_{L^\infty L^1}\big)^{-s},$$

*in the scaling $N\varepsilon^{d-1} \equiv 1$, where $|f_0|_{L^\infty L^1}$ denotes the $L^\infty(\mathbf{R}^d_x, L^1(\mathbf{R}^d_v))$ norm of $f_0$, and $\kappa_d$ denotes the volume of the unit ball in $\mathbf{R}^d$.*

*Proof.* — We have by definition

$$\mathcal{Z}_{s+1} = \int_{\mathbf{R}^{2d(s+1)}} 1\!\!1_{Z_{s+1} \in \mathcal{D}_{s+1}} \Big( \prod_{i=1}^{s} 1\!\!1_{|x_i - x_{s+1}| > \varepsilon} \Big) f_0^{\otimes (s+1)}(Z_{s+1})\, dZ_{s+1}\,.$$

By Fubini, we deduce

$$\mathcal{Z}_{s+1} = \int_{\mathbf{R}^{2ds}} \left( \int_{\mathbf{R}^{2d}} \Big( \prod_{1 \leq i \leq s} 1\!\!1_{|x_i - x_{s+1}| > \varepsilon} \Big) f_0(z_{s+1}) dz_{s+1} \right) 1\!\!1_{Z_s \in \mathcal{D}_s} f_0^{\otimes s}(Z_s) dZ_s\,.$$

Since

$$\int_{\mathbf{R}^{2d}} \Big( \prod_{1\leq i\leq s} \mathbb{1}_{|x_i - x_{s+1}|>\varepsilon} \Big) f_0(z_{s+1}) dz_{s+1} \geq \|f_0\|_{L^1} - \kappa_d s \varepsilon^d |f_0|_{L^\infty L^1} \,,$$

we deduce from the above, by nonnegativity of $f_0^{\otimes s}$ and the fact that $\|f_0\|_{L^1} = 1$ the lower bound

$$\mathcal{Z}_{s+1} \geq \mathcal{Z}_s \big( 1 - \kappa_d s \varepsilon^d |f_0|_{L^\infty L^1} \big) \,,$$

implying by induction

$$\mathcal{Z}_N \geq \mathcal{Z}_{N-s} \prod_{j=N-s}^{N-1} (1 - j\varepsilon^d \kappa_d |f_0|_{L^\infty L^1}) \geq \mathcal{Z}_{N-s} \big( 1 - \varepsilon \kappa_d |f_0|_{L^\infty L^1} \big)^s \,,$$

where we used $s \leq N$ and the scaling $N\varepsilon^{d-1} \equiv 1$. That proves the lemma. □

**6.1.3. Characterization of admissible Boltzmann initial data.** — The aim of this paragaph is to prove Proposition 6.1.1.

Let us start by proving the following statement, which provides examples of admissible Boltzmann initial data, in terms of tensor products.

***Proposition 6.1.2***. — *Given* $f_0$ *satisfying* (6.1.6)*, the data* $F_0 = (f_0^{\otimes s})_{s\geq 1}$ *is admissible in the sense of Definition* 6.1.1*.*

*Proof.* — Let us define, with notation (6.1.7),

$$f_{0,N}^{(N)} := \mathcal{Z}_N^{-1} \mathbb{1}_{Z_N \in \mathcal{D}_N} f_0^{\otimes N}(Z_N)$$

and let $F_{0,N} := \big( f_{0,N}^{(s)} \big)_{s\leq N}$ be the set of its marginals. In a first step we prove they satisfy uniform bounds as in (6.1.2). In a second step, we prove the local uniform convergence to zero of $f_{0,N}^{(s)} - f_0^{\otimes s}$ in $\Omega_s$, as in (6.1.3).

*First step.* We have clearly

$$f_{0,N}^{(s)}(Z_s) \leq \mathcal{Z}_N^{-1} \mathbb{1}_{Z_s \in \mathcal{D}_s} f_0^{\otimes s}(Z_s) \int_{\mathbf{R}^{2d(N-s)}} \prod_{s+1\leq i<j\leq N} \mathbb{1}_{|x_i - x_j|>\varepsilon} \prod_{s+1\leq i\leq N} f_0(z_i)\, dZ_{(s+1,N)} \,,$$

where we have used the notation

$$dZ_{(s+1,N)} := dz_{s+1} \dots dz_N \,.$$

This gives

$$\begin{aligned} f_{0,N}^{(s)}(Z_s) &\leq \mathcal{Z}_N^{-1} \mathcal{Z}_{N-s} \mathbb{1}_{Z_s \in \mathcal{D}_s} f_0^{\otimes s}(Z_s) \\ &\leq \big( 1 - \varepsilon \kappa_d |f_0|_{L^\infty L^1} \big)^{-s} \mathbb{1}_{Z_s \in \mathcal{D}_s} f_0^{\otimes s}(Z_s) \,, \end{aligned}$$

the second inequality by Lemma 6.1.2.

By $2x + \ln(1-x) \geq 0$ for $x \in [0, 1/2]$, there holds

$$(1 - \varepsilon \kappa_d |f_0|_{L^\infty L^1})^{-s} \leq e^{2s\varepsilon\kappa_d |f_0|_{L^\infty L^1}} \,, \qquad \text{if } 2\varepsilon\kappa_d |f_0|_{L^\infty L^1} < 1 \,, \tag{6.1.8}$$

so that for $N$ larger than some $N_0$ (equivalently, for $\varepsilon$ small enough),

$$e^{s\mu_0'} \big| f_{0,N}^{(s)} \big|_{\varepsilon,s,\beta_0} \leq e^{s(\mu_0' + 2\varepsilon\kappa_d |f_0|_{L^\infty L^1})} \big| \mathbb{1}_{Z_s \in \mathcal{D}_s} f_0^{\otimes s}(Z_s) \big|_{\varepsilon,s,\beta_0} \leq \Big( e^{2\varepsilon\kappa_d |f_0|_{L^\infty L^1}} |f_0|_{0,1,\beta_0} \Big)^s \,.$$

Therefore, for any $\mu_0' < \mu_0$ and for $\varepsilon$ sufficiently small,

$$\sup_{N\geq N_1} \|F_{0,N}\|_{\varepsilon,\beta_0,\mu_0'} < \infty \,,$$

which of course implies the uniform bound $\sup_{N\geq 1} \|F_{0,N}\|_{\varepsilon,\beta_0,\mu_0'} < \infty$.

*Second step.* We compute for $s \leq N$ :

$$f_{0,N}^{(s)} = \mathcal{Z}_N^{-1} 1\!\!1_{Z_s\in\mathcal{D}_s} f_0^{\otimes s} \int_{\mathbf{R}^{2d(N-s)}} \prod_{s+1\leq i<j\leq N} 1\!\!1_{|x_i-x_j|>\varepsilon} \prod_{i\leq s<j} 1\!\!1_{|x_i-x_j|>\varepsilon} \prod_{s+1\leq i\leq N} f_0(z_i)\, dZ_{(s+1,N)} \,.$$

We deduce, by symmetry,

$$f_{0,N}^{(s)} = \mathcal{Z}_N^{-1} 1\!\!1_{Z_s\in\mathcal{D}_s} f_0^{\otimes s}\Big(\mathcal{Z}_{N-s} - \mathcal{Z}^\flat_{(s+1,N)}\Big) \tag{6.1.9}$$

with the notation

$$\mathcal{Z}^\flat_{(s+1,N)} = \int_{\mathbf{R}^{2d(N-s)}} \Big(1 - \prod_{i\leq s<j} 1\!\!1_{|x_i-x_j|>\varepsilon}\Big) \prod_{s+1\leq i<j\leq N} 1\!\!1_{|x_i-x_j|>\varepsilon} \prod_{s+1\leq i\leq N} f_0(z_i)\, dZ_{(s+1,N)} \,,$$

so that $\mathcal{Z}^\flat_{(s+1,N)}$ is a function of $X_s$.

From there, the difference $1\!\!1_{Z_s\in\mathcal{D}_s} f_0^{\otimes s} - f_{0,N}^{(s)}$ decomposes as a sum:

$$1\!\!1_{Z_s\in\mathcal{D}_s} f_0^{\otimes s} - f_{0,N}^{(s)} = \Big(1 - \mathcal{Z}_N^{-1}\mathcal{Z}_{N-s}\Big) 1\!\!1_{Z_s\in\mathcal{D}_s} f_0^{\otimes s} + \mathcal{Z}_N^{-1}\mathcal{Z}^\flat_{(s+1,N)} 1\!\!1_{Z_s\in\mathcal{D}_s} f_0^{\otimes s} \,. \tag{6.1.10}$$

By Lemma 6.1.2, there holds $1-\mathcal{Z}_N^{-1}\mathcal{Z}_{N-s} \to 0$ as $N\to\infty$, for fixed $s$. Since $f_0^{\otimes s}$ is uniformly bounded in $\Omega_s$, this implies that the first term in the right-hand side of (6.1.10) tends to 0 as $N\to\infty$, uniformly in $\Omega_s$. Besides, by

$$0 \leq 1 - \prod_{i\leq s<j} 1\!\!1_{|x_i-x_j|>\varepsilon} \leq \sum_{\substack{1\leq i\leq s\\ s+1\leq j\leq N}} 1\!\!1_{|x_i-x_j|<\varepsilon} \,,$$

we bound

$$\mathcal{Z}^\flat_{(s+1,N)} \leq \sum_{1\leq k\leq s} \int_{\mathbf{R}^{2d(N-s)}} \Big(\sum_{s+1\leq j\leq N} 1\!\!1_{|x_k-x_j|<\varepsilon}\Big) \prod_{s+1\leq i<j\leq N} 1\!\!1_{|x_i-x_j|>\varepsilon} \prod_{s+1\leq i\leq N} f_0(z_i)\, dZ_{(s+1,N)} \,.$$

Given $1\leq k\leq s$, there holds by symmetry and Fubini,

$$\begin{aligned}
\int_{\mathbf{R}^{2d(N-s)}} \Big(\sum_{s+1\leq j\leq N} 1\!\!1_{|x_k-x_j|<\varepsilon}\Big) &\prod_{s+1\leq i<j\leq N} 1\!\!1_{|x_i-x_j|>\varepsilon} \prod_{s+1\leq i\leq N} f_0(z_i)\, dZ_{(s+1,N)} \\
&\leq (N-s) \int_{\mathbf{R}^{2d}} 1\!\!1_{|x_i-x_{s+1}|<\varepsilon} f_0(z_{s+1}) dz_{s+1} \\
&\qquad \times \int_{\mathbf{R}^{2d(N-s-1)}} \prod_{s+2\leq i<j\leq N} 1\!\!1_{|x_k-x_j|>\varepsilon} \prod_{s+2\leq i\leq N} f_0(z_i)\, dZ_{(s+2,N)} \\
&= (N-s) \int_{\mathbf{R}^{2d}} 1\!\!1_{|x_i-x_{s+1}|<\varepsilon} f_0(z_{s+1}) dz_{s+1} \,\times\, \mathcal{Z}_{N-s-1} \,,
\end{aligned}$$

so that

$$\mathcal{Z}^\flat_{(s+1,N)} \leq s(N-s)\varepsilon^d \kappa_d |f_0|_{L^\infty L^1} \mathcal{Z}_{N-s-1} \,, \tag{6.1.11}$$

where $|f_0|_{L^\infty L^1}$ denotes the $L^\infty(\mathbf{R}^d_x, L^1(\mathbf{R}^d_v))$ norm of $f_0$. By Lemma 6.1.2, we obtain

$$\mathcal{Z}_N^{-1}\mathcal{Z}^\flat_{(s+1,N)} \leq \varepsilon s \kappa_d |f_0|_{L^\infty L^1} \big(1 - \varepsilon\kappa_d |f_0|_{L^\infty L^1}\big)^{-(s+1)} \,,$$

and the upper bound tends to 0 as $N\to\infty$, for fixed $s$. This implies convergence to 0, uniformly in $\Omega_s$, of the second term in the right-hand side of (6.1.10).

We thus proved the uniform convergence $f_{0,N}^{(s)} - 1\!\!1_{Z_s \in \mathcal{D}_s} f_0^{\otimes s} \to 0$ in $\Omega_s$, and hence $f_{0,N}^{\otimes s} \to f_0^{\otimes s}$ holds locally uniformly in $\Omega_s$. We conclude that $f_{0,N}^{(s)}$ converges locally uniformly to tensor products $f_0^{\otimes s}$ in $\Omega_s$.

Proposition 6.1.2 is proved. □

By Proposition 6.1.2, tensor products $(f_0^{\otimes s})_{s\geq 1}$, with $f_0$ satisfying (6.1.6), are admissible Boltzmann data. It is easy to generalize that result (see Proposition 6.1.4 below) to the convex hull of the set of tensor products. We shall actually also show the converse: all admissible Boltzmann data belong to the convex hull of tensor products, and that will enable us to deduce Proposition 6.1.1.

We first remark that given a Boltzmann datum $F_0$, and an associated BBGKY datum $F_{0,N}$, there holds

$$\|F_0\|_{0,\beta_0,\mu_0} < \infty \,, \tag{6.1.12}$$

with $\beta_0$ and $\mu_0$ as in (6.1.2). Indeed, let $C_0 = \sup\limits_{N\geq 1} \|F_{0,N}\|_{\varepsilon,\beta_0,\mu_0} < \infty$. Given $s$ and $Z_s \in \Omega_s$, for $\varepsilon$ small enough, $1\!\!1_{Z_s\in\mathcal{D}_s} = 1$. Besides, by (6.1.4) there holds the pointwise convergence $f_{0,N}^{(s)}(Z_s) \to f_0^{(s)}(Z_s)$. Hence taking the limit $\varepsilon \to 0$ in the left-hand side of the inequality $e^{s\mu_0+\beta_0 E_\varepsilon(Z_s)}|f_{0,N}^{(s)}(Z_s)| \leq C_0$, we find (6.1.12).

The Hewitt-Savage theorem reveals the specific role played by tensor products: the set of families $F_0 = (f_0^{(s)})_{s\geq 1}$ of marginals (6.1.1) satisfying the uniform bound (6.1.12) is the convex hull of tensorized initial data, as described in the following statement. We define $\mathcal{P} = \mathcal{P}(\mathbf{R}^{2d})$ be the set of continuous densities of probability in $\mathbf{R}^{2d}$ :

$$\mathcal{P} := \big\{ h \in C^0(\mathbf{R}^{2d};\mathbf{R})\,, \quad h \geq 0\,, \quad \int_{\mathbf{R}^{2d}} h(z)dz = 1 \big\}\,. \tag{6.1.13}$$

***Proposition 6.1.3***. — *Given* $F_0 = (f_0^{(s)})_{s\geq 1}$ *a family of marginals* (6.1.1) *satisfying the uniform bound* (6.1.12) *with constants* $\beta_0 > 0$ *and* $\mu_0 \in \mathbf{R}$, *there exists a probability measure* $\pi$ *over the set* $\mathcal{P}$, *with*

$$\operatorname{supp} \pi \subset \big\{ g \in \mathcal{P},\, |g|_{0,1,\beta_0} \leq e^{-\mu_0} \big\}\,, \tag{6.1.14}$$

*such that the following representation holds:*

$$f_0^{(s)} = \int_{\mathcal{P}} g^{\otimes s} d\pi(g)\,, \qquad s \geq 1\,. \tag{6.1.15}$$

*Proof*. — Given a family $F_0$ satisfying (6.1.1) and (6.1.12), the existence of $\pi$ satisfying (6.1.15) is granted by the Hewitt-Savage theorem [**26**]. The goal is then to prove the inclusion (6.1.14). Assume by contradiction that, for some $\alpha > 0$,

$$\pi(A_\alpha) = \kappa_\alpha > 0\,, \quad \text{where } A_\alpha := \big\{ g \in \mathcal{P}(\mathbf{R}^{2d}), \quad |g|_{0,1,\beta_0} \geq e^{-\mu_0} + \alpha \big\}\,. \tag{6.1.16}$$

We then have by (6.1.15)

$$f_0^{(s)} \geq \int_{A_\alpha} g^{\otimes s} d\pi(g),$$

hence by $f_0^{(s)} \leq e^{-s\mu_0}\|F_0\|_{0,\beta_0,\mu_0}$, we infer that $\|F_0\|_{0,\beta_0,\mu_0} \geq \kappa_\alpha(1+\alpha e^{\mu_0})^s$, which cannot hold for some $\alpha > 0$ and all $s$, since $1+\alpha e^{\mu_0} > 1$. Hence (6.1.16) does not hold, which proves the result. □

We now give the generalization of Proposition 6.1.2 that will be useful in the proof of Proposition 6.1.1. Let $\pi$ be a probability measure on $\mathcal{P}$ satisfying (6.1.14) for some $\beta_0 > 0$ and some $\mu_0 \in \mathbf{R}$. Next we define

$$\pi^{(s)} := \int_{\mathcal{P}} h^{\otimes s} d\pi(h)\,. \tag{6.1.17}$$

In the case when $\pi = \delta_{f_0}$, then (6.1.17) reduces to the tensor product (6.1.5)-(6.1.6).

In the general case, we define

$$\begin{aligned} \mathcal{Z}_N(h) &:= \int_{\mathbf{R}^{2dN}} \mathbb{1}_{Z_N \in \mathcal{D}_N} h^{\otimes N}(Z_N)\, dZ_N\,, \quad h \in \mathcal{P}\,, \\ \pi_N^{(N)} &:= \int_{\mathcal{P}} \frac{1}{\mathcal{Z}_N(h)} h^{\otimes N} d\pi(h)\,. \end{aligned} \tag{6.1.18}$$

generalizing (6.1.7).

The following result is an obvious generalization of Lemma 6.1.2.

***Lemma 6.1.3***. — *Given $\pi$ satisfying* (6.1.14) *and $h \in \operatorname{supp} \pi$, the family of partition functions $\mathcal{Z}_s$ defined in* (6.1.18) *satisfies for $1 \leq s \leq N$ the bound*

$$1 \leq \mathcal{Z}_N(h)^{-1} \mathcal{Z}_{N-s}(h) \leq \left(1 - \varepsilon C_d e^{-\mu_0} \beta_0^{-1/2}\right)^{-s},$$

*where $C_d$ depends only on $d$.*

The next statement generalizes Proposition 6.1.2. Its proof is an immediate extension of the proof of Proposition 6.1.2 thanks to the dominated convergence theorem, using the obvious bound $\mathbb{1}_{Z_s \in \mathcal{D}_s} h^{\otimes s}(Z_s) \leq e^{-s\mu_0}$.

***Proposition 6.1.4***. — *Given $\pi$ satisfying* (6.1.14), *the data $(\pi^{(s)})_{s \geq 1}$, with $\pi^{(s)}$ defined in* (6.1.17), *is admissible in the sense of Definition* 6.1.1.

*It is obtained for instance from the BBGKY data $(\pi_N^{(s)})_{s \leq N}$ defined by (6.1.18).*

*Proof of Proposition* 6.1.1. — We already observed in (6.1.12) that admissible Boltzmann data are bounded families of marginals. Conversely, given a bounded family of marginals $F_0$, by Proposition 6.1.3 representation (6.1.15) holds. Then, by Proposition 6.1.4, $F_0$ is an admissible Boltzmann datum. This proves Proposition 6.1.1. □

Combining Propositions 6.1.1 and 6.1.3, we see that all admissible Boltzmann data are built on tensor products, in the sense that given an admissible Boltzmann datum, representation (6.1.15) holds for some $\pi$ satisfying (6.1.14).

## 6.2. Main result: Convergence of the BBGKY hierarchy to the Boltzmann hierarchy

### 6.2.1. Statement of the result. —

Our main result is a *weak convergence* result, in the sense of convergence of observables, or averages with respect to the momentum variables. Moreover, since the marginals are defined in $\mathcal{D}_s$, we must also eliminate, in the convergence, the diagonals in physical space. Let us give a precise definition of the convergence we shall be considering.

***Definition 6.2.1* (Convergence).** — *Given a sequence $(h_N^s)_{1\leq s\leq N}$ of functions $h_N^s \in C^0(\mathcal{D}_s;\mathbf{R})$, a sequence $(h^s)_{s\geq 1}$ of functions $h^s \in L^\infty(\Omega_s;\mathbf{R})$, we say that $(h_N^s)$ converges on average and locally uniformly outside the diagonals to $(h^s)$, and we denote*

$$(h_N^s)_{1\leq s\leq N} \stackrel{\sim}{\longrightarrow} (h^s)_{1\leq s}\,,$$

*when for any fixed $s$, any test function $\varphi_s \in \mathcal{C}_c^0(\mathbf{R}^{ds};\mathbf{R})$, there holds*

$$I_{\varphi_s}\big(h_N^s - h^s\big)(X_s) := \int_{\mathbf{R}^{ds}} \varphi_s(V_s)\big(h_N^s - h^s\big)(Z_s)dV_s \longrightarrow 0\,, \quad \textit{as } N\to\infty\,,$$

*in $L^\infty_{loc}\big(\big\{X_s \in \mathbf{R}^{ds},\ x_i \neq x_j \text{ for } i\neq j\big\}\,.\big)$*

With regard to spatial variables, this notion of convergence is similar to the convergence in the sense of Chacon.

We remark that local uniform convergence in $\Omega_s$ implies convergence in the sense of Definition 6.2.1:

***Lemma 6.2.2.*** — *Given $(f_N^{(s)})_{1\leq s\leq N}$ a bounded sequence in $\mathbf{X}_{\varepsilon,\boldsymbol{\beta},\boldsymbol{\mu}}$ with the notation of Definition* 5.2.2, *if $f_N^{(s)} \to f^{(s)}$ for fixed $s$, uniformly in $t\in[0,T]$ and locally uniformly in $\Omega_s$, then there holds $f_N^{(s)} \stackrel{\sim}{\longrightarrow} f^{(s)}$, uniformly in $t\in[0,T]$.*

*Proof.* — Let $K_s$ be compact in $\big\{X_s \in \mathbf{R}^{ds},\ x_i \neq x_j \text{ for } i\neq j\big\}$. There holds

$$\big|I_{\varphi_s}\big(f_N^{(s)} - f^{(s)}\big)(X_s)\big| \leq \|\varphi_s\|_{L^1(\mathbf{R}^d)}\mathrm{supess}_{V_s\in\operatorname{supp}\varphi_s}\big|\big(f_N^{(s)} - f^{(s)}\big)(X_s,V_s)\big|\,.$$

The set $K_s\times\operatorname{supp}\varphi_s$ is compact in $\Omega_s$. Hence the above upper bound converges to 0 as $N\to\infty$, in the space $C([0,T],L^\infty(K_s))$. □

We can now state our main result.

***Theorem 8* (Convergence).** — *Let $\beta_0>0$ and $\mu_0\in\mathbf{R}$ be given. There is a time $T>0$ such that the following holds. Let $F_0$ in $\mathbf{X}_{0,\beta_0,\mu_0}$ be an admissible Boltzmann datum, with associated family of BBGKY datum $(F_{0,N})_{N\geq 1}$, in $\mathbf{X}_{\varepsilon,\beta_0,\mu_0}$. Let $F$ and $F_N$ be the solutions to the Boltzmann and BBGKY hierarchy produced by $F_0$ and $F_{0,N}$ respectively. There holds*

$$F_N \stackrel{\sim}{\longrightarrow} F\,, \tag{6.2.1}$$

*uniformly on $[0,T]$,.*

*In particular, if $F_0 = (f_0^{\otimes s})_{s\geq 1}$, then the first marginal $f_N^{(1)}$ converges to the solution $f$ of the Boltzmann equation* (2.1.1) *with initial data $f_0$.*

*Finally in the case when $F_0 = (f_0^{\otimes s})_{s\geq 1}$ with $f_0$ Lipschitz, then the convergence* (6.2.1) *holds at a rate $O(\varepsilon^\alpha)$ for any $\alpha < (d-1)/(d+1)$.*

Solutions to the Boltzmann hierarchy issued from tensorized initial data are themselves tensorized. For such data, the Boltzmann hierarchy then reduces to the nonlinear Boltzmann equation (2.1.1), and Theorem 8 describes an asymptotic form of propagation of chaos, in the sense that an initial property of independence is propagated in time, in the limit. This corresponds to Theorem 5 stated in the Introduction page 17.

**6.2.2. About the proof of Theorem 8: outline of Chapter 7 and Part IV.** —

The formal derivation presented in Chapter 4, Section 4.4, fails because of a number of incorrect arguments:

- Since mild solutions to the BBGKY hierarchy are defined by the Duhamel formula (4.3.6) where the solution itself occurs in the source term, we need some precise information on the convergence to take limits directly in (4.3.6).
- The irreversibility inherent to the Boltzmann hierarchy appears in the limiting process as an arbitrary choice of the time direction (encoded in the distinction between pre-collisional and post-collisional particles), and more precisely as an arbitrary choice of the initial time, which is the only time for which one has a complete information on the family of marginals $F_{0,N}$. This specificity of the initial time does not appear clearly in (4.3.6).
- The heuristic argument which allows to neglect pathological trajectories, meaning trajectories for which the reduced dynamics with $s$-particles does not coincide with the free transport ($\mathbf{T}_s \neq \mathbf{S}_s$), requires to be quantified. Indeed we have more or less to repeat the operation infinitely many times, since mild solutions are defined by a loop process; moreover, the question of the stability with respect to micro-translations in space must be analyzed.
- Because of the conditioning, the initial data are not so smooth. The operations such as infinitesimal translations on the arguments require therefore a careful treatment.

To overcome the two first difficulties, the idea is to start from the iterated Duhamel formula, which allows to express any marginal $f_N^{(s)}(t, Z_s)$ in terms of the initial data $F_{0,N}$. By successive integrations in time, we have indeed the following representation of $f_N^{(s)}$:

$$f_N^{(s)}(t) = \sum_{k=0}^{\infty} \int_0^t \int_0^{t_1} \dots \int_0^{t_{k-1}} \mathbf{T}_s(t-t_1)\mathcal{C}_{s,s+1}\mathbf{T}_{s+1}(t_1-t_2)\mathcal{C}_{s+1,s+2}\dots \\ \dots \mathbf{T}_{s+k}(t_k) f_N^{(s+k)}(0)\, dt_k \dots dt_1 \tag{6.2.1}$$

where by convention $f_N^{(j)}(0) \equiv 0$ for $j > N$.

Using a **dominated convergence argument**, we shall first reduce (in Chapter 7) to the study of a functional

- defined as a finite sum of terms (independent of $N$),
- where the energies of the particles are assumed to be bounded (namely $E_0(Z_{s+k}) \leq R^2$),
- and where the collision times are supposed to be well separated (namely $|t_j - t_{j+1}| \geq \delta$).

The reason for the two last assumptions is essentially technical, and will appear more clearly in the next step.

The heart of the proof, in Part IV, is then to prove the term by term convergence, dealing with pathological trajectories. Let us recall that each collision term is defined as an integral with respect to positions and velocities. The main idea consists then in proving that we cannot build pathological trajectories if we exclude at each step a small domain of integration. The explicit construction of this "bad set" lies on

– a very simple geometrical lemma which ensures that two particles of size $\varepsilon$ have not collided in the past provided that their relative velocity does not belong to a small subset of $\mathbf{R}^d$ (see Lemma 12.2.1),
– scattering estimates which tell us how these properties of the transport are modified when a particle is deviated by a collision (see Lemma 12.2.3).

This construction, which is the technical part of the proof, will be detailed in Chapter 12. The conclusion of the convergence proof is presented in Chapters 13 and14.

# CHAPTER 7

# STRATEGY OF THE CONVERGENCE PROOF

The goal of this chapter is to use dominated convergence arguments to reduce the proof of Theorem 8 stated page 51 to the term-by-term convergence of some functionals involving a finite (uniformly bounded) number of marginals (Section 7.1). In order to further simplify the convergence analysis, we shall modify these functionals by eliminating some small domains of integration in the time and velocity variables corresponding to pathological dynamics, namely by removing large energies in Section 7.2 and clusters of collision times in Section 7.3.

We consider therefore families of initial data: Boltzmann initial data $F_0 = (f_0^{(s)})_{s\in\mathbf{N}} \in \mathbf{X}_{0,\beta_0,\mu_0}$ and for each $N$, BBGKY initial data $F_{N,0} = (f_{N,0}^{(s)})_{1\leq s\leq N} \in \mathbf{X}_{\varepsilon,\beta_0,\mu_0}$ such that

$$\sup_N \|F_{N,0}\|_{\varepsilon,\beta_0,\mu_0} = \sup_N \sup_{s\leq N} \sup_{Z_s\in\mathcal{D}_s} \big(\exp(\beta_0 E_0(Z_s)+\mu_0 s) f_{N,0}^{(s)}(Z_s)\big) < +\infty\,.$$

We then associate the respective unique mild solutions of the hierarchies

$$f^{(s)}(t) = \mathbf{S}_s(t) f_0^{(s)} + \int_0^t \mathbf{S}_s(t-\tau)\mathcal{C}^0_{s,s+1} f^{(s+1)}(\tau)\,d\tau$$

and

$$f_N^{(s)}(t) = \mathbf{T}_s(t) f_{N,0}^{(s)} + \int_0^t \mathbf{T}_s(t-\tau)\mathcal{C}_{s,s+1} f_N^{(s+1)}(\tau)\,d\tau\,.$$

In terms of the initial datum, they can be rewritten

$$\begin{aligned} f^{(s)}(t,Z_s) = \sum_{k=0}^{\infty} \int_0^t \int_0^{t_1} \dots \int_0^{t_{k-1}} &\mathbf{S}_s(t-t_1)\mathcal{C}^0_{s,s+1}\mathbf{S}_{s+1}(t_1-t_2)\mathcal{C}^0_{s+1,s+2}\dots \\ &\dots \mathbf{S}_{s+k}(t_k) f_0^{(s+k)}\,dt_k\dots dt_1 \end{aligned}$$

and

$$\begin{aligned} f_N^{(s)}(t,Z_s) = \sum_{k=0}^{\infty} \int_0^t \int_0^{t_1} \dots \int_0^{t_{k-1}} &\mathbf{T}_s(t-t_1)\mathcal{C}_{s,s+1}\mathbf{T}_{s+1}(t_1-t_2)\mathcal{C}_{s+1,s+2}\dots \\ &\dots \mathbf{T}_{s+k}(t_k) f_{N,0}^{(s+k)}\,dt_k\dots dt_1\,. \end{aligned}$$

The observables we are interested in (recall the definition of convergence provided in Definition 6.2.1) are the following:

$$I_s(t)(X_s) := \int \varphi_s(V_s) f_N^{(s)}(t,Z_s)dV_s \quad \text{and} \quad I_s^0(t)(X_s) := \int \varphi_s(V_s) f^{(s)}(t,Z_s)dV_s\,,$$

and they therefore involve infinite sums, as there may be infinitely many particles involved (the sum over $n$ is unbounded).

### 7.1. Reduction to a finite number of collision times

Due to the uniform bounds derived in Chapter 5, the dominated convergence theorem implies that it is enough to consider finite sums of elementary functions

$$
\begin{aligned}
f_N^{(s,k)}(t) &:= \int_0^t \int_0^{t_1} \dots \int_0^{t_{k-1}} \mathbf{T}_s(t-t_1)\mathcal{C}_{s,s+1}\mathbf{T}_{s+1}(t_1-t_2)\mathcal{C}_{s+1,s+2}\dots \\
&\qquad\qquad \dots \mathbf{T}_{s+k}(t_k) f_{N,0}^{(s+k)}\, dt_k \dots dt_1 \\
f^{(s,k)}(t) &:= \int_0^t \int_0^{t_1} \dots \int_0^{t_{n-1}} \mathbf{S}_s(t-t_1)\mathcal{C}_{s,s+1}\mathbf{S}_{s+1}(t_1-t_2)\mathcal{C}_{s+1,s+2}\dots \\
&\qquad\qquad \dots \mathbf{S}_{s+k}(t_k) f_0^{(s+k)}\, dt_k \dots dt_1 \, ,
\end{aligned}
\tag{7.1.1}
$$

and the associate elementary observables :

$$
I_{s,k}(t)(X_s) := \int \varphi_s(V_s) f_N^{(s,k)}(t,Z_s)dV_s\, , \quad \text{and} \quad I^0_{s,k}(t)(X_s) := \int \varphi_s(V_s) f^{(s,k)}(t,Z_s)dV_s\, ,
\tag{7.1.2}
$$

and therefore to study the term-by-term convergence (for any fixed $k$), as expressed by the following statement.

***Proposition 7.1.1***. — *Fix $\beta_0 > 0$ and $\mu_0 \in \mathbf{R}$. With the notation of Theorems 6 and 7 page 39, for each given $s \in \mathbf{N}^*$ and $t \in [0,T]$ there is a constant $C>0$ such that for each $n \in \mathbf{N}^*$,*

$$
\Big\| I_s(t) - \sum_{k=0}^n I_{s,k}(t) \Big\|_{L^\infty(\mathbf{R}^{ds})} \leq C \|\varphi_s\|_{L^\infty(\mathbf{R}^{ds})} \left(\frac12\right)^n \|F_{N,0}\|_{\varepsilon,\beta_0,\mu_0}
$$

*and*

$$
\Big\| I^0_s(t) - \sum_{k=0}^n I^0_{s,k}(t) \Big\|_{L^\infty(\mathbf{R}^{ds})} \leq C \|\varphi_s\|_{L^\infty(\mathbf{R}^{ds})} \left(\frac12\right)^n \|F_0\|_{0,\beta_0,\mu_0}\, ,
$$

*uniformly in $N$ and $t \leq T$, in the Boltzmann-Grad scaling $N\varepsilon^{d-1} \equiv 1$.*

*Proof*. — We use the notation of Chapter 5. Using the continuity estimate (5.3.3) we have

$$
\sup_{t\in[0,T]} \Big\| \int_0^t \mathbf{T}(-t')\mathbf{C}_N \mathbf{T}(t') G_N(t')\, dt' \Big\|_{\varepsilon,\beta(t),\mu(t)} \leq \frac12 |\!|\!| G_N |\!|\!|_{\varepsilon,\boldsymbol{\beta},\boldsymbol{\mu}}\, .
\tag{7.1.3}
$$

Recalling the definition of the Hamiltonian

$$
E_0(Z_s) := \sum_{1\leq i\leq s} \frac{|v_i|^2}{2}
$$

we then deduce that

$$
\begin{aligned}
e^{\beta(t)E_0(Z_s)+s\mu(t)} \Big\| \sum_{k=n+1}^\infty \int_0^t \int_0^{t_1} \dots \int_0^{t_{k-1}} & \mathbf{T}_s(t-t_1)\mathcal{C}_{s,s+1}\mathbf{T}_{s+1}(t_1-t_2)\mathcal{C}_{s+1,s+2}\dots \\
\dots \mathbf{T}_{s+k}(t_k) f_{N,0}^{(s+k)}\, dt_k \dots dt_1 \Big\|_{L^\infty} & \leq C\left(\frac12\right)^n |\!|\!| F_N |\!|\!|_{\varepsilon,\boldsymbol{\beta},\boldsymbol{\mu}}\, .
\end{aligned}
\tag{7.1.4}
$$

Combining this estimate together with the uniform bound on $|\!|\!| F_N |\!|\!|_{\varepsilon,\boldsymbol{\beta},\boldsymbol{\mu}}$ given in Theorem 6 leads to the first statement in Proposition 7.1.1. The second statement is established exactly in an analogous way, using estimate (5.3.4) together with the uniform bound obtained in Theorem 7. □

## 7.2. Energy truncation

We introduce a parameter $R > 0$ and define

(7.2.1)
$$\begin{aligned} f^{(s,k)}_{N,R}(t) &:= \sum_{k=0}^{n} \int_0^t \int_0^{t_1} \dots \int_0^{t_{k-1}} \mathbf{T}_s(t-t_1)\mathcal{C}_{s,s+1}\mathbf{T}_{s+1}(t_1-t_2)\mathcal{C}_{s+1,s+2}\dots \\ &\qquad \dots \mathbf{T}_{s+k}(t_k)1\!\!1_{E_0(Z_{s+k})\leq R^2} f^{(s+k)}_{N,0}\, dt_k \dots dt_1 \,, \\ f^{(s,k)}_{R}(t) &:= \sum_{k=0}^{n} \int_0^t \int_0^{t_1} \dots \int_0^{t_{k-1}} \mathbf{S}_s(t-t_1)\mathcal{C}^0_{s,s+1}\mathbf{S}_{s+1}(t_1-t_2)\mathcal{C}^0_{s+1,s+2}\dots \\ &\qquad \dots \mathbf{S}_{s+k}(t_k)1\!\!1_{E_0(Z_{s+k})\leq R^2} f^{(s+k)}_{0}\, dt_k \dots dt_1 \end{aligned}$$

and the corresponding observables

(7.2.2)
$$I^R_{s,k}(t)(X_s) := \int \varphi_s(V_s) f^{(s,k)}_{N,R}(t,Z_s)dV_s \quad \text{and} \quad I^{0,R}_{s,k}(t)(X_s) := \int \varphi_s(V_s) f^{(s,k)}_{R} dV_s \,.$$

Using the bounds derived in Chapter 5 we find easily that $\sum_k (I_{s,k} - I^R_{s,k})(t)$ and $\sum_k (I^0_{s,k} - I^{0,R}_{s,k})(t)$ can be made arbitrarily small when $R$ is large. More precisely the following result holds.

***Proposition 7.2.1***. — *Fix $\beta_0 > 0$ and $\mu_0 \in \mathbf{R}$. Let $s \in \mathbf{N}^*$ and $t \in [0,T]$ be given. There are two nonnegative constants $C, C'$ such that for each $n$,*

$$\Big\| \sum_{k=0}^{n} (I_{s,k} - I^R_{s,k})(t) \Big\|_{L^\infty(\mathbf{R}^{ds})} \leq C \|\varphi_s\|_{L^\infty(\mathbf{R}^{ds})} e^{-C'\beta_0 R^2} \|F_{N,0}\|_{\varepsilon,\beta_0,\mu_0} \,,$$

*and*

$$\Big\| \sum_{k=0}^{n} (I^0_{s,k} - I^{0,R}_{s,k})(t) \Big\|_{L^\infty(\mathbf{R}^{ds})} \leq C \|\varphi_s\|_{L^\infty(\mathbf{R}^{ds})} e^{-C'\beta_0 R^2} \|F_0\|_{0,\beta_0,\mu_0} \,.$$

*Proof*. — Let $0 < \beta_0' < \beta_0$ be given, and define the associate functions $\boldsymbol{\beta}'$ and $\boldsymbol{\beta}$ as in Theorem 6 stated in Chapter 5. Choose $\beta_0' < \beta_0$ so that

$$C_d\big(1 + (\beta_0' - \lambda T)^{-\frac12}\big) e^{-\mu_0 + \lambda T} (\beta_0' - \lambda T)^{-\frac{d}{2}} = \frac{2\lambda}{3} \,.$$

(to be compared with (5.4.10) for $\beta_0$).

Then according to the results of Chapter 5 and similarly to (7.1.4) we know that

$$\begin{aligned} \int_0^t \int_0^{t_1} \dots \int_0^{t_{k-1}} \mathbf{T}_s(t-t_1)\mathcal{C}_{s,s+1}\mathbf{T}_{s+1}(t_1-t_2)\dots \mathbf{T}_{s+k}(t_k)1\!\!1_{E_0(Z_{s+k})\geq R^2} f^{(s+k)}_{N,0}\, dt_k \dots dt_1 \\ \leq C \left(\frac{3}{2}\right)^{-k} e^{-\boldsymbol{\beta}'(T)E_0(Z_s) - s\mu_0(T)} \|G_{N,0,s}\|_{\varepsilon,\beta_0',\mu_0} \,, \end{aligned}$$

where we have defined

$$G_{N,0,s} := (g^{s+k}_{N,0})_{0\leq k\leq N-s} \,, \quad \text{with} \quad g^{s+k}_{N,0}(Z_{s+k}) := 1\!\!1_{E_0(Z_{s+k})\geq R^2}\, f^{(s+k)}_{N,0}(Z_{s+k}) \,.$$

The result then follows from the fact that

$$\|G_{N,0,s}\|_{\varepsilon,\beta_0',\mu_0} \leq C e^{(\beta_0' - \beta_0)R^2} \|F_{N,0}\|_{\varepsilon,\beta_0,\mu_0} \,.$$

The argument is identical for $I^0_{s,n}(t) - I^{0,R}_{s,n}(t)$. □

***Remark 7.2.1***. — *It is useful to notice that the collision operators preserve the bound on high energies, in the sense that*

$$\mathcal{C}_{s,s+1} 1\!\!1_{E_0(Z_{s+1})\le R^2} \equiv 1\!\!1_{E_0(Z_s)\le R^2}\, \mathcal{C}_{s,s+1} 1\!\!1_{E_0(Z_{s+1})\le R^2}$$
$$\mathcal{C}^0_{s,s+1} 1\!\!1_{E(Z_{s+1})\le R^2} \equiv 1\!\!1_{E(Z_s)\le R^2}\, \mathcal{C}^0_{s,s+1} 1\!\!1_{E(Z_{s+1})\le R^2}\,.$$

## 7.3. Time separation

We choose another small parameter $\delta > 0$ and further restrict the study to the case when $t_i - t_{i+1} \geq \delta$. That is, we define

$$\mathcal{T}_k(t) := \Big\{ T_k = (t_1,\dots,t_k)\,/\, t_i < t_{i-1} \quad \text{with} \quad t_{k+1} = 0 \quad \text{and} \quad t_0 = t \Big\}\,,$$

$$\mathcal{T}_{k,\delta}(t) := \Big\{ T_k \in \mathcal{T}_k(t)\,/\, t_i - t_{i+1} \geq \delta \Big\}\,,$$

and

$$\begin{aligned}
I^{R,\delta}_{s,k}(t)(X_s) &:= \int \varphi_s(V_s) \int_{\mathcal{T}_{k,\delta}(t)} \mathbf{T}_s(t-t_1)\mathcal{C}_{s,s+1}\mathbf{T}_{s+1}(t_1-t_2)\mathcal{C}_{s+1,s+2} \\
&\qquad \dots \mathcal{C}_{s+k-1,s+k}\mathbf{T}_{s+k}(t_k-t_{k+1}) 1\!\!1_{E_0(Z_{s+k})\le R^2} f^{(s+k)}_{N,0}\, dT_k dV_s\,, \\
I^{0,R,\delta}_{s,k}(t)(X_s) &:= \int \varphi_s(V_s) \int_{\mathcal{T}_{k,\delta}(t)} \mathbf{S}_s(t-t_1)\mathcal{C}^0_{s,s+1}\mathbf{S}_{s+1}(t_1-t_2)\mathcal{C}^0_{s+1,s+2} \\
&\qquad \dots \mathcal{C}^0_{s+k-1,s+k}\mathbf{S}_{s+k}(t_k-t_{k+1}) 1\!\!1_{E_0(Z_{s+k})\le R^2} f^{(s+k)}_{0}\, dT_k dV_s\,.
\end{aligned} \tag{7.3.1}$$

Again applying the continuity bounds for the transport and collision operators, the error on the functions $\displaystyle\sum_k (I^R_{s,k} - I^{R,\delta}_{s,k})(t)$ and $\displaystyle\sum_k (I^{0,R}_{s,k} - I^{0,R,\delta}_{s,k})(t)$ can be estimated as follows.

***Proposition 7.3.1***. — *Let $s \in \mathbf{N}^*$ and $t \in [0,T]$ be given. There is a constant $C$ such that for each $n$ and $R$,*

$$\Big\| \sum_{k=0}^n (I^R_{s,k} - I^{R,\delta}_{s,k})(t) \Big\|_{L^\infty(\mathbf{R}^{ds})} \leq C n^2 \frac{\delta}{T} \|\varphi\|_{L^\infty(\mathbf{R}^{ds})} \|F_{N,0}\|_{\varepsilon,\beta_0,\mu_0}$$

*and*

$$\Big\| \sum_{k=0}^n (I^{0,R}_{s,k} - I^{0,R,\delta}_{s,k})(t) \Big\|_{L^\infty(\mathbf{R}^{ds})} \leq C\delta n^2 \frac{\delta}{T} \|\varphi\|_{L^\infty(\mathbf{R}^{ds})} \|F_0\|_{0,\beta_0,\mu_0}\,.$$

## 7.4. Reformulation in terms of pseudo-trajectories

Putting together Propositions 7.1.1, 7.2.1 and 7.3.1 we obtain the following result.

***Corollary 7.4.1***. — *With the notation of Theorem 9, given $s \in \mathbf{N}^*$ and $t \in [0,T]$, there are two positive constants $C$ and $C'$ such that for each $n \in \mathbf{N}^*$,*

$$\Big\| I_s(t) - \sum_{k=0}^n I^{R,\delta}_{s,k}(t) \Big\|_{L^\infty(\mathbf{R}^{ds})} \leq C(2^{-n} + e^{-C'\beta_0 R^2} + n^2 \frac{\delta}{T}\delta) \|\varphi\|_{L^\infty(\mathbf{R}^{ds})} \|F_{N,0}\|_{\varepsilon,\beta_0,\mu_0}\,.$$

In the same way as in (4.3.4) we now decompose the Boltzmann collision operators (4.4.3) into

$$\mathcal{C}^0_{s,s+1} = \mathcal{C}^{0,+}_{s,s+1} - \mathcal{C}^{0,-}_{s,s+1} \, ,$$

where the index + corresponding to post-collisional configurations and the index − to pre-collisional configurations. By definition of the collision cross-section for hard spheres, we have

$$\begin{aligned}
\big(\mathcal{C}^{0,-,m}_{s,s+1} f^{(s+1)}\big)(Z_s) &:= \int_{\mathbf{S}_1^{d-1}\times\mathbf{R}^d} b(v_{s+1}-v_m,\omega) f^{(s+1)}(Z_s, x_m, v_{s+1})\, d\omega dv_{s+1} \\
&= \int_{\mathbf{S}_1^{d-1}\times\mathbf{R}^d} ((v_{s+1}-v_m)\cdot\omega)_- f^{(s+1)}(Z_s, x_m, v_{s+1})\, d\omega dv_{s+1} \quad \text{and} \\
\big(\mathcal{C}^{0,+,m}_{s,s+1} f^{(s+1)}\big)(Z_s) &:= \int_{\mathbf{S}_1^{d-1}\times\mathbf{R}^d} b(v_{s+1}-v_m,\omega) f^{(s+1)}(z_1,\dots,x_m,v_m^*,\dots,z_s,x_m,v_{s+1}^*)\, d\omega dv_{s+1} \\
&= \int_{\mathbf{S}_1^{d-1}\times\mathbf{R}^d} ((v_{s+1}-v_m)\cdot\omega)_+ f^{(s+1)}(z_1,\dots,x_m,v_m^*,\dots,z_s,x_m,v_{s+1}^*)\, d\omega dv_{s+1} \, .
\end{aligned}$$

The elementary BBGKY and Boltzmann observables we are interested in can therefore be decomposed as

$$\begin{aligned}
I^{R,\delta}_{s,k}(t)(X_s) &= \sum_{J,M} \Big(\prod_{i=1}^k j_i\Big) I^{R,\delta}_{s,k}(t,J,M)(X_s) \quad \text{and} \\
I^{0,R,\delta}_{s,k}(t)(X_s) &= \sum_{J,M} I^{0,R,\delta}_{s,k}(t,J,M)(X_s)
\end{aligned} \tag{7.4.1}$$

where the *elementary functionals* $I^{R,\delta}_{s,k}(t,J,M)$ are defined by

$$\begin{aligned}
I^{R,\delta}_{s,k}(t,J,M)(X_s) &:= \int \varphi_s(V_s) \int_{\mathcal{T}_{k,\delta}(t)} \mathbf{T}_s(t-t_1)\mathcal{C}^{j_1,m_1}_{s,s+1}\mathbf{T}_{s+1}(t_1-t_2)\mathcal{C}^{j_2,m_2}_{s+1,s+2} \\
&\qquad\qquad \dots \mathbf{T}_{s+k}(t_k-t_{k+1}) 1\!\!1_{E_0(Z_{s+k})\leq R^2} f^{(s+k)}_{N,0}\, dT_k dV_s \, , \\
I^{0,R,\delta}_{s,k}(t,J,M)(X_s) &:= \int \varphi_s(V_s) \int_{\mathcal{T}_{k,\delta}(t)} \mathbf{S}_s(t-t_1)\mathcal{C}^{0,j_1,m_1}_{s,s+1}\mathbf{S}_{s+1}(t_1-t_2)\mathcal{C}^{0,j_2,m_2}_{s+1,s+2} \\
&\qquad\qquad \dots \mathbf{S}_{s+k}(t_k-t_{k+1}) 1\!\!1_{E_0(Z_{s+k})\leq R^2} f^{(s+k)}_{0}\, dT_k dV_s \, ,
\end{aligned} \tag{7.4.2}$$

with

$$J := (j_1,\dots,j_k) \in \{+,-\}^k \quad \text{and} \quad M := (m_1,\dots,m_k) \quad \text{with} \quad m_i \in \{1,\dots,s+i-1\} \, .$$

Each one of the functionals $I^{R,\delta}_{s,k}(t,J,M)$ and $I^{0,R,\delta}_{s,k}(t,J,M)$ defined in (7.4.2) can be viewed as the observable associated with some dynamics, which of course is not the actual dynamics in physical space since

- the total number of particles is not conserved;
- the distribution does even not remain nonnegative because of the sign of loss collision operators.

This explains the terminology of "pseudo-trajectories" we choose to describe the process.

In this formulation, the characteristics associated with the operators $\mathbf{T}_{s+i}(t_i-t_{i+1})$ and $\mathbf{S}_{s+i}(t_i-t_{i+1})$ are followed *backwards* in time between two consecutive times $t_i$ and $t_{i+1}$, and collision terms (associated with $\mathcal{C}_{s+i,s+i+1}$ and $\mathcal{C}^0_{s+i,s+i+1}$) are seen as *source terms*, in which, in the words of Lanford [**34**], "additional particles" are "adjoined" to the marginal.

The main heuristic idea is that for the BBGKY hierarchy, in the time interval $[t_{i+1}, t_i]$ between two collisions $\mathcal{C}_{s+i-1,s+i}$ and $\mathcal{C}_{s+i,s+i+1}$, the particles should not interact in general so trajectories should correspond to the free flow $\mathbf{S}_{s+i}$. On the other hand at a collision time $t_i$, the velocities of the two particles in interaction are liable to change. This is depicted in Figure 3.

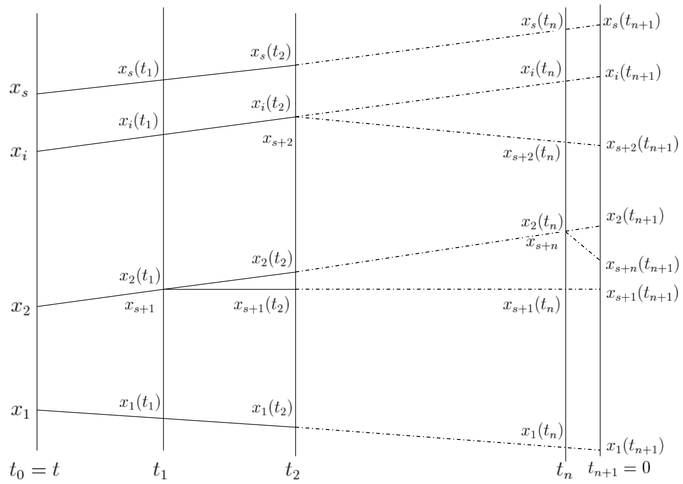


FIGURE 3. Pseudo-trajectories

At this stage however, we still cannot study directly the convergence of $I^{R,\delta}_{s,k}(t, J, M) - I^{0,R,\delta}_{s,k}(t, J, M)$ since the transport operators $\mathbf{T}_k$ do not coincide everywhere with the free transport operators $\mathbf{S}_k$, which means – in terms of pseudo-trajectories – that there are recollisions. We shall thus prove that these recollisions arise only for a few pathological pseudo-trajectories, which can be eliminated by additional truncations of the domains of integration. This is the goal of Part IV.

# PART III

# THE CASE OF SHORT RANGE POTENTIALS

# CHAPTER 8

# TWO-PARTICLE INTERACTIONS

In the case when the microscopic interaction between particles is governed by a short-range repulsive potential, collisions are no more instantaneous and pointwise, and they possibly involve more than two particles. Our analysis in Chapter 11 shows however that the low density limit $N\varepsilon^{d-1} \equiv 0$ requires only a description of two-particle interactions, at the exclusion of more complicated interactions.

In this chapter we therefore study precisely, following the lines of [**13**], the Hamiltonian system (1.2.1) for $N = 2$. The study of the reduced motion is carried out in Section 8.1, while the scattering map is introduced in Section 8.2, and the cross-section, which will play in important role in the Boltzmann hierarchy, is described in Section 8.3.

## 8.1. Reduced motion

We first define a notion of pre- and post-collisional particles, by analogy with the dynamics of hard spheres.

***Definition 8.1.1***. — *Two particles $z_1, z_2$ are said to be pre-collisional if their distance is $\varepsilon$ and decreasing:*

$$|x_1 - x_2| = \varepsilon\,, \qquad (v_1 - v_2)\cdot(x_1 - x_2) < 0\,.$$

*Two particles $z_1, z_2$ are said to be post-collisional if their distance is $\varepsilon$ and increasing:*

$$|x_1 - x_2| = \varepsilon\,, \qquad (v_1 - v_2)\cdot(x_1 - x_2) > 0\,.$$

We consider here only two-particle systems, and show in Lemma 8.1.2 that, if $z_1$ and $z_2$ are pre-collisional at time $t_-$, then there exists a post-collisional configuration $z_1', z_2'$, attained at $t_+ > t_-$. Since $\nabla\Phi(x/\varepsilon)$ vanishes on $\{|x| \geq \varepsilon\}$, the particles $z_1$ and $z_2$ travel at constant velocities $v_1'$ and $v_2'$ for ulterior $(t > t_+)$ times.

Momentarily changing back the macroscopic scales of (1.2.1) to the microscopic scales of (1.0.3) by defining $\tau := (t - t_-)/\varepsilon$ and $y(\tau) := x(\tau)/\varepsilon$, $w(\tau) = v(\tau)$, we find that the two-particle dynamics is governed by the equations

$$\left\{\begin{aligned} &\frac{dy_1}{d\tau} = w_1\,, \quad \frac{dy_2}{d\tau} = w_2\,,\\ &\frac{dw_1}{d\tau} = -\nabla\Phi\,(y_1 - y_2) = -\frac{dw_2}{d\tau}\,, \end{aligned}\right. \tag{8.1.1}$$

whence the conservations

$$\frac{d}{d\tau}(w_1+w_2)=0\,,\qquad \frac{d}{d\tau}\left(\frac14(w_1+w_2)^2+\frac14(w_1-w_2)^2+\Phi(y_1-y_2)\right)=0\,. \tag{8.1.2}$$

From (8.1.2) we also deduce that the center of mass has a uniform, rectilinear motion:

$$(y_1+y_2)(\tau)=(y_1+y_2)(0)+\tau(w_1+w_2)\,, \tag{8.1.3}$$

and that pre- and post-collisional velocities are linked by the classical relations

$$w_1'+w_2'=w_1+w_2,\quad |w_1'|^2+|w_2'|^2=|w_1|^2+|w_2|^2\,. \tag{8.1.4}$$

A consequence of (8.1.1) is that $(\delta y,\delta w):=(y_1-y_2,w_1-w_2)$ solves

$$\frac{d}{d\tau}\delta y=\delta w\,,\qquad \frac{d}{d\tau}\delta w=-2\nabla\Phi(\delta y)\,. \tag{8.1.5}$$

In the following we denote by $\phi_t:\mathbf{R}^{2d}\to\mathbf{R}^{2d}$ the flow of (8.1.5).

We notice that, $\Phi$ being radial, there holds

$$\frac{d}{d\tau}(\delta y\wedge\delta w)=\delta w\wedge\delta w-2\delta y\wedge\nabla\Phi(\delta y)=0\,,$$

implying that, if the initial angular momentum $\delta y_0\wedge\delta w_0$ is non-zero, then $\delta y$ remains for all times in the hyperplane orthogonal to $\delta y_0\wedge\delta w_0$. In this hyperplane, introducing polar coordinates $(\rho,\varphi)$ in $\mathbf{R}_+\times\mathbf{S}_1^1$, such that

$$\delta y=\rho e_\rho\quad\text{and}\quad \delta w=\dot\rho e_\rho+\rho\dot\varphi e_\varphi$$

the conservations of energy and angular momentum take the form

$$\begin{aligned}\frac12(\dot\rho^2+(\rho\dot\varphi)^2)+2\Phi(\rho)&=\frac12|\delta w_0|^2\,,\\ \rho^2|\dot\varphi|&=|\delta y_0\wedge\delta w_0|\,,\end{aligned}$$

implying $\rho>0$ for all times, and

$$\dot\rho^2+\Psi(\rho,\mathcal{E}_0,\mathcal{J}_0)=\mathcal{E}_0\,,\qquad \Psi:=\frac{\mathcal{E}_0\mathcal{J}_0^2}{\rho^2}+4\Phi(\rho)\,, \tag{8.1.6}$$

where we have defined

$$\mathcal{E}_0:=|\delta w_0|^2\quad\text{and}\quad \mathcal{J}_0:=|\delta y_0\wedge\delta w_0|/|\delta w_0|=:\sin\alpha\,, \tag{8.1.7}$$

which are respectively (twice) the energy and the impact parameter, $\pi-\alpha$ being the angle between $\delta w_0$ and $\delta y_0$ (notice that $\alpha\geq\pi/2$ for pre-collisional situations). In the limit case when $\alpha=0$, the movement is confined to a line since $\dot\varphi\equiv0$.

We consider the sets corresponding to pre- and post-collisional configurations:

$$\mathcal{S}^\pm:=\big\{(\delta y,\delta w)\in\mathbf{S}_1^{d-1}\times\mathbf{R}^d\,/\,\pm\delta y\cdot\delta w>0\big\}\,. \tag{8.1.8}$$

In polar coordinates pre-collisional configurations correspond to $\rho=1$ and $\dot\rho<0$ while post-collisional configurations correspond to $\rho=1$ and $\dot\rho>0$.

***Lemma 8.1.2* (Description of the reduced motion).** — *For the differential equation* (8.1.5) *with pre-collisional datum $(\delta y_0,\delta w_0)\in\mathcal{S}^-$, there holds $|\delta y(\tau)|\geq\rho_*$ for all $\tau\geq0$, with the notation*

$$\rho_*=\rho_*(\mathcal{E}_0,\mathcal{J}_0):=\max\big\{\rho\in(0,1)\,/\,\Psi(\rho,\mathcal{E}_0,\mathcal{J}_0)=\mathcal{E}_0\big\}\,, \tag{8.1.9}$$

*and for* $\tau_*$ *defined by*

(8.1.10) $$\tau_* := 2\int_{\rho_*}^{1} \left(\mathcal{E}_0 - \Psi(\rho, \mathcal{E}_0, \mathcal{J}_0)\right)^{-1/2} d\rho\,,$$

*the configuration is post-collisional* ($\rho = 1$, $\dot\rho > 0$) *at* $\tau = \tau_*$.

*Proof.* — Solutions to (8.1.6) satisfy $\dot\rho = \iota(\rho)\big(\mathcal{E}_0 - \Psi(\rho)\big)^{1/2}$, with $\iota(\rho) = \pm 1$, possibly changing values only on $\{\Psi = \mathcal{E}_0\}$, by Darboux's theorem (a derivative function satisfies the intermediate value theorem). The initial configuration being pre-collisional, there holds initially $\iota = -1$, corresponding to a decreasing radius. The existence of $\rho_*$ satisfying (8.1.9) is then easily checked: we have $|\delta y_0| = 1$ and $\delta y_0 \cdot \delta w_0 \neq 0$, so there holds $\Psi(1, \mathcal{E}_0, \mathcal{J}_0) < \mathcal{E}_0$, and $\Psi$ is increasing as $\rho$ is decreasing. The set $\{\tau \geq 0, \rho(\tau) \geq \rho_*\}$ is closed by continuity. It is also open: since $\Phi$ is nonincreasing, then $\partial_\rho \Psi \neq 0$ everywhere and in particular at $(\rho_*, \mathcal{E}_0, \mathcal{J}_0)$. So $\mathcal{E}_0 - \Psi$ changes sign at $\rho_*$, which forces, by (8.1.6), the sign function $\iota$ to jump from $-$ to $+$ as $\rho$ reaches the value $\rho_*$ from above. This proves $\rho \geq \rho_*$ by connexity. The minimal radius $\rho = \rho_*$ is attained at $\tau_*/2$, where $\tau_*$ is defined by (8.1.10), the integral being finite since $\partial_\rho \Psi$ does not vanish. Assume finally that for all $\tau > 0$, there holds $\rho(\tau) < 1$. Then on $[\tau_*/2, +\infty)$, $\rho$ is increasing and bounded, hence converges to a limit radius, which contradicts the definition of $\rho_*$. This proves $\rho = 1$ at $\tau = \tau_*$, a time at which $\dot\rho > 0$, since $\iota$ has jumped exactly once, by definition of $\rho_*$. $\square$

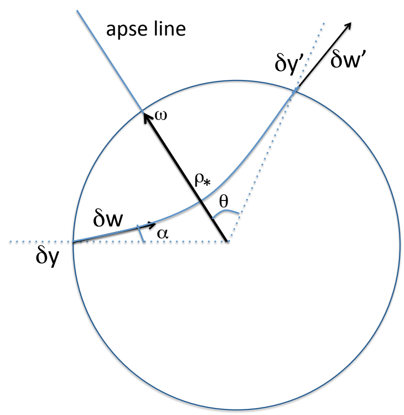


FIGURE 4. Reduced dynamics

The reduced dynamics is pictured on Figure 4, where the half-deflection angle $\theta$ is the integral of the angle $\varphi$ as a function of $\rho$ over $[\rho_*, 1]$ :

(8.1.11) $$\theta = \int_{\rho_*}^{1} \frac{\mathcal{E}_0^{1/2}\mathcal{I}_0}{\rho^2} \left(\mathcal{E}_0 - \Psi(\rho, \mathcal{E}_0, \mathcal{I}_0)\right)^{-1/2} d\rho\,,$$

With the initialization choice $\varphi_0 = 0$, the post-collisional configuration is $(\rho, \varphi)(\tau_*) = (1, 2\theta)$; it can be deduced from the pre-collisional configuration by symmetry with respect to the apse line, which by definition is the line through the origin and the point of closest approach $(\delta y(\tau_*/2), \delta w(\tau_*/2))$. The direction of this line is denoted $\omega \in \mathbf{S}_1^{d-1}$.

## 8.2. Scattering map

We shall now define a microscopic scattering map that sends pre- to post-collisional configurations:

$$(\delta y_0, \delta w_0) \in \mathcal{S}^- \mapsto (\delta y(\tau_*), \delta w(\tau_*)) = \phi_{\tau_*}(\delta y_0, \delta w_0) \in \mathcal{S}^+ .$$

By uniqueness of the trajectory of (8.1.5) issued from $(\delta y_0, \delta w_0)$ (a consequence of the regularity assumption on the potential, via the Cauchy-Lipschitz theorem), the scattering is one-to-one. It is also clearly onto.

Back in the macroscopic variables, we now define a corresponding scattering operator for the two-particle dynamics. In this view, we introduce the sets

$$\mathcal{S}_\varepsilon^\pm := \Big\{ (z_1, z_2) \in \mathbf{R}^{4d} \, / \, |x_1 - x_2| = \varepsilon \, , \, \pm (x_1 - x_2) \cdot (v_1 - v_2) > 0 \Big\} .$$

We define, as in (8.1.7),

$$\mathcal{E}_0 = |v_1 - v_2|^2 \quad \text{and} \quad \mathcal{J}_0 \; := \frac{|(x_1 - x_2) \wedge (v_1 - v_2)|}{\varepsilon |v_1 - v_2|} =: \sin \alpha \, . \tag{8.2.1}$$

***Definition 8.2.1* (Scattering operator).** — *The* scattering operator *is defined as*

$$\sigma_\varepsilon : (x_1, v_1, x_2, v_2) \in \mathcal{S}_\varepsilon^- \mapsto (x_1', v_1', x_2', v_2') \in \mathcal{S}_\varepsilon^+ ,$$

*where*

$$\begin{aligned}
x_1' &:= \frac12 (x_1 + x_2) + \frac{\varepsilon \tau_*}{2} (v_1 + v_2) + \frac{\varepsilon}{2} \delta y(\tau_*) = -x_1 + \omega \cdot (x_1 - x_2) \omega + \frac{\varepsilon \tau_*}{2} (v_1 + v_2) \, , \\
x_2' &:= \frac12 (x_1 + x_2) + \frac{\varepsilon \tau_*}{2} (v_1 + v_2) - \frac{\varepsilon}{2} \delta y(\tau_*) = -x_2 - \omega \cdot (x_1 - x_2) \omega + \frac{\varepsilon \tau_*}{2} (v_1 + v_2) \, , \\
v_1' &:= \frac12 (v_1 + v_2) + \frac12 \delta w(\tau_*) = v_1 - \omega \cdot (v_1 - v_2) \, \omega \, , \\
v_2' &:= \frac12 (v_1 + v_2) - \frac12 \delta w(\tau_*) = v_2 + \omega \cdot (v_1 - v_2) \, \omega \, ,
\end{aligned} \tag{8.2.2}$$

*where $\tau_*$ is the microscopic interaction time, as defined in Lemma* 8.1.2, *$(\delta y(\tau_*), \delta w(\tau_*))$ is the microscopic post-collisional configuration: $(\delta y(\tau_*), \delta w(\tau_*)) = \phi_{\tau_*}((x_1 - x_2)/\varepsilon, v_1 - v_2)$, and $\omega$ is the direction of the apse line. Denoting by $\nu := (x_1 - x_2)/|x_1 - x_2|$ we also define*

$$\sigma_0(\nu, v_1, v_2) := (\nu', v_1', v_2') .$$

The above description of $(x_1', v_1')$ and $(x_2', v_2')$ in terms of $\omega$ is deduced from the identities

$$\delta v(\tau_*) = \delta v_0 - 2\omega \cdot \delta v_0 \, \omega \quad \text{and} \quad \delta y(\tau_*) = -\delta y_0 + 2\omega \cdot \delta y_0 \, \omega$$

in the reduced microscopic coordinates.

By $\partial_\rho \Psi \neq 0$ in $(0,1)$ and the implicit function theorem, the map $(\mathcal{E}, \mathcal{J}) \to \rho_*(\mathcal{E}, \mathcal{J})$ is $C^2$ just like $\Psi$. Similarly, $\tau_* \in C^2$. By Definition 8.2.1 and $C^1$ regularity of $\nabla \Phi$ (Assumption 1.2.1), this implies that the scattering operator $\sigma_\varepsilon$ is $C^1$, just like the flow map $\phi$ of the two-particle scattering. The scattering $\sigma_\varepsilon$ is also bijective, for the same reason that the microsopic scattering is bijective.

***Proposition 8.2.1*.** — *Let $R > 0$ be given and consider*

$$\mathcal{S}_{\varepsilon,R}^\pm := \Big\{ (z_1, z_2) \in \mathbf{R}^{4d} \; / \, |x_1 - x_2| = \varepsilon \, , \, |(v_1, v_2)| = R \, , \pm (v_1 - v_2) \cdot (x_1 - x_2) > 0 \Big\} .$$

*The scattering operator $\sigma_\varepsilon$ is a bijection from $\mathcal{S}_{\varepsilon,R}^-$ to $\mathcal{S}_{\varepsilon,R}^+$.*

*The macroscopic time of interaction $\varepsilon\tau_*$, where $\tau_*$ is defined in* (8.1.10), *is uniformly bounded on compact sets of* $\mathbf{R}^+ \setminus \{0\} \times [0,1]$, *as a function of* $\mathcal{E}_0$ *and* $\mathcal{J}_0$.

*Proof.* — We already know that $\sigma_\varepsilon$ is a bijection from $\mathcal{S}^-_\varepsilon$ to $\mathcal{S}^+_\varepsilon$. By (8.1.4), it also preserves the velocity bound. Hence $\sigma_\varepsilon$ is bijective $\mathcal{S}^-_{\varepsilon,R} \to \mathcal{S}^+_{\varepsilon,R}$. Now given $\mathcal{E}_0 > 0$ and $\mathcal{J}_0 \in [0,1]$, we shall show that $\tau_*$ can be bounded by a constant depending only on $\mathcal{E}_0$. Since $\Phi(\rho_*) \le \mathcal{E}_0/4$, then $\rho_* \ge \Phi^{-1}(\mathcal{E}_0/4)$. Let us then define $i_0 \in (0,1)$ by

$$i_0 := \frac{1}{2\sqrt{2}}\Phi^{-1}\big(\frac{\mathcal{E}_0}{4}\big)\,,$$

so that $\rho_*^2 \ge 8i_0^2$.

On the one hand it is easy to see, after a change of variable in the integral, using

$$\frac{d}{d\rho}(\mathcal{E}_0 - \Psi(\mathcal{E}_0,\mathcal{J}_0,\rho)) = \frac{2\mathcal{E}_0\mathcal{J}_0^2}{\rho^3} - 4\Phi'(\rho) \ge \frac{2\mathcal{E}_0\mathcal{J}_0^2}{\rho^3} \ge 2\mathcal{E}_0\mathcal{J}_0^2\,,$$

that there holds the bound

$$\tau_* \le \frac{1}{\mathcal{E}_0\mathcal{J}_0^2}\int_0^{\mathcal{E}_0(1-\mathcal{J}_0^2)} \frac{dy}{\sqrt{y}} \le \frac{2\sqrt{1-\mathcal{J}_0^2}}{\mathcal{J}_0^2\sqrt{\mathcal{E}_0}}\,.$$

– So if $\mathcal{J}_0 \ge i_0$, we find that

$$\tau_* \le \frac{2}{\sqrt{\mathcal{E}_0}i_0^2} = \frac{16}{\sqrt{\mathcal{E}_0}\big(\Phi^{-1}\big(\frac{\mathcal{E}_0}{4}\big)\big)^2}\,.$$

– On the other hand for $\mathcal{J}_0 \le i_0$ we define $\gamma := \Phi^{-1}(\mathcal{E}_0/8)$ and we cut the integral defining $\tau_*$ into two parts:

$$\tau_* = \tau_*^{(1)} + \tau_*^{(2)} \quad \text{with} \quad \tau_*^{(1)} = 2\int_{\rho_*}^{\gamma} (\mathcal{E}_0 - \Psi(\mathcal{E}_0,\mathcal{J}_0,\rho))^{-1/2}\, d\rho\,.$$

Notice that since $\rho_*^2 \ge 8i_0^2$ and $\mathcal{J}_0 \le i_0$, then $\mathcal{E}_0/4 - \mathcal{E}_0\mathcal{J}_0^2/4\rho_*^2 \ge 7\mathcal{E}_0/32 \ge \mathcal{E}_0/8$ so

$$\rho_* = \Phi^{-1}\Big(\frac{\mathcal{E}_0}{4} - \frac{\mathcal{E}_0\mathcal{J}_0^2}{4\rho_*^2}\Big) \le \Phi^{-1}\Big(\frac{\mathcal{E}_0}{8}\Big) = \gamma\,.$$

The first integral $\tau_*^{(1)}$ is estimated using the fact that $\Phi'$ does not vanish outside 1 as stated in Assumption 1.2.1: defining

$$M(\Phi) := \inf_{i_0 \le \rho \le \gamma} |\Phi'(\rho)| > 0\,,$$

we find that on $[i_0,\gamma]$,

$$\frac{d}{d\rho}(\mathcal{E}_0 - \Psi(\mathcal{E}_0,\mathcal{J}_0,\rho)) = \frac{2\mathcal{E}_0\mathcal{J}_0^2}{\rho^3} - 4\Phi'(\rho) \ge 4M(\Phi)$$

so

$$\tau_*^{(1)} \le \frac{\big(\mathcal{E}_0/2 - \mathcal{E}_0\mathcal{J}_0^2/\gamma^2\big)^{\frac12}}{M(\Phi)} \le \frac{\sqrt{\mathcal{E}_0}}{\sqrt{2}M(\Phi)}\,.$$

For the second integral we estimate simply

$$\tau_*^{(2)} \le \frac{2}{\big(\mathcal{E}_0/2 - \mathcal{E}_0\mathcal{J}_0^2/\gamma^2\big)^{\frac12}} \le \frac{2}{\big(\mathcal{E}_0/2 - \mathcal{E}_0/8\big)^{\frac12}} = \frac{4\sqrt{2}}{\sqrt{3\mathcal{E}_0}}\,.$$

The result follows. □

***Remark 8.2.2.*** — *If* $\Phi$ *is of the type* $\dfrac{1}{\rho^s}\exp(-\dfrac{1}{1-\rho^2})$ *then the proof of Proposition* 8.2.1 *shows that* $\tau_*$ *may be bounded from above by a constant of the order of* $C/\sqrt{e_0}(1+\log e_0)$ *if* $\mathcal{E}_0 \ge e_0$.

## 8.3. Scattering cross-section and the Boltzmann collision operator

The scattering operator in Definition 8.2.1 is parametrized by the impact parameter and the two ingoing (or outgoing) velocities. However in the Boltzmann limit the impact parameter cannot be observed: the observed quantity is the *deflection angle* or *scattering angle*, defined as the angle between ingoing and outgoing relative velocities. The next paragraph defines that angle as well as the scattering cross-section, and the following paragraph defines the Boltzmann collision operators using that formulation.

**8.3.1. Scattering cross-section.** — With notation from the previous paragraphs, the deflection angle is equal to $\pi - 2\Theta$ where $\Theta := \alpha + \theta$, the angle $\alpha$ being defined in (8.2.1) and $\theta$ being defined in (8.1.11), so that

$$\Theta = \Theta(\mathcal{E}_0, \mathcal{J}_0) := \arcsin \mathcal{J}_0 + \mathcal{J}_0 \int_{\rho_*}^1 \frac{d\rho}{\sqrt{1 - \frac{4\Phi(\rho)}{\mathcal{E}_0} - \frac{\mathcal{J}_0^2}{\rho^2}}} \cdot$$

The following result, and its proof, are due to [**39**]:

***Lemma 8.3.1.*** — *Under Assumption* 1.2.1*, assume moreover that for all* $\rho \in (0,1)$*,*

$$\rho\Phi''(\rho) + 2\Phi'(\rho) \geq 0 \,. \tag{8.3.1}$$

*Then for all* $\mathcal{E}_0 > 0$*, the function* $\mathcal{J}_0 \mapsto \Theta(\mathcal{E}_0, \mathcal{J}_0) \in [0, \pi/2]$ *satisfies* $\Theta(\mathcal{E}_0, 0) = 0$ *and is strictly monotonic:* $\partial_{\mathcal{J}_0}\Theta > 0$ *for all* $\mathcal{J}_0 \in (0,1)$*. Moreover, it satisfies*

$$\lim_{\mathcal{J}_0 \to 0} \partial_{\mathcal{J}_0}\Theta \in (0, \infty] \quad \textit{and} \quad \lim_{\mathcal{J}_0 \to 1} \partial_{\mathcal{J}_0}\Theta = 0 \,.$$

*Proof.* — An energy $\mathcal{E}_0 > 0$ being fixed, the limiting values $\Theta(\mathcal{E}_0, 0) = 0$ and $\Theta(\mathcal{E}_0, 1) = \pi/2$ are found by direct computation. To prove monotonicity, the main idea of [**39**] is to use the change of variable

$$\sin^2\varphi := \frac{4\Phi(\rho)}{\mathcal{E}_0} + \frac{\mathcal{J}_0^2}{\rho^2}$$

which yields

$$\Theta(\mathcal{E}_0, \mathcal{J}_0) = \arcsin \mathcal{J}_0 + \int_{\arcsin \mathcal{J}_0}^{\frac{\pi}{2}} \frac{\sin\varphi}{\frac{\mathcal{J}_0}{\rho} - \frac{2\rho\Phi'(\rho)}{\mathcal{E}_0\mathcal{J}_0}}\, d\varphi \,.$$

Computing the derivative of this expression with respect to $\mathcal{J}_0$ gives

$$\begin{aligned} \frac{\partial\Theta}{\partial\mathcal{J}_0}(\mathcal{E}_0, \mathcal{J}_0) = &\frac{1}{\sqrt{1-\mathcal{J}_0^2}}\left(1 - \frac{\mathcal{E}_0\mathcal{J}_0^2}{\mathcal{E}_0\mathcal{J}_0^2 - \Phi'(1)}\right) \\ &+ \int_{\arcsin \mathcal{J}_0}^{\frac{\pi}{2}} \frac{\mathcal{E}_0^2\mathcal{J}_0^2\rho^4\sin\varphi}{(\mathcal{J}_0^2\mathcal{E}_0 - \rho^3\Phi'(\rho))^3}\left(\rho\Phi''(\rho) + 2\Phi'(\rho) + \frac{\rho^3}{\mathcal{E}_0\mathcal{J}_0^2}(\Phi'(\rho))^2\right) d\varphi \end{aligned}$$

where $\varphi$ is defined by

$$\sin^2\varphi = \frac{\mathcal{J}_0^2}{\rho^2} + \frac{2\Phi(\rho)}{\mathcal{E}_0} \cdot$$

In view of the formula giving $\partial_{\mathcal{J}_0}\Theta$, it turns out assumption (8.3.1) implies $\partial_{\mathcal{J}_0}\Theta > 0$ for all $\mathcal{J}_0 \in (0,1)$, and also the limits

$$\lim_{\mathcal{J}_0 \to 0} \partial_{\mathcal{J}_0}\Theta \in (0, \infty] \quad \text{and} \lim_{\mathcal{J}_0 \to 1} \partial_{\mathcal{J}_0}\Theta = 0$$

as soon as $\Phi'(1) = 0$ (if not then $\lim\limits_{\mathcal{J}_0 \to 1} \partial_{\mathcal{J}_0}\Theta = \infty$). The result follows. □

***Remark 8.3.2.*** — *Note that one can construct examples that violate assumption* (8.3.1) *and for which monotonicity fails, regardless of convexity properties of the potential* $\Phi$ *(*[**39**]*).*

By Lemma 8.3.1, for each $\mathcal{E}_0$ we can locally invert the map $\Theta(\mathcal{E}_0, \cdot)$, and thus define $\mathcal{J}_0$ as a smooth function of $\mathcal{E}_0$ and $\Theta$. This enables us to define a scattering cross-section (or *collision kernel*), as follows.

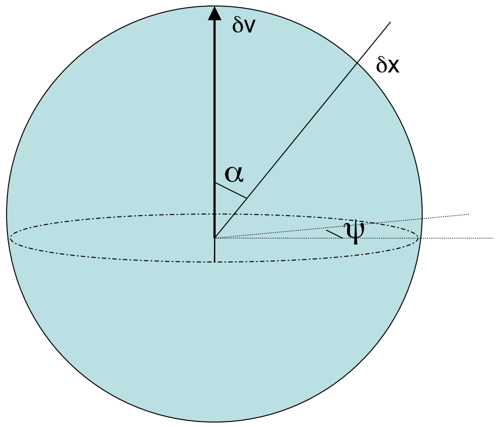


FIGURE 5. Spherical coordinates

For fixed $x_1$, we denote $d\sigma_1$ the surface measure on the sphere $\{y \in \mathbf{R}^d, |y - x_1| = \varepsilon\}$, to which $x_2$ belongs. We can parametrize the sphere by $(\alpha, \psi)$, with $\psi \in \mathbf{S}_1^{d-2}$, where $\alpha$ is the angle defined in (8.2.1). There holds

$$d\sigma_1 = \varepsilon^{d-1} (\sin\alpha)^{d-2} d\alpha d\psi\,.$$

The direction of the apse line is $\omega = (\Theta, \psi)$, so that, denoting $d\omega$ the surface measure on the unit sphere, there holds

$$d\omega = (\sin\Theta)^{d-2} d\Theta d\psi\,. \tag{8.3.2}$$

By definition of $\alpha$ in (8.2.1), there holds

$$(x_1 - x_2) \cdot (v_1 - v_2) = \varepsilon |v_1 - v_2| \cos\alpha\,,$$

so that

$$\begin{aligned}\frac{1}{\varepsilon}\,(x_1 - x_2) \cdot (v_1 - v_2)\, d\sigma_1 &= \varepsilon^{d-1} |v_1 - v_2| \cos\alpha\, (\sin\alpha)^{d-2}\, d\alpha d\psi \\ &= \varepsilon^{d-1} |v_1 - v_2|\, \mathcal{J}_0^{d-2} d\mathcal{J}_0 d\psi\,,\end{aligned}$$

where in the second equality we used the definition of $\mathcal{J}_0$ in (8.2.1). This gives

$$\frac{1}{\varepsilon}\,(x_1 - x_2) \cdot (v_1 - v_2)\, d\sigma_1 = \varepsilon^{d-1} |v_1 - v_2| \mathcal{J}_0^{d-2} \partial_\Theta \mathcal{J}_0\, d\Theta d\psi\,, \tag{8.3.3}$$

wherever $\partial_\Theta \mathcal{J}_0$ is defined, that is, according to Lemma 8.3.1, for $\mathcal{J}_0 \in [0, 1)$.

***Definition 8.3.3***. — *The scattering cross-section is defined for $|w| > 0$ and $\Theta \in (0, \pi/2]$ by $\mathcal{J}_0^{d-2} \partial_\Theta \mathcal{J}_0 (\sin\Theta)^{2-d}$. In the following we shall use the notation*

$$b(w, \Theta) := |w| \mathcal{J}_0^{d-2} \partial_\Theta \mathcal{J}_0 (\sin\Theta)^{2-d}\,, \tag{8.3.4}$$

*and abusing notation we shall write $b(w, \Theta) = b(w, \omega)$.*

By Lemma 8.3.1, the cross-section $b$ is a locally bounded function of the relative velocities and scattering angle.

**8.3.2. Scattering cross-section.** — The relevance of $b$ is made clear in the derivation of the Boltzmann hierarchy, where we shall use the identity

$$\frac{1}{\varepsilon}(x_1 - x_2) \cdot (v_1 - v_2)\, d\sigma_1 = \varepsilon^{d-1} b(v_1 - v_2, \omega) d\omega \,, \tag{8.3.5}$$

derived from (8.3.2), (8.3.3) and Definition 8.3.3. As in Chapter 4 (see in particular Paragraph 4.4), we can formally derive the Boltzmann collision operators using this formulation: we thus define

$$\begin{aligned} \mathcal{C}^0_{s,s+1} f^{(s+1)}(t, Z_s) &:= \sum_{i=1}^{s} \int \mathbb{1}_{\nu\cdot(v_{s+1}-v_i)>0}\, \nu \cdot (v_{s+1} - v_i) \\ &\times \Big( f^{(s+1)}(t, x_1, v_1, \dots, x_i, v_i^*, \dots, x_s, v_s, x_i, v_{s+1}^*) - f^{(s+1)}(t, Z_s, x_i, v_{s+1}) \Big) d\nu dv_{s+1} \,, \end{aligned} \tag{8.3.6}$$

where $(v_i^*, v_{s+1}^*)$ is obtained from $(v_i, v_{s+1})$ by applying the inverse scattering operator $\sigma_0^{-1}$:

$$\sigma_0^{-1}\big(\nu, v_i, v_{s+1}\big) = \big(\nu, v_i^*, v_{s+1}^*\big) \,.$$

This can also be written using the cross-section:

$$\begin{aligned} \mathcal{C}^0_{s,s+1} f^{(s+1)}(t, Z_s) &:= \sum_{i=1}^{s} \int b(v_1 - v_2, \omega) \\ &\times \Big( f^{(s+1)}(t, x_1, v_1, \dots, x_i, v_i^*, \dots, x_s, v_s, x_i, v_{s+1}^*) - f^{(s+1)}(t, Z_s, x_i, v_{s+1}) \Big) d\omega dv_{s+1} \,. \end{aligned} \tag{8.3.7}$$

***Remark 8.3.4***. — *It is not possible to define an integrable cross-section if the potential is not compactly supported, no matter how fast it might be decaying. This issue is related to the occurrence of grazing collisions and discussed in particular in* [**46**]*, Chapter* 1*, Section* 1.4*. However it is still possible to study the limit towards the Boltzmann equation, if one is ready to change the formulation of the Boltzmann equation by renouncing to the cross-section formulation* ([**39**]).

*The question of the convergence to Boltzmann in the case of long-range potentials is a challenging open problem; it was considered by L. Desvillettes and M. Pulvirenti in* [**16**] *and L. Desvillettes and V. Ricci in* [**17**].

# CHAPTER 9

# TRUNCATED MARGINALS AND THE BBGKY HIERARCHY

Our starting point in this first part is the Liouville equation (1.2.2) satisfied by the $N$-particle distribution function $f_N$. We reproduce here equation (1.2.2):

$$\partial_t f_N + \sum_{1\leq i\leq N} v_i \cdot \nabla_{x_i} f_N - \sum_{1\leq i\neq j\leq N} \frac{1}{\varepsilon}\nabla\Phi\left(\frac{x_i-x_j}{\varepsilon}\right)\cdot\nabla_{v_i} f_N = 0\,. \tag{9.0.1}$$

The arguments of $f_N$ in (9.0.1) are $(t,Z_N)\in\mathbf{R}_+\times\Omega_N$, where we recall that

$$\Omega_N := \Big\{ Z_N \in \mathbf{R}^{2dN}\,,\ \forall i\neq j\,,\ x_i\neq x_j\Big\}\,.$$

As recalled in Part II, Chapter 4, the classical strategy to obtain a kinetic equation is to write the evolution equation for the first marginal of the distribution function $f_N$, namely

$$f_N^{(1)}(t,z_1) := \int_{\mathbf{R}^{2d(N-1)}} f_N(t,z_1,z_2,\dots,z_N)\,dz_2\dots dz_N\,,$$

which leads to the study of the hierarchy of equations involving all the marginals of $f_N$

$$f_N^{(s)}(t,Z_s) := \int_{\mathbf{R}^{2d(N-s)}} f_N(t,Z_s,z_{s+1},\dots,z_N)\,dz_{s+1}\cdots dz_N\,. \tag{9.0.2}$$

In Section 9.1 it is shown that due to the presence of the potential, and contrary to the hard-spheres case described in Paragraph 4.2, it is necessary to truncate those marginals away from the set $\Omega_N$. An equation for the *truncated marginals* is derived in weak form in Section 9.2. In order to introduce adequate collision operators, the notion of cluster is introduced and described in Section 9.3, following the work of F. King [**30**]. Then collision operators are introduced in Section 9.4, and finally the integral formulation of the equation is written in Section 9.5.

## 9.1. Truncated marginals

From (9.0.1), we deduce by integration that the *untruncated marginals* defined in (9.0.2) solve

$$\begin{aligned}\partial_t f_N^{(s)}(t,Z_s) &+ \sum_{i=1}^s v_i\cdot\nabla_{x_i} f_N^{(s)}(t,Z_s) - \frac{1}{\varepsilon}\sum_{\substack{i,j=1\\ i\neq j}}^s \nabla\Phi\left(\frac{x_i-x_j}{\varepsilon}\right)\cdot\nabla_{v_i} f_N^{(s)}(t,Z_s)\\ &= \frac{N-s}{\varepsilon}\sum_{i=1}^s\int\nabla\Phi\left(\frac{x_i-x_{s+1}}{\varepsilon}\right)\cdot\nabla_{v_i} f_N^{(s+1)}(t,Z_s,z_{s+1})\,dz_{s+1}\,.\end{aligned} \tag{9.1.1}$$

There are several differences between (9.1.1) and the BBGKY hierarchy for hard spheres (4.3.2)-(4.3.3). One is that the transport operator in the left-hand side of (9.1.1) involves a force term. Another is that the integral term in the right-hand side of (9.1.1) involves velocity derivatives. Also, that integral term is a linear integral operator acting on higher-order marginals, just like (4.3.2), but, contrary to (4.3.2), is *not* spatially localized, in the sense that the integral in $x_{s+1}$ is over the whole ball $B(x_i, \varepsilon)$, as opposed to an integral over a sphere in (4.3.2).

This leads us to distinguish spatial configurations in which interactions do take place from spatial configurations in which particles are pairwise at a distance greater than $\varepsilon$, by truncating off the interaction domain $\big\{Z_N,\, |x_i - x_j| \le \varepsilon \text{ for some } i \neq j\big\}$ in the integrals defining the marginals. For the resulting truncated marginals, collision operators will appear as integrals over a piece of the boundary of the interaction domain, just like in the case of hard spheres. The scattering operator of Chapter 8 (Section 8.2) will then play the role that the boundary condition plays in the case of hard spheres in Chapter 4.

Suitable quantities to be studied are therefore not the marginals defined in (9.0.2) but rather the *truncated marginals*

$$\widetilde{f}_N^{(s)}(t, Z_s) := \int_{\mathbf{R}^{2d(N-s)}} f_N(t, Z_s, z_{s+1}, \dots, z_N) \prod_{\substack{i \in \{1,\dots,s\} \\ j \in \{s+1,\dots,N\}}} 1\!\!1_{|x_i - x_j| > \varepsilon}\, dz_{s+1} \cdots dz_N \,, \tag{9.1.2}$$

where $|\cdot|$ denotes the euclidean norm. Notice that

$$(\widetilde{f}_N^{(1)} - f_N^{(1)})(t, z_1) = \int_{\mathbf{R}^{2d(N-1)}} f_N(t, z_1, z_2, \dots, z_N)\big(1 - \prod_{j \in \{2,\dots,N\}} 1\!\!1_{|x_1 - x_j| > \varepsilon}\big)\, dz_2 \cdots dz_N$$

so that

$$\|(\widetilde{f}_N^{(1)} - f_N^{(1)})(t)\|_{L^\infty(\mathbf{R}^{2d})} \le C(N-1)\varepsilon^d \|f_N^{(2)}(t)\|_{L^\infty(\Omega_2)} \,. \tag{9.1.3}$$

We therefore expect both functions to have the same asymptotic behaviour in the Boltzmann-Grad limit $N\varepsilon^{d-1} \equiv 1$. This is indeed proved in Lemma 11.1.2.

Given $1 \le i < j \le N$, we recall that $dZ_{(i,j)}$ denotes the $2d(j-i+1)$-dimensional Lebesgue measure $dz_i dz_{i+1} \dots dz_j$, and $dX_{(i,j)}$ the $d(j-i+1)$-dimensional Lebesgue measure $dx_i dx_{i+1} \dots dx_j$. We also define

$$\mathcal{D}_N^s := \Big\{ X_N \in \mathbf{R}^{dN},\, \forall (i,j) \in [1,s] \times [s+1, N] \,,\, |x_i - x_j| > \varepsilon \Big\} \,, \tag{9.1.4}$$

where $[1,s]$ is short for $[1,s] \cap \mathbf{N} = \{k \in \mathbf{N},\, 1 \le k \le s\}$. Then the truncated marginals (9.1.2) may be formulated as follows:

$$\widetilde{f}_N^{(s)}(t, Z_s) = \int_{\mathbf{R}^{2d(N-s)}} f_N(t, Z_s, z_{s+1}, \dots, z_N) 1\!\!1_{X_N \in \mathcal{D}_N^s}\, dZ_{s+1,N} \,. \tag{9.1.5}$$

The key in introducing the truncated marginals (9.1.5), following King [**30**], is that it allows for a derivation of a hierarchy that is similar to the case of hard spheres. The main drawback is that contrary to the hard-spheres case in (4.2.3), truncated marginals are not actual *marginals*, in the sense that

$$\widetilde{f}_N^{(s)}(Z_s) \neq \int_{\mathbf{R}^{2d}} 1\!\!1_{X_{s+1} \in B} \widetilde{f}_N^{(s+1)}(Z_s, z_{s+1})\, dz_{s+1} \,, \tag{9.1.6}$$

for any $B \subset \mathbf{R}^{d(s+1)}$, in particular if $B = \mathbf{R}^{d(s+1)}$, simply because $\mathcal{D}_N^s$ is *not* included in $\mathcal{D}_N^{s+1}$. Indeed, conditions $|x_j - x_{s+1}| > \varepsilon$, for $j \le s$, hold for $X_N \in \mathcal{D}_N^s$, but not necessarily for $X_N \in \mathcal{D}_N^{s+1}$. Furthermore, $\mathcal{D}_N^s$ intersects all the $\mathcal{D}_N^{s+m}$, for $m \in [1, N-s]$. A consequence is the existence of higher-order interactions between truncated marginals, as seen below in (9.4.8). Proposition 10.3.1 in Chapter 10 states however that these higher-order interactions are negligible in the Boltzmann-Grad limit.

## 9.2. Weak formulation of Liouville's equation

Our goal in this section is to find the weak formulation of the system of equations satisfied by the family of truncated marginals $\big(\widetilde f_N^{(s)}\big)_{s\in[1,N]}$ defined above in (9.1.5). The strategy will be similar to that followed in Chapter 4 in the hard-spheres case. From now on we assume that $f_N$ decays at infinity in the velocity variable.

Given a smooth, compactly supported function $\phi$ defined on $\mathbf{R}_+\times\mathbf{R}^{2ds}$ and satisfying the symmetry assumption (1.1.1), we have

$$\begin{aligned}\int_{\mathbf{R}_+\times\mathbf{R}^{2dN}}\Big(\partial_t f_N+\sum_{i=1}^N v_i\cdot\nabla_{x_i}f_N-\frac1\varepsilon\sum_{i=1}^N\sum_{j\neq i}\nabla\Phi\left(\frac{x_i-x_j}{\varepsilon}\right)\cdot\nabla_{v_i}f_N\Big)(t,Z_N)\\ \times\phi(t,Z_s)1\!\!1_{X_N\in\mathcal{D}_N^s}\,dZ_Ndt=0\,.\end{aligned}\tag{9.2.1}$$

Note that in the above double sum in $i$ and $j$, all the terms vanish except when $(i,j)\in[1,s]^2$ and when $(i,j)\in[s+1,N]^2$, by assumption on the support of $\Phi$.

We now use integrations by parts to derive from (9.2.1) the weak form of the equation in the marginals $\widetilde f_N^{(s)}$. On the one hand an integration by parts in the time variable gives

$$\begin{aligned}\int_{\mathbf{R}_+\times\mathbf{R}^{2dN}}\partial_t f_N(t,Z_N)\phi(t,Z_s)1\!\!1_{X_N\in\mathcal{D}_N^s}\,dZ_Ndt=&-\int_{\mathbf{R}^{2dN}}f_N(0,Z_N)\phi(0,Z_s)1\!\!1_{X_N\in\mathcal{D}_N^s}\,dZ_N\\&-\int_{\mathbf{R}_+\times\mathbf{R}^{2dN}}f_N(t,Z_N)\partial_t\phi(t,Z_s)1\!\!1_{X_N\in\mathcal{D}_N^s}\,dZ_Ndt\,,\end{aligned}$$

hence, by definition of $\widetilde f_N^{(s)}$,

$$\begin{aligned}\int_{\mathbf{R}_+\times\mathbf{R}^{2dN}}\partial_t f_N(t,Z_N)\phi(t,Z_s)1\!\!1_{X_N\in\mathcal{D}_N^s}\,dZ_Ndt=&-\int_{\mathbf{R}^{2ds}}\widetilde f_N^{(s)}(0,Z_s)\phi(0,Z_s)\,dZ_s\\&-\int_{\mathbf{R}_+\times\mathbf{R}^{2ds}}\widetilde f_N^{(s)}(t,Z_s)\partial_t\phi(t,Z_s)\,dZ_sdt\,.\end{aligned}$$

Now let us compute

$$\sum_{i=1}^N\int_{\mathbf{R}^{2dN}}v_i\cdot\nabla_{x_i}f_N(t,Z_N)\phi(t,Z_s)1\!\!1_{X_N\in\mathcal{D}_N^s}\,dZ_N=\int_{\mathbf{R}^{2dN}}\mathrm{div}_{X_N}\big(V_N\,f_N(t,Z_N)\big)\phi(t,Z_s)1\!\!1_{X_N\in\mathcal{D}_N^s}\,dZ_N$$

using Green's formula. The boundary of $\mathcal{D}_N^s$ is made of configurations with at least one pair $(i,j)$, satisfying $1\le i\le s$ and $s+1\le j\le N$, with $|x_i-x_j|=\varepsilon$.

Let us define, for any couple $(i,j)\in[1,N]^2$,

$$\begin{aligned}\Sigma_N^s(i,j):=\Big\{X_N\in\mathbf{R}^{dN},\quad&|x_i-x_j|=\varepsilon\\&\text{and}\quad\forall(k,\ell)\in[1,s]\times[s+1,N]\setminus\{i,j\},\ |x_k-x_\ell|>\varepsilon\Big\}\,.\end{aligned}\tag{9.2.2}$$

We notice that $\Sigma_N^s(i,j)$ is a submanifold of $\big\{X_N\in\mathbf{R}^{dN},\ |x_i-x_j|=\varepsilon\big\}$, which is a smooth, codimension 1 manifold of $\mathbf{R}^{dN}$ (locally isomorphic to the space $\mathbf{S}_\varepsilon^d\times\mathbf{R}^{d(N-1)}$), and we denote by $d\sigma_N^{i,j}$

its surface measure, induced by the Lebesgue measure. Configurations with more than one *collisional* pair, i.e., $(i,j)$ and $(i',j')$ with $1 \leq i,i' \leq s$, $s+1 \leq j,j' \leq N$, with $|x_i - x_j| = |x_{i'} - x_{j'}| = \varepsilon$, and $\{i,j\} \neq \{i',j'\}$, are subsets of submanifols of $\mathbf{R}^{dN}$ of codimension at least two, and therefore contribute nothing to the boundary terms. Denoting $n^{i,j}$ the outward normal to $\Sigma_N^s(i,j)$ we therefore obtain by Green's formula:

$$\begin{aligned}
&\sum_{i=1}^N \int_{\mathbf{R}_+\times\mathbf{R}^{2dN}} v_i\cdot\nabla_{x_i} f_N(t,Z_N)\phi(t,Z_s)\mathbb{1}_{X_N\in\mathcal{D}_N^s}\, dZ_N\, dt \\
&\qquad = -\sum_{i=1}^s \int_{\mathbf{R}_+\times\mathbf{R}^{2dN}} f_N(t,Z_N) v_i\cdot\nabla_{x_i}\phi(t,Z_s)\mathbb{1}_{X_N\in\mathcal{D}_N^s}\, dZ_N dt \\
&\qquad\quad + \sum_{1\leq i\neq j\leq N} \int_{\mathbf{R}_+\times\mathbf{R}^{dN}\times\Sigma_N^s(i,j)} n^{i,j}\cdot V_N\, f_N(t,Z_N)\phi(t,Z_s)\, d\sigma_N^{i,j} dV_N dt\,.
\end{aligned}$$

By symmetry (1.1.1) and recalling that $\nu^{i,j} = (x_i - x_j)/|x_i - x_j|$ this gives

$$\begin{aligned}
&\sum_{i=1}^N \int_{\mathbf{R}_+\times\mathbf{R}^{2dN}} v_i\cdot\nabla_{x_i} f_N(t,Z_N)\phi(t,Z_s)\mathbb{1}_{X_N\in\mathcal{D}_N^s}\, dZ_N\, dt \\
&\quad = -\sum_{i=1}^s \int_{\mathbf{R}_+\times\mathbf{R}^{2dN}} f_N(t,Z_N) v_i\cdot\nabla_{x_i}\phi(t,Z_s)\mathbb{1}_{X_N\in\mathcal{D}_N^s}\, dZ_N dt \\
&\quad\quad + (N-s)\sum_{i=1}^s \int_{\mathbf{R}_+\times\mathbf{R}^{dN}\times\Sigma_N^s(i,j)} \frac{\nu^{i,s+1}}{\sqrt{2}}\cdot(v_{s+1}-v_i)\, f_N(t,Z_N)\phi(t,Z_s)\, d\sigma_N^{i,j} dV_N dt\,,
\end{aligned}$$

so finally by definition of $\widetilde{f}_N^{(s)}$, we obtain

$$\begin{aligned}
&\sum_{i=1}^N \int_{\mathbf{R}_+\times\mathbf{R}^{2dN}} v_i\cdot\nabla_{x_i} f_N(t,Z_N)\phi(t,Z_s)\mathbb{1}_{X_N\in\mathcal{D}_N^s}\, dZ_N\, dt \\
&\quad = -\sum_{i=1}^s \int_{\mathbf{R}_+\times\mathbf{R}^{2ds}} \widetilde{f}_N^{(s)}(t,Z_s) v_i\cdot\nabla_{x_i}\phi(t,Z_s)\, dZ_s dt \\
&\quad\quad + (N-s)\sum_{i=1}^s \int_{\mathbf{R}_+\times\mathbf{R}^{dN}\times\Sigma_N^s(i,j)} \frac{\nu^{i,s+1}}{\sqrt{2}}\cdot(v_{s+1}-v_i)\, f_N(t,Z_N)\phi(t,Z_s)\, d\sigma_N^{i,j} dV_N dt\,.
\end{aligned}\tag{9.2.3}$$

Now let us consider the contribution of the potential in (9.2.1). We split the sum as follows:

$$\begin{aligned}
&\frac{1}{\varepsilon}\sum_i\sum_{j\neq i} \int_{\mathbf{R}_+\times\mathbf{R}^{2dN}} \nabla\Phi\left(\frac{x_i-x_j}{\varepsilon}\right)\cdot\nabla_{v_i} f_N(t,Z_N)\phi(t,Z_s)\mathbb{1}_{X_N\in\mathcal{D}_N^s}\, dZ_N dt \\
&\quad = \frac{1}{\varepsilon}\sum_{\substack{i,j=1\\ j\neq i}}^s \int_{\mathbf{R}_+\times\mathbf{R}^{2dN}} \nabla\Phi\left(\frac{x_i-x_j}{\varepsilon}\right)\cdot\nabla_{v_i} f_N(t,Z_N)\phi(t,Z_s)\mathbb{1}_{X_N\in\mathcal{D}_N^s}\, dZ_N dt \\
&\quad\quad + \frac{1}{\varepsilon}\sum_{\substack{i,j=s+1\\ j\neq i}}^N \int_{\mathbf{R}_+\times\mathbf{R}^{2dN}} \nabla\Phi\left(\frac{x_i-x_j}{\varepsilon}\right)\cdot\nabla_{v_i} f_N(t,Z_N)\phi(t,Z_s)\mathbb{1}_{X_N\in\mathcal{D}_N^s}\, dZ_N dt\,.
\end{aligned}$$

We notice that the second term in the right-hand side vanishes identically. It follows that

$$\frac{1}{\varepsilon}\sum_i\sum_{j\neq i}\int_{\mathbf{R}_+\times\mathbf{R}^{2dN}}\nabla\Phi\left(\frac{x_i-x_j}{\varepsilon}\right)\cdot\nabla_{v_i}f_N(t,Z_N)\phi(t,Z_s)\mathbb{1}_{X_N\in\mathcal{D}_N^s}\,dZ_Ndt$$
$$=-\frac{1}{\varepsilon}\sum_{\substack{i,j=1\\ j\neq i}}^s\int_{\mathbf{R}_+\times\mathbf{R}^{2ds}}\nabla\Phi\left(\frac{x_i-x_j}{\varepsilon}\right)\cdot\nabla_{v_i}\phi(t,Z_s)\widetilde{f}_N^{(s)}(t,Z_s)\,dZ_sdt$$

so in the end we obtain

$$\int_{\mathbf{R}_+\times\mathbf{R}^{2ds}}\widetilde{f}_N^{(s)}(t,Z_s)\Big(\partial_t\phi+\mathrm{div}_{X_s}\,(V_s\phi)-\frac{1}{\varepsilon}\sum_{\substack{i,j=1\\ j\neq i}}^s\nabla\Phi\left(\frac{x_i-x_j}{\varepsilon}\right)\cdot\nabla_{v_i}\phi\Big)(t,Z_s)\,dZ_sdt$$
$$(9.2.4)\qquad=-\int_{\mathbf{R}^{2ds}}\widetilde{f}_N^{(s)}(0,Z_s)\phi(0,Z_s)\,dZ_s$$
$$-(N-s)\sum_{i=1}^s\int_{\mathbf{R}_+\times\mathbf{R}^{dN}\times\Sigma_N^s(i,s+1)}\frac{\nu^{i,s+1}}{\sqrt{2}}\cdot(v_{s+1}-v_i)\,f_N(t,Z_N)\phi(t,Z_s)\,d\sigma_N^{i,s+1}dV_Ndt\,.$$

***Remark 9.2.1***. — *Using the weak form of Liouville's equation, we see that configurations in which there would be two pre-or post-collisional pairs, can be neglected (they occur as a boundary integral on a zero measure subset of $\partial\mathcal{D}_N^s$) .*

## 9.3. Clusters

We want to analyze the second term on the right-hand side of (9.2.4). We notice that in the space integration the variables $x_{s+2},\dots,x_N$ are integrated over $\mathbf{R}^{d(N-s-1)}$ (with the restriction that they must be at a distance at least $\varepsilon$ from $X_s$) whereas $x_{s+1}$ must lie in the sphere centered at $x_i$ and of radius $\varepsilon$. It is therefore natural to try to express that contribution in terms of the marginal $\widetilde{f}_N^{(s+1)}(Z_{s+1})$. However as pointed out in (9.1.6),

$$\int\widetilde{f}_N^{(s+1)}(Z_{s+1})\,dz_{s+1}\neq\widetilde{f}_N^{(s)}(Z_s)\,.$$

The difference between those two terms is that on the one hand

$$\forall X_N\in\mathcal{D}_N^{s+1}\,,\qquad\text{one has }|x_j-x_{s+1}|>\varepsilon\text{ for all }j\geq s+2\,,$$

which is not the case for $X_N\in\mathcal{D}_N^s$, and on the other hand

$$\forall X_N\in\mathcal{D}_N^{s}\,,\qquad\text{one has }|x_j-x_{s+1}|>\varepsilon\text{ for all }j\leq s\,,$$

a condition which does not appear in the definition of $\mathcal{D}_N^{s+1}$.

This leads to the following definition.

***Definition 9.3.1*** (**$\varepsilon$-closure**). — *Given a subset $X_N=\{x_1,\dots,x_N\}$ of $\mathbf{R}^{dN}$ and an integer $s$ in $[1,N]$, the $\varepsilon$-closure $E(X_s,X_N)$ of $X_s$ in $X_N$ is defined as the intersection of all subsets $Y$ of $X_N$ which contain $X_s$ and satisfy the separation condition*

$$(9.3.1)\qquad\forall y\in Y\,,\quad\forall x\in X_N\setminus Y\,,\quad|x-y|>\varepsilon\,.$$

*We denote $|E(X_s,X_N)|$ the cardinal of $E(X_s,X_N)$.*

Now let us introduce the following notation, useful in situations where $X_N$ belongs to $\Sigma_N^s(i,s+1)$, defined in (9.2.2).

***Notation 9.3.2.*** — *If $X_{s+m} = E(X_s, X_{s+m})$ and if for some integers $j_0 \le s < k_0 \le s+m$, there holds $|x_j - x_k| > \varepsilon$ for all $(j,k) \in [1,s] \times [s+1, s+m] \setminus \{(j_0,k_0)\}$, then we say that $E(X_s, X_{s+m})$ has a weak link at $(j_0,k_0)$, and we denote $X_{s+m} = E_{\langle j_0,k_0\rangle}(X_s, X_{s+m})$.*

Moreover the following notion, following King [**30**], will turn out to be very useful.

***Definition 9.3.3*** **(Cluster)**. — *A cluster of base $X_s = \{x_1, \dots, x_s\}$ and length $m$ is any point $\{x_{s+1}, \dots, x_{s+m}\}$ in $\mathbf{R}^{dm}$ such that $E(X_s, X_{s+m}) = X_{s+m}$. We denote $\Delta_m(X_s)$ the set of all such clusters.*

The proof of the following lemma is completely elementary.

***Lemma 9.3.4.*** — *The following equivalences hold, for $m \ge 1$ :*

$$\Big(E(X_s, X_N) = X_{s+m}\Big) \iff \Big(E(X_s, X_{s+m}) = X_{s+m} \text{ and } X_N \in \mathcal{D}_N^{s+m}\Big)\,, \tag{9.3.2}$$

$$\begin{pmatrix} E(X_s, X_N) = X_{s+m} \\ X_N \in \Sigma_N^s(i, s+1) \end{pmatrix} \iff \begin{pmatrix} E_{\langle i,s+1\rangle}(X_s, X_{s+m}) = X_{s+m} \\ X_N \in \mathcal{D}_N^{s+m} \\ |x_i - x_{s+1}| = \varepsilon \end{pmatrix}, \tag{9.3.3}$$

*as well as the implication, for $m \ge 2$,*

$$\Big(E_{\langle i,s+1\rangle}(X_s, X_{s+m}) = X_{s+m}\Big) \implies \Big(\{x_{s+2}, \dots, x_{s+m}\} \in \Delta_{m-1}(x_{s+1})\Big)\,. \tag{9.3.4}$$

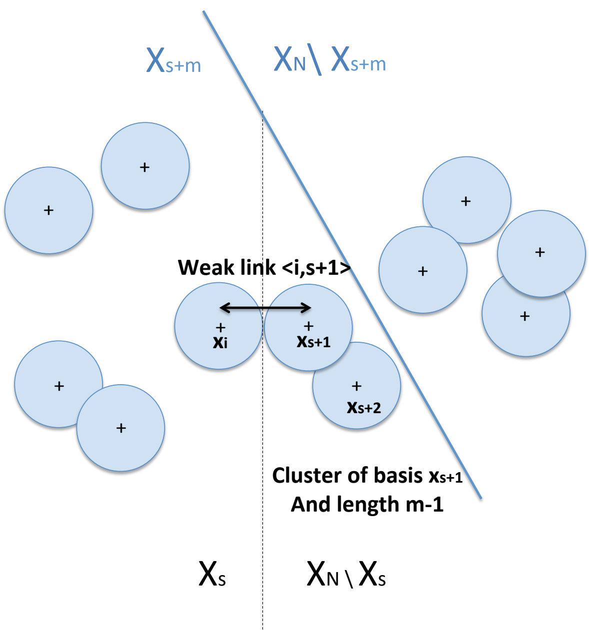


FIGURE 6. Clusters with weak links

## 9.4. Collision operators

With the help of the notions introduced in Section 9.3, we now can reformulate the boundary integral in (9.2.4).

Given $1 \leq s \leq N-1$ and $X_N$ in $\Sigma_N^s(i,s+1)$, there holds $|x_{s+1}-x_i| = \varepsilon$, so that $x_{s+1}$ belongs to $E(X_s,X_N)$, implying $|E(X_s,X_N)| \geq s+1$. We decompose $\Sigma_N^s(i,s+1)$ into a disjoint union over the possible cardinals of the $\varepsilon$-closure of $X_s$ in $X_N$ :

$$\Sigma_N^s(i,s+1) = \bigcup_{1\leq m\leq N-s} \Big(\Sigma_N^s(i,s+1) \bigcap \big\{Y_N,\ |E(Y_s,Y_N)| = s+m\big\}\Big)\,, \tag{9.4.1}$$

implying

$$\begin{aligned}&\int_{\mathbf{R}^{dN}\times\Sigma_N^s(i,s+1)} \nu^{i,s+1}\cdot(v_{s+1}-v_i)\, f_N(Z_N)\phi(Z_s)\, d\sigma_N^{i,s+1} dV_N \\ &= \sum_{1\leq m\leq N-s} \int_{\mathbf{R}^{dN}\times\Sigma_N^s(i,s+1)} 1\!\!1_{|E(X_s,X_N)|=s+m}\, \nu^{i,s+1}\cdot(v_{s+1}-v_i)\, f_N(Z_N)\phi(Z_s)\, d\sigma_N^{i,s+1} dV_N\,.\end{aligned}$$

By assumption of symmetry (1.1.1) for $f_N$ and $\phi$, if $|E(X_s,X_N)| = s+m$, we can index the particles so that $E(X_s,X_N) = X_{s+m}$ : we obtain

$$\begin{aligned}&\int_{\mathbf{R}^{dN}\times\Sigma_N^s(i,s+1)} 1\!\!1_{|E(X_s,X_N)|=s+m}\, \nu^{i,s+1}\cdot(v_{s+1}-v_i)\, f_N(Z_N)\phi(Z_s)\, d\sigma_N^{i,s+1} dV_N \\ &\quad = C_{N-s-1}^{m-1} \int_{\mathbf{R}^{dN}\times\Sigma_N^s(i,s+1)} 1\!\!1_{E(X_s,X_N)=X_{s+m}} \nu^{i,s+1}\cdot(v_{s+1}-v_i)\, f_N(Z_N)\phi(Z_s)\, d\sigma_N^{i,s+1} dV_N\,.\end{aligned} \tag{9.4.2}$$

We use equivalence (9.3.3) from Lemma 9.3.4 and Fubini's theorem to write

$$\begin{aligned}&\int_{\mathbf{R}^{dN}\times\Sigma_N^s(i,s+1)} 1\!\!1_{E(X_s,X_N)=X_{s+m}} \nu^{i,s+1}\cdot(v_{s+1}-v_i) f_N(Z_N)\phi(Z_s) d\sigma_N^{i,s+1} dV_N \\ &\quad = \sqrt{2}\int_{S_\varepsilon(x_i)\times\mathbf{R}^d} \nu^{i,s+1}\cdot(v_{s+1}-v_i)\phi(Z_s) \\ &\qquad\qquad \times\Big(\int_{\mathbf{R}^{2d(m-1)}} 1\!\!1_{E_{\langle i,s+1\rangle}(X_s,X_{s+m})=X_{s+m}} f_N^{(s+m)}(Z_{s+m}) dZ_{(s+1,s+m)}\Big)\, d\sigma_i(x_{s+1})\,,\end{aligned}$$

with $d\sigma_i$ the surface measure on $S_\varepsilon(x_i) := \big\{x\in\mathbf{R}^d,\ |x-x_i| = \varepsilon\big\}$. With (9.3.4), if $m\geq 2$, then the above integral over $\mathbf{R}^{2d(m-1)}$ appears as an integral over $\Delta_{m-1}(x_{s+1})$. We also remark that in the case $m=1$, we have a simple description of $E_{\langle i,s+1\rangle}(X_s,X_{s+1}) = X_{s+1}$ :

$$\Big(1\!\!1_{E_{\langle i,s+1\rangle}(X_s,X_{s+1})=X_{s+1}} \neq 0\Big) \iff \begin{pmatrix} |x_i-x_{s+1}|\leq\varepsilon & \\ |x_j-x_{s+1}|>\varepsilon & \text{for } j\in[1,s]\setminus\{i\}\end{pmatrix}. \tag{9.4.3}$$

This leads to the following definition of the collision term of order $m\geq 1$, for $s+m\leq N$ : we define

$$\begin{aligned}\mathcal{C}_{s,s+m}\widetilde{f}_N^{(s+m)}(Z_s) := m C_{N-s}^m \sum_{i=1}^s \int_{S_\varepsilon(x_i)\times\mathbf{R}^d} & \nu^{s+1,i}\cdot(v_{s+1}-v_i) \\ & \times G_{\langle i,s+1\rangle}^{(m-1)}(f_N^{(s+m)})(Z_{s+1})\, d\sigma_i(x_{s+1}) dv_{s+1}\,,\end{aligned} \tag{9.4.4}$$

where for $m=1$, by (9.4.3):

$$G_{\langle i,s+1\rangle}^{(0)}(\widetilde{f}_N^{(s+1)})(Z_{s+1}) := \Big(\prod_{\substack{1\leq j\leq s\\ j\neq i}} 1\!\!1_{|x_{s+1}-x_j|>\varepsilon}\Big)\widetilde{f}_N^{(s+1)}(Z_{s+1})\,, \tag{9.4.5}$$

and for $m \geq 2$ :

$$
\begin{aligned}
&G^{(m-1)}_{\langle i,s+1\rangle}(\widetilde{f}^{(s+m)}_N)(Z_{s+1}) \\
&\qquad := \int_{\Delta_{m-1}(x_{s+1})\times\mathbf{R}^{d(m-1)}} \mathbb{1}_{E_{\langle i,s+1\rangle}(X_s,X_{s+m})=X_{s+m}} \widetilde{f}^{(s+m)}_N(Z_{s+m})dZ_{(s+2,s+m)}\,.
\end{aligned} \tag{9.4.6}
$$

The complex-looking indicator function $\mathbb{1}_{E_{\langle i,s+1\rangle}(X_s,X_{s+m})=X_{s+m}}$ will, in the estimates of the next chapters, be simply bounded from above by one. This will be the case for instance in an estimate showing that higher-order collision operators (9.4.6) are negligible in the thermodynamical limit; this estimate is (10.3.2) in Proposition 10.3.1. One should notice on the other hand that the operator $\mathcal{C}_{s,s+1}$ is very similar to the corresponding collision operator (4.3.2) in the hard-spheres situation.

With $(N-s)C^{m-1}_{N-s-1} = mC^m_{N-s}$, we can now reformulate (9.2.4) into

$$
\begin{aligned}
&\int_{\mathbf{R}_+\times\mathbf{R}^{2ds}} \widetilde{f}^{(s)}_N(t,Z_s)\Big(\partial_t\phi + \mathrm{div}_{X_s}\,(V_s\phi) - \frac{1}{\varepsilon}\sum_{\substack{i,j=1\\ j\neq i}}^{s} \nabla\Phi\left(\frac{x_i-x_j}{\varepsilon}\right)\cdot\nabla_{v_i}\phi\Big)(t,Z_s)\,dZ_s dt \\
&\qquad + \int_{\mathbf{R}^{2ds}} \widetilde{f}^{(s)}_N(0,Z_s)\phi(0,Z_s)\,dZ_s = \sum_{m=1}^{N-s}\int_{\mathbf{R}^+\times\mathbf{R}^{2ds}} \phi(t,Z_s)\mathcal{C}_{s,s+m}\widetilde{f}^{(s+m)}_N(t,Z_s)\,dtdZ_s\,,
\end{aligned} \tag{9.4.7}
$$

so that $\widetilde{f}^{(s)}_N$ appears as a (formal) weak solution to

$$
\partial_t\widetilde{f}^{(s)}_N + \sum_{1\leq i\leq s} v_i\cdot\nabla_{x_i}\widetilde{f}^{(s)}_N - \frac{1}{\varepsilon}\sum_{1\leq i\neq j\leq s}\nabla\Phi\left(\frac{x_i-x_j}{\varepsilon}\right)\cdot\nabla_{v_i}\widetilde{f}^{(s)}_N = \sum_{m=1}^{N-s}\mathcal{C}_{s,s+m}\widetilde{f}^{(s+m)}_N\,. \tag{9.4.8}
$$

## 9.5. Mild solutions

We now define the integral formulation of (9.4.8). Denote by $\mathbf{\Phi}_s(t)$ the $s$-particle Hamiltonian flow, and by $\mathbf{H}_s$ the associated solution operator:

$$
\mathbf{H}_s(t): \qquad f\in C^0(\Omega_s;\mathbf{R})\mapsto f(\mathbf{\Phi}_s(-t,\cdot))\in C^0(\Omega_s;\mathbf{R})\,. \tag{9.5.1}
$$

The time-integrated form of equation (9.4.8) is

$$
\widetilde{f}^{(s)}_N(t,Z_s) = \mathbf{H}_s(t)\widetilde{f}^{(s)}_N(0,Z_s) + \sum_{m=1}^{N-s}\int_0^t \mathbf{H}_s(t-\tau)\mathcal{C}_{s,s+m}\widetilde{f}^{(s+m)}_N(\tau,Z_s)\,d\tau\,. \tag{9.5.2}
$$

The *total flow* and *total collision* operators $\mathbf{H}$ and $\mathbf{C}_N$ are defined on finite sequences $G_N = (g_s)_{1\leq s\leq N}$ as follows:

$$
\begin{cases}
\forall s\leq N\,,\ \big(\mathbf{H}(t)G_N\big)_s := \mathbf{H}_s(t)g_s\,,\\
\forall\, s\leq N-1\,,\ \big(\mathbf{C}_N G_N\big)_s := \displaystyle\sum_{m=1}^{N-s}\mathcal{C}_{s,s+m}g_{s+m}\,, \quad \big(\mathbf{C}_N G_N\big)_N := 0\,.
\end{cases} \tag{9.5.3}
$$

We define *mild solutions* to the BBGKY hierarchy (9.5.2) to be solutions of

$$
\widetilde{F}_N(t) = \mathbf{H}(t)\widetilde{F}_N(0) + \int_0^t \mathbf{H}(t-\tau)\mathbf{C}_N\widetilde{F}_N(\tau)\,d\tau\,, \qquad \widetilde{F}_N = (\widetilde{f}^{(s)}_N)_{1\leq s\leq N}\,. \tag{9.5.4}
$$

***Remark 9.5.1***. — *At this stage, the use of weak formulations could seem a little bit suspicious since they are used essentially as a technical artifice to go from the Liouville equation* (1.2.2) *to the mild form of the BBGKY hierarchy* (9.5.2)*. In particular, this allows to ignore pathological trajectories as mentioned in Remark* 9.2.1*. Nevertheless, the existence of mild solutions to the BBGKY hierarchy* *provides the existence of weak solutions to the BBGKY hierarchy, and in particular to the Liouville equation (which is nothing else than the last equation of the hierarchy). The classical uniqueness result for kinetic transport equations then implies that the object we consider, that is the family of truncated marginals, is uniquely determined (almost everywhere).*

## 9.6. The limiting Boltzmann hierarchy

The limit of the BBGKY collision operators (9.4.4) was obtained formally in Section 8.3.2, following the formal derivation of the hard-spheres case in Paragraph 4.4, assuming higher-order interactions can be neglected. We recall the form of the collision operator as given in (8.3.7):

$$\begin{aligned}\mathcal{C}^0_{s,s+1} f^{(s+1)}(t, Z_s) := \sum_{i=1}^{s} \int & b(v_1 - v_2, \omega) \\ \times \Big( f^{(s+1)}(t, x_1, v_1, \dots, x_i, v_i^*, \dots, x_s, v_s, x_i, v_{s+1}^*) & - f^{(s+1)}(t, Z_s, x_i, v_{s+1}) \Big) d\omega dv_{s+1} \, .\end{aligned}$$

where $(v_i^*, v_{s+1}^*)$ is obtained from $(v_i, v_{s+1})$ by applying the inverse scattering operator $\sigma_0^{-1}$ defined in Definition 8.2.1 and $b(w, \omega)$ is the cross-section given by Definition 8.3.3.

The asymptotic dynamics are therefore governed by the following integral form of the Boltzmann hierarchy:

$$f^{(s)}(t) = \mathbf{S}_s(t) f_0^{(s)} + \int_0^t \mathbf{S}_s(t-\tau) \mathcal{C}^0_{s,s+1} f^{(s+1)}(\tau) \, d\tau \, , \tag{9.6.1}$$

where $\mathbf{S}_s(t)$ denotes the $s$-particle free-flow.

Similarly to (4.3.7), we can define the total Boltzmann flow and collision operators $\mathbf{S}$ and $\mathbf{C}$ as follows:

$$\begin{cases} \forall s \geq 1 \, , \ \big(\mathbf{S}(t) G\big)_s := \mathbf{S}_s(t) g_s \, , \\ \forall \, s \geq 1 \, , \ \big(\mathbf{C}^{\mathbf{0}} G\big)_s := \mathcal{C}^0_{s,s+1} g_{s+1} \, , \end{cases} \tag{9.6.2}$$

so that *mild solutions* to the Boltzmann hierarchy (9.6.1) are solutions of

$$F(t) = \mathbf{S}(t) F(0) + \int_0^t \mathbf{S}(t-\tau) \mathbf{C}^{\mathbf{0}} F(\tau) \, d\tau \, , \qquad F = (f^{(s)})_{s \geq 1} \, . \tag{9.6.3}$$

Note that if $f^{(s)}(t, Z_s) = \displaystyle\prod_{i=1}^{s} f(t, z_i)$ (meaning $f^{(s)}(t)$ is *tensorized*) then $f$ satisfies the Boltzmann equation (2.1.1)-(2.1.2), with the cross-section $b(w, \omega)$ given by Definition 8.3.3.

# CHAPTER 10

# CLUSTER ESTIMATES AND UNIFORM A PRIORI ESTIMATES

In view of proving the existence of mild solutions to the BBGKY hierarchy (9.5.2), we need continuity estimates on the linear collision operators $\mathcal{C}_{s,s+m}$ defined in (9.4.4)-(9.4.5)-(9.4.6), and the total collision operator $\mathbf{C}_N$ defined in (9.5.3).

We first note that, by definition, the operator $\mathcal{C}_{s,s+m}$ involves only configurations with clusters of length $m$. Classical computations of statistical mechanics, presented in Section 10.1, show that the probability of finding such clusters is exponentially decreasing with $m$.

It is then natural to introduce functional spaces encoding the decay with respect to energy and the growth with respect to the order of the marginal (see Section 10.2, where norms are introduced, generalizing the norms introduced in Chapter 5 for the hard spheres case). In these appropriate functional spaces, we can establish uniform continuity estimates for the BBGKY collision operators (Section 10.3). These will enable us in Section 10.4 to obtain directly uniform bounds for the hierarchy as in Chapter 5.

## 10.1. Cluster estimates

A point $X_s \in \mathbf{R}^{ds}$ being given, we recall that $\Delta_m(X_s)$ is the set of all clusters of base $X_s$ and length $m$ (this notation is introduced in Definition 9.3.3 page 76).

***Lemma 10.1.1***. — *For any symmetric function $\varphi$ on $\mathbf{R}^{Nd}$, any $s \in [1, N-1]$, any $X_s \in \mathbf{R}^{ds}$, the following identity holds:*

$$\begin{aligned}\int_{\mathbf{R}^{(N-s)d}} \varphi(X_N) dX_{(s+1,N)} &= \int_{\mathbf{R}^{d(N-s)}} \mathbb{1}_{X_N \in \mathcal{D}_N^s} \varphi(X_N)\, dX_{(s+1,N)} \\ &+ \sum_{m=1}^{N-s} C_{N-s}^m \int_{\Delta_m(X_s)} \Big( \int_{\mathbf{R}^{d(N-s-m)}} \mathbb{1}_{X_N \in \mathcal{D}_N^{s+m}} \varphi(X_N)\, dX_{(s+m+1,N)} \Big) dX_{(s+1,s+m)} \,,\end{aligned} \tag{10.1.1}$$

*implying, for $\zeta > 0$,*

$$\frac{1}{m!} \int_{\Delta_m(X_s)} dX_{(s+1,s+m)} \leq \zeta^{-m} \exp\big(\zeta \kappa_d (s+m) \varepsilon^d\big) \tag{10.1.2}$$

*and*

$$\sum_{m\geq 1} \frac{\zeta^{m+1} \exp\big(-\zeta\kappa_d(m+1)\varepsilon^d\big)}{m!} \int_{\Delta_m(x_1)} dX_{(2,m+1)} \leq \zeta\big(1-\exp\big(-\zeta\kappa_d\varepsilon^d\big)\big)\,, \tag{10.1.3}$$

*where* $\kappa_d$ *is the volume of the unit ball in* $\mathbf{R}^d$*.*

*Proof.* — The first identity (10.1.1) is obtained by a simple partitioning argument, which extends the splitting used to define $\mathcal{C}_{s,s+m}$ in (9.4.4) in the previous chapter. We recall that, given any $X_s \in \mathbf{R}^{ds}$, the family

$$\Big\{(x_{s+1},\dots,x_N)\,,\ |E(X_s,X_N)| = s+m\Big\} \quad \text{for } 0\leq m\leq N-s\,,$$

is a partition of $\mathbf{R}^{(N-s)d}$. Then we use the symmetry assumption, as we did in (9.4.2), to find

$$\int_{\mathbf{R}^{(N-s)d}} \varphi(X_N)dX_{(s+1,N)} = \sum_{0\leq m\leq N-s} C^m_{N-s} \int_{\mathbf{R}^{(N-s)d}} 1\!\!1_{E(X_s,X_N)=X_{s+m}}\varphi(X_N)dX_{(s+1,N)}\,.$$

It then suffices to use equivalence (9.3.2) from Lemma 9.3.4, noting that the set of all $(x_{s+1},\dots,x_{s+m})$ in $\mathbf{R}^{md}$ such that $E(X_s,X_{s+m}) = X_{s+m}$ coincides with $\Delta_m(X_s)$. This proves (10.1.1).

Estimates (10.1.2) and (10.1.3) come from the counterpart of (10.1.1) at the grand canonical level, i.e. when the activity $\zeta^{-1} \sim e^\mu$ is fixed, rather than the total number $N$ of particles (we refer to Remark 5.2.3 for comments on this terminology).

For any bounded $\Lambda\subset\mathbf{R}^d$, the associated grand-canonical ensemble for $n$ non-interacting particles is defined as the probability measure with density

$$\varphi_n(X_n) := \frac{\zeta^n\exp(-\zeta|\Lambda|)}{n!}\prod_{1\leq i\leq n} 1\!\!1_{x_i\in\Lambda}\,.$$

The $s$-point correlation function $g_s$ and the truncated $s$-point correlation function $\widetilde g_s$ are defined by

$$g_s(X_s) := \sum_{n=s}^\infty \frac{n!}{(n-s)!}\int_{\mathbf{R}^{(n-s)d}} \varphi_n(X_n)dX_{(s+1,n)}\,,$$

$$\widetilde g_s(X_s) := \sum_{n=s}^\infty \frac{n!}{(n-s)!}\int_{\mathbf{R}^{(n-s)d}} 1\!\!1_{X_n\in\mathcal{D}^s_n}\varphi_n(X_n)dX_{(s+1,n)}\,.$$

We compute

$$\int_{\mathbf{R}^{(n-s)d}} \varphi_n(X_n)dX_{(s+1,n)} = \zeta^s\exp\big(-\zeta|\Lambda|\big)\frac{(\zeta|\Lambda|)^{n-s}}{n!}\prod_{1\leq i\leq s} 1\!\!1_{x_i\in\Lambda}\,,$$

so that

$$g_s(X_s) = \zeta^s\exp\big(-\zeta|\Lambda|\big)\sum_{k=0}^\infty\frac{(\zeta|\Lambda|)^k}{k!}\prod_{1\leq i\leq s}1\!\!1_\Lambda(x_i) = \zeta^s\prod_{1\leq i\leq s}1\!\!1_{x_i\in\Lambda}\,. \tag{10.1.4}$$

Similarly, by definition of $\mathcal{D}^s_n$ in (9.1.4),

$$\int_{\mathbf{R}^{(n-s)d}} 1\!\!1_{X_n\in\mathcal{D}^s_n}\prod_{s+1\leq j\leq n}1\!\!1_{x_i\in\Lambda}\,dX_{(s+1,n)} = \big|\Lambda\cap{}^cB_\varepsilon(X_s)\big|\,,$$

where we denote $B_\varepsilon(X_s) := \bigcup\limits_{1\leq i\leq s} B_\varepsilon(x_i)$, with $B_\varepsilon(x_i) := \big\{y\in\mathbf{R}^d,\ |y-x_i|\leq\varepsilon\big\}$. This implies

$$\widetilde g_s(X_s) = \zeta^s\exp\big(-\zeta|\Lambda|\big)\sum_{n\geq s}\frac{\big(\zeta|\Lambda\cap{}^cB_\varepsilon(X_s)|\big)^{n-s}}{(n-s)!}\prod_{1\leq i\leq s}1\!\!1_{x_i\in\Lambda}\,.$$

Since $|\Lambda| - |\Lambda \cap {}^c B_\varepsilon(X_s)| = |\Lambda \cap B_\varepsilon(X_s)|$, we obtain

$$\widetilde{g}_s(X_s) = \zeta^s \exp\big(-\zeta|\Lambda \cap B_\varepsilon(X_s)|\big)\,. \tag{10.1.5}$$

Besides, by (10.1.1),

$$\begin{aligned}g_s(X_s) &= \widetilde{g}_s(X_s)\\ &\quad + \sum_{n=s}^{\infty}\sum_{m=1}^{n-s} \frac{n!C_{n-s}^m}{(n-s)!}\int_{\Delta_m(X_s)}\Big(\int_{\mathbf{R}^{(n-s-m)d}} 1\!\!1_{X_n\in\mathcal{D}_n^{s+m}}\, g_s(X_n)\,dX_{(s+m+1,n)}\Big)\,dX_{(s+1,s+m)}\,.\end{aligned}$$

By Fubini, we get

$$\begin{aligned}&\sum_{n=s}^{\infty}\sum_{m=1}^{n-s} \frac{n!C_{n-s}^m}{(n-s)!}\int_{\Delta_m(X_s)}\Big(\int_{\mathbf{R}^{(n-s-m)d}} 1\!\!1_{X_n\in\mathcal{D}_n^{s+m}}\varphi_n(X_n)\,dX_{(s+m+1,n)}\Big)\,dX_{(s+1,s+m)}\\
&\quad = \sum_{n=s}^{\infty}\sum_{m=1}^{n-s} \frac{n!}{(k-s)!(n-k)!}\int_{\Delta_{k-s}(X_s)}\Big(\int_{\mathbf{R}^{(n-k)d}} 1\!\!1_{X_n\in\mathcal{D}_n^{k}}\varphi_n(X_n)\,dX_{(k+1,n)}\Big)\,dX_{(s+1,k)}\\
&\quad = \sum_{k=s+1}^{\infty}\frac{1}{(k-s)!}\sum_{n=k}^{\infty} \frac{n!}{(n-k)!}\int_{\Delta_{k-s}(X_s)}\Big(\int_{\mathbf{R}^{(n-k)d}} 1\!\!1_{X_n\in\mathcal{D}_n^{k}}\varphi_n(X_n)\,dX_{(k+1,n)}\Big)\,dX_{(s+1,k)}\\
&\quad = \sum_{k=s+1}^{\infty}\frac{1}{(k-s)!}\int_{\Delta_{k-s}(X_s)}\widetilde{g}_k(X_k)dX_{(s+1,k)}\,.\end{aligned}$$

We have proved that

$$g_s(X_s) = \widetilde{g}_s(X_s) + \sum_{k=s+1}^{\infty}\frac{1}{(k-s)!}\int_{\Delta_{k-s}(X_s)} g_k(X_k)dX_{(s+1,k)}\,. \tag{10.1.6}$$

We now show how identities (10.1.4)-(10.1.5)-(10.1.6) imply the bounds (10.1.2)-(10.1.3).

We first retain only the contribution of $k = s+m$ in the right-hand side of (10.1.6). We have

$$\zeta^s \geq \frac{1}{m!}\int_{\Delta_m(X_s)}\zeta^{s+m}\exp\big(-\zeta|\Lambda\cap B_\varepsilon(X_{s+m})|\big)\,dX_{(s+1,s+m)}\,,$$

and now $|\Lambda \cap B_\varepsilon(X_{s+m})| \leq \kappa_d\varepsilon^d(s+m)$ implies (10.1.2).

We finally fix an integer $K \geq 2$ and choose $s = 1$ in (10.1.6). Then

$$\zeta - \zeta\exp\big(-\zeta|\Lambda\cap B_\varepsilon(x_1)|\big) \geq \sum_{k=2}^{K}\int_{\Delta_{k-1}(x_1)}\zeta^k\exp\big(-\zeta|B_\varepsilon(X_k)|\big)\,dX_{(2,k)}\,,$$

and bounding the volumes of balls from above, we find

$$\zeta\big(1-\exp(-\zeta\kappa_d\varepsilon^d)\big) \geq \sum_{k=1}^{K-1}\frac{\zeta^{k+1}}{k!}\exp\big(-\zeta\kappa_d(k+1)\varepsilon^d\big)\int_{\Delta_k(x_1)}dX_{(2,k+1)}\,.$$

It then suffices to let $K\to\infty$ to find (10.1.3). This ends the proof of Lemma 10.1.1. □

## 10.2. Functional spaces

To show the convergence of the series defining mild solutions (9.5.2) to the BBGKY hierarchy, we need to introduce some norms on the space of sequences $(\widetilde{f}^{(s)})_{s\geq 1}$. Given $\varepsilon > 0$, $\beta > 0$, an integer $s \geq 1$, and a continuous function $g_s : \Omega_s \to \mathbf{R}$, we let

$$|g_s|_{\varepsilon,s,\beta} := \sup_{Z_s \in \Omega_s} \Big( |g_s(Z_s)| \exp\big(\beta E_\varepsilon(Z_s)\big)\Big) \tag{10.2.1}$$

where for $\varepsilon > 0$, the function $E_\varepsilon$ is the $s$-particle Hamiltonian

$$E_\varepsilon(Z_s) := \sum_{1\le i\le s} \frac{|v_i|^2}{2} + \sum_{1\le i<k\le s} \Phi_\varepsilon(x_i - x_k)\,, \quad \text{with} \quad \Phi_\varepsilon(x) := \Phi\Big(\frac{x}{\varepsilon}\Big)\,. \tag{10.2.2}$$

Notice that this norm does coincide with its counterpart defined in Paragraph 5.2 in the limit described in Remark 1.0.1.

***Definition 10.2.1***. — *For $\varepsilon > 0$ and $\beta > 0$, we denote $X_{\varepsilon,s,\beta}$ the Banach space of continuous functions $\Omega_s \to \mathbf{R}$ with finite $|\cdot|_{\varepsilon,s,\beta}$ norm.*

By Assumption 1.2.1, for $\varepsilon > 0$ (and $\beta > 0$) there holds $\exp(\beta E_\varepsilon(Z_s)) \to \infty$ as $Z_s$ approaches $\partial\Omega_s$. This implies for $g_s \in X_{\varepsilon,s,\beta}$ the existence of an extension by continuity: $\bar g_s \in C^0(\mathbf{R}^{2ds};\mathbf{R})$ such that $\bar g_s \equiv 0$ on $\partial\Omega_s$, and $\bar g_s \equiv g$ on $\Omega_s$.

For sequences of functions $G = (g_s)_{s\ge1}$, with $g_s : \Omega_s \to \mathbf{R}$, we let for $\varepsilon > 0$, $\beta > 0$, $\mu \in \mathbf{R}$,

$$\|G\|_{\varepsilon,\beta,\mu} := \sup_{s\ge1} \Big( |g_s|_{\varepsilon,s,\beta} \exp(\mu s)\Big)\,.$$

***Definition 10.2.2***. — *For $\varepsilon \ge 0$, $\beta > 0$, and $\mu \in \mathbf{R}$, we denote $\mathbf{X}_{\varepsilon,\beta,\mu}$ the Banach space of sequences $G = (g_s)_{s\ge1}$, with $g_s \in X_{\varepsilon,s,\beta}$ and $\|G\|_{\varepsilon,\beta,\mu} < \infty$.*

As in (5.2.4), he following inclusions hold:

$$\text{if } \beta' \le \beta \text{ and } \mu' \le \mu\,, \text{ then} \quad X_{\varepsilon,s,\beta'} \subset X_{\varepsilon,s,\beta}\,, \quad \mathbf{X}_{\varepsilon,\beta',\mu'} \subset \mathbf{X}_{\varepsilon,\beta,\mu}\,. \tag{10.2.3}$$

Finally similarly to Definition 5.2.4 we define norms of time-dependent functions as follows.

***Definition 10.2.3***. — *Given $T > 0$, a positive function $\boldsymbol{\beta}$ and a real valued function $\boldsymbol{\mu}$ defined on $[0,T]$ we denote $\mathbf{X}_{\varepsilon,\boldsymbol{\beta},\boldsymbol{\mu}}$ the space of functions $G : t \in [0,T] \mapsto G(t) = (g_s(t))_{1\le s} \in \mathbf{X}_{\varepsilon,\beta(t),\mu(t)}$, such that for all $Z_s \in \mathbf{R}^{2ds}$, the map $t \in [0,T] \mapsto g_s(t,Z_s)$ is measurable, and*

$$|\!|\!|G|\!|\!|_{\varepsilon,\boldsymbol{\beta},\boldsymbol{\mu}} := \sup_{0\le t\le T} \|G(t)\|_{\varepsilon,\beta(t),\mu(t)} < \infty\,. \tag{10.2.4}$$

Notice that the following conservation of energy properties hold, as for (5.3.1):

$$|\mathbf{H}_s(t) g_s|_{\varepsilon,s,\beta} = |g_s|_{\varepsilon,s,\beta} \quad \text{and} \quad \|\mathbf{H}(t) G_N\|_{\varepsilon,\beta,\mu} = \|G_N\|_{\varepsilon,\beta,\mu}\,, \tag{10.2.5}$$

for all parameters $\beta > 0$, $\mu \in \mathbf{R}$, and for all $g_s \in X_{\varepsilon,s,\beta}$, $G_N = (g_s)_{1\le s\le N} \in \mathbf{X}_{\varepsilon,\beta,\mu}$, and all $t \ge 0$.

## 10.3. Continuity estimates

We now establish bounds, in the above defined functional spaces, for the collision operators defined in (9.4.4)-(9.4.6), and for the total collision operator $\mathbf{C}_N$ defined in (9.5.3).

Notice that in the case when $m = 1$ the estimates are the same as in Chapter 5: in particular thanks to (10.2.5) the following bound holds:

$$e^{s(\mu_0-\lambda t)} \Big| \int_0^t \mathbf{H}_s(t-\tau) \mathcal{C}_{s,s+1} g_{s+1}(\tau)\, d\tau \Big|_{\varepsilon,s,\beta_0-\lambda t} \le \bar c(\beta_0,\mu_0,\lambda,T) |\!|\!|G_N|\!|\!|_{\varepsilon,\boldsymbol{\beta},\boldsymbol{\mu}}\,, \tag{10.3.1}$$

for all $G_N = (g_{s+1})_{1\leq s\leq N} \in \mathbf{X}_{\varepsilon,\boldsymbol{\beta},\boldsymbol{\mu}}$, with $\bar c(\beta_0,\mu_0,\lambda,T)$ computed explicitly in (5.4.9).

The following statement is the analogue of Proposition 5.4.1 in the hard spheres case, but in the present situation higher order correlations must be taken into account.

***Proposition 10.3.1***. — *Given $\beta > 0$ and $\mu \in \mathbf{R}$, for $m \geq 1$ and $1 \leq s \leq N-m$, the collision operators $\mathcal{C}_{s,s+m}$ satisfy the bounds, for all $G_N = (g_s)_{1\leq s\leq N} \in \mathbf{X}_{\varepsilon,\beta,\mu}$,*

$$\big|\mathcal{C}_{s,s+m} g_{s+m}(Z_s)\big| \leq \varepsilon^{m-1} C_d e^{m\kappa_d} (\beta/C_d)^{-\frac{md}{2}} \Big(s\beta^{-\frac12} + \sum_{1\leq i\leq s} |v_i|\Big) e^{-\beta E_\varepsilon(Z_s)} |g_{s+m}|_{\varepsilon,s+m,\beta}\,, \tag{10.3.2}$$

*for some $C_d > 0$ depending only on $d$.*

*If $\varepsilon < C_d e^{\mu}\beta^{\frac d2}$, then for all $0 < \beta' < \beta$ and $\mu' < \mu$, the total collision operator $\mathbf{C}_N$ satisfies the bound*

$$\|\mathbf{C}_N G_N\|_{\varepsilon,\beta',\mu'} \leq C_d(1+\beta^{-\frac12})\Big(\frac{1}{\beta-\beta'} + \frac{1}{\mu-\mu'}\Big)\|G_N\|_{\varepsilon,\beta,\mu}\,. \tag{10.3.3}$$

Considering the case $m > 1$ in (10.3.2), for which the upper bound is $O(\varepsilon)$, we see that higher-order interactions are negligible in the Boltzmann-Grad limit (provided (10.3.2,) can be summed over $m$, which is possible for $\varepsilon$ small enough).

*Proof*. — We shall only consider the case $m \geq 2$, as the case $m = 1$ is dealt with exactly as in the proof of Proposition 5.4.1. From the definition of $G^{(m-1)}_{\langle i,s+1\rangle}$ in (9.4.6), we obtain

$$\big|G^{(m-1)}_{\langle i,s+1\rangle}(g_{s+m})(Z_{s+1})\big| \leq |g_{s+m}|_{\varepsilon,s+m,\beta} \int_{\Delta_{m-1}(x_{s+1})\times\mathbf{R}^{d(m-1)}} \exp\big(-\beta E_\varepsilon(Z_{s+m})\big) dZ_{(s+2,s+m)}\,,$$

where the norm $|\cdot|_{\varepsilon,s,\beta}$ is defined in (10.2.1), and the Hamiltonian $E_\varepsilon$ is defined in (10.2.2). For the collision operator defined in (9.4.4), this implies the bound

$$|\mathcal{C}_{s,s+m} g_{s+m}(Z_s)| \leq m C^m_{N-s} |g_{s+m}|_{\varepsilon,s+m,\beta} \times \sum_{1\leq i\leq s} I_{i,m}(V_s) \times J_{i,m}(X_s)\,, \tag{10.3.4}$$

where $I_{i,m}$ is the velocity integral

$$I_{i,m}(V_s) := \int_{\mathbf{R}^{dm}} \big(|v_{s+1}| + |v_i|\big) \exp\Big(-\frac{\beta}{2}\sum_{j=1}^{s+m} |v_j|^2\Big) dV_{(s+1,s+m)}\,,$$

and $J_{i,m}$ is the spatial integral

$$J_{i,m}(X_s) := \int_{S_\varepsilon(x_i)\times\Delta_{m-1}(x_{s+1})} \exp\Big(-\beta\sum_{1\leq j<k\leq s+m} \Phi_\varepsilon(x_j-x_k)\Big) d\sigma(x_{s+1}) dX_{(s+2,s+m)}\,.$$

The velocity integral is a product of Gaussian integrals and can be exactly computed, as in the hard-spheres case:

$$I_{i,m}(V_s) \leq (\beta/C_d)^{-\frac{md}{2}} \Big(|v_i| + \beta^{-\frac12}\Big) \exp\Big(-\frac{\beta}{2}\sum_{1\leq j\leq s} |v_j|^2\Big)\,. \tag{10.3.5}$$

For the spatial integral, there holds

$$\begin{aligned} J_{i,m}(X_s) &\leq \exp\Big(-\beta\sum_{1\leq j<k\leq s} \Phi_\varepsilon(x_j-x_k)\Big) |S_\varepsilon(x_i)| \times \sup_x \int_{\Delta_{m-1}(x)} dX_{(1,m-1)} \\ &\leq \exp\Big(-\beta\sum_{1\leq j<k\leq s} \Phi_\varepsilon(x_j-x_k)\Big) \times \kappa_d\varepsilon^{d-1} \times \Big((m-1)!\,\varepsilon^{(m-1)d}\exp(m\kappa_d)\Big)\,, \end{aligned}$$

where in the last bound we used identity (10.1.2) from Lemma 10.1.1 with $s = 1$ and $\zeta = \varepsilon^{-d}$. This implies

$$|\mathcal{C}_{s,s+m}g_{s+m}(Z_s)| \leq C_d\varepsilon^{m-1}\big((N-s)\varepsilon^{d-1}\big)^m e^{m\kappa_d}(\beta/C_d)^{-\frac{md}{2}}\Big(s\beta^{-\frac{1}{2}} + \sum_{1\leq i\leq s}|v_i|\Big) \\ \times e^{-\beta E_\varepsilon(Z_s)}|g_{s+m}|_{\varepsilon,s+m,\beta}\,.$$

In the Boltzmann-Grad scaling $N\varepsilon^{d-1} \equiv 1$, this gives (10.3.2). Above and in the following, $C_d$ denotes a positive constant which depends only on $d$, and which may change from line to line.

We turn to the proof of (10.3.3), which is similar to the proof of (**??**) up to the control of higher correlations. From the pointwise inequality (**??**) we deduce for the above velocity integral $I_{i,m}(V_s)$ the bound, for $0 < \beta' < \beta$,

$$\sum_{1\leq i\leq s}\exp\Big((\beta'/2)\sum_{1\leq j\leq s}|v_j|^2\Big)I_{i,m}(V_s) \leq C_d(\beta/C_d)^{-\frac{md}{2}}\Big(s\beta^{-\frac{1}{2}} + s^{\frac{1}{2}}(\beta-\beta')^{-\frac{1}{2}}\Big)\,.$$

From the above bound in $J_{i,m}(X_s)$, we deduce immediately, for $0 < \beta' < \beta$,

$$\max_{1\leq i\leq s}\exp\Big(\beta'\sum_{1\leq j<k\leq s}\Phi_\varepsilon(x_j - x_k)\Big)J_{i,m}(X_s) \leq \kappa_d(m-1)!\,e^{m\kappa_d}\varepsilon^{md-1}\,.$$

With (10.3.4), these bounds yield, in the Boltzmann-Grad scaling,

$$e^{\beta' E_\varepsilon(Z_s)+\mu' s}\big|\mathcal{C}_{s,s+m}g_{s+m}(Z_s)\big| \leq \varepsilon^{m-1}C_d(\beta/C_d)^{-\frac{md}{2}}e^{m\kappa_d}e^{\mu' s}\big(s\beta^{-\frac{1}{2}} + s^{\frac{1}{2}}(\beta-\beta')^{-\frac{1}{2}}\big) \\ \times |g_{s+m}|_{\varepsilon,s+m,\beta}\,.$$

Summing over $m$, we finally obtain, for $\mathbf{C}_N$ defined in (9.5.3),

$$\|\mathbf{C}_N G_N\|_{\varepsilon,\beta',\mu'} \leq C_d\|G_N\|_{\varepsilon,\beta,\mu}\sup_{1\leq s\leq N}\Big(\big(s\beta^{-\frac{1}{2}} + s^{\frac{1}{2}}(\beta-\beta')^{-\frac{1}{2}}\big)e^{-(\mu-\mu')s}\Big) \\ \times \sum_{1\leq m\leq N-s} e^{-m(\mu-\kappa_d)}\varepsilon^{m-1}\,(\beta/C_d)^{-\frac{md}{2}}\,.$$

If $\varepsilon$ is small enough so that $\varepsilon e^{\kappa_d-\mu}(C_d/\beta)^{d/2} < 1$, then the above series is convergent, and

$$\sum_{1\leq m\leq N-s} e^{-m(\mu-\kappa_d)}\varepsilon^{m-1}(C_d/\beta)^{md/2} \leq \frac{e^{\kappa_d-\mu}(C_d/\beta)^{d/2}}{1-\varepsilon e^{\kappa_d-\mu}(C_d/\beta)^{d/2}}\,.$$

We conclude as in the proof of Proposition 5.4.1. Proposition 10.3.1 is proved. □

***Remark 10.3.1.*** — *We do not use the extra decay provided by the contribution of the potential in the exponential of the Hamiltonian. This is quite obvious in the bound for $J_{i,m}(X_s)$ in the proof of Proposition* 10.3.1, *where we bound $e^{-\beta\sum_{1\leq j<k\leq s+m}\Phi_\varepsilon(x_j-x_k)}$ by $e^{-\beta\sum_{1\leq j<k\leq s}\Phi_\varepsilon(x_j-x_k)}$. Then, we might be tempted to replace $E_\varepsilon$ by the free Hamiltonian $E_0$ in the definition of the functional spaces. The kinetic energy, however, is not a conserved quantity, so that in $X_{0,s,\beta}$ there is no analogue of* (10.2.5).

This leads to the following lemma, which is the key to the proof of the uniform bound stated in Theorem 9 in the next paragraph. It is the analogue of Lemma 5.4.3.

***Lemma 10.3.2.*** — *Let $\beta_0 > 0$ and $\mu_0 \in \mathbf{R}$ be given. There is $T > 0$ depending only on $\beta_0$ and $\mu_0$ such that for an appropriate choice of $\lambda$ in $(0, \beta_0/T)$, there holds for all $t \in [0, T]$*

$$\Big\|\int_0^t \mathbf{H}(t-\tau)\mathbf{C}_N G_N(\tau)\,d\tau\Big\|_{\varepsilon,\beta_0-\lambda t,\mu_0-\lambda t} \leq \frac{1}{2}\,|||G_N|||_{\varepsilon,\boldsymbol{\beta},\boldsymbol{\mu}}\,. \tag{10.3.6}$$

*Proof.* — We follow closely the proof of Lemma 5.4.3. The difference is that here we take into account higher-order collision operators $\mathcal{C}_{s,s+m}$, with $m \geq 2$. Using notation (5.4.5), Estimate (10.3.2) from Proposition 10.3.1 gives

$$\begin{aligned}
&e^{\beta_0^\lambda(t)E_\varepsilon(Z_s)}\big|\mathcal{C}_{s,s+m}g_{s+m}(t',Z_s)\big| \\
&\leq \varepsilon^{m-1}C_d e^{m\kappa_d}(2\pi/\beta_0^\lambda(t'))^{md/2}|g_{s+m}(t')|_{\varepsilon,s+m,\beta_0^\lambda(t')}\Big(s(\beta_0^\lambda(t'))^{-d/2}+\sum_{1\leq i\leq s}|v_i|\Big)e^{\lambda(t'-t)E_\varepsilon(Z_s)}\,.
\end{aligned}$$

Using also (5.4.6) with $s+1$ replaced by $s+m$, we get

$$\begin{aligned}
&\Big\|\int_0^t \mathbf{H}(t-t')\mathbf{C}_N G_N(t')\,dt'\Big\|_{\varepsilon,\beta_0^\lambda(t),\mu_0^\lambda(t)} \\
&\qquad\leq |\!|\!| G_N|\!|\!|_{\varepsilon,\boldsymbol{\beta},\boldsymbol{\mu}}\Big(\sum_{1\leq m\leq N-s}C_m\Big)\sup_{Z_s\in\mathbf{R}^{2ds}}\int_0^t\overline{C}(t,t',Z_s)\,dt'\,,
\end{aligned}\tag{10.3.7}$$

where $C_m := C_d\varepsilon^{m-1}e^{m(\kappa_d-\mu_0^\lambda(T))}(C_d/\beta_0^\lambda(T))^{md/2}$, and $\overline{C}$ is defined in (5.4.7) and satisfies (5.4.8) which we recall here:

$$\sup_{Z_s\in\mathbf{R}^{2ds}}\int_0^t\overline{C}(\tau,t,Z_s)\,d\tau\leq\frac{C_d}{\lambda}\Big(1+\frac{1}{(\beta_0^\lambda(T))^{d/2}}\Big)\,,\tag{10.3.8}$$

Under the assumption that

$$\varepsilon_0 e^{\kappa_d-\mu_0^\lambda(T)}(2\pi/\beta_0^\lambda(T))^{d/2}<1/2\,,\tag{10.3.9}$$

we find

$$\sum_{1\leq m\leq N-s}C_m\leq 2C_d e^{-\mu_0^\lambda(T)}(\beta_0^\lambda(T))^{-d/2}\,.\tag{10.3.10}$$

The upper bounds in (10.3.8) and (10.3.10) are independent of $s$, and their product is equal to $2\bar{c}(\beta_0,\mu_0,\lambda,T)$. It then suffices to choose $\lambda$ so that $2\bar{c}(\beta_0,\mu_0,\lambda,T)\leq 1/2$ and taking the supremum in $s$ in (10.3.7) then yields the result. □

## 10.4. Uniform bounds for the BBGKY and Boltzmann hierarchies

The results of the previous section enable us, exactly as in the hard spheres case page 39, to deduce directly the following bounds on the BBGKY hierarchy defined in (9.5.4) page 78.

***Theorem 9* (Uniform bounds for the BBGKY hierarchy).** — *Let $\beta_0>0$ and $\mu_0\in\mathbf{R}$ be given. There is a time $T>0$ as well as two nonincreasing functions $\boldsymbol{\beta}>0$ and $\boldsymbol{\mu}$ defined on $[0,T]$, satisfying $\boldsymbol{\beta}(0)=\beta_0$ and $\boldsymbol{\mu}(0)=\mu_0$, such that in the Boltzmann-Grad scaling $N\varepsilon^{d-1}\equiv 1$, any family of initial marginals $\widetilde{F}_N(0)=\big(\widetilde{f}_N^{(s)}(0)\big)_{1\leq s\leq N}$ in $\mathbf{X}_{\varepsilon,\beta_0,\mu_0}$ gives rise to a unique solution $\widetilde{F}_N(t)=(\widetilde{f}_N^{(s)}(t))_{1\leq s\leq N}$ in $\mathbf{X}_{\varepsilon,\boldsymbol{\beta},\boldsymbol{\mu}}$ to the BBGKY hierarchy* (9.5.4) *satisfying the following bound:*

$$|\!|\!|\widetilde{F}_N|\!|\!|_{\varepsilon,\boldsymbol{\beta},\boldsymbol{\mu}}\leq 2\|\widetilde{F}_N(0)\|_{\varepsilon,\beta_0,\mu_0}\,.$$

In the case of the Boltzmann hierarchy associated with the collision operator (8.3.6), the same existence result as in Theorem 7 holds, again with the same proof.

***Theorem 10* (Existence for the Boltzmann hierarchy).** — *Assume the potential satisfies Assumption* 1.2.1. *Let* $\beta_0 > 0$ *and* $\mu_0 \in \mathbf{R}$ *be given. There are a time* $T > 0$ *as well as two nonincreasing functions* $\boldsymbol{\beta} > 0$ *and* $\boldsymbol{\mu}$ *defined on* $[0, T]$, *satisfying* $\boldsymbol{\beta}(0) = \beta_0$ *and* $\boldsymbol{\mu}(0) = \mu_0$, *such that any family of initial marginals* $F(0) = \big(f^{(s)}(0)\big)_{s\geq 1}$ *in* $\mathbf{X}_{0,\beta_0,\mu_0}$ *gives rise to a unique solution* $F(t) = (f^{(s)}(t))_{s\geq 1}$ *in* $\mathbf{X}_{0,\boldsymbol{\beta},\boldsymbol{\mu}}$ *to the Boltzmann hierarchy* (5.0.2), *satisfying the following bound:*

$$|\!|\!| F |\!|\!|_{0,\boldsymbol{\beta},\boldsymbol{\mu}} \leq 2 \|F(0)\|_{0,\beta_0,\mu_0} \, .$$

# CHAPTER 11

# CONVERGENCE RESULT AND STRATEGY OF PROOF

The main goal of this chapter is to reduce the proof of Theorem 5 stated page 17 to the term-by-term convergence of some functionals involving a finite (uniformly bounded) number of marginals with only first-order collisions, bounded energies and a finite number of collision times, exactly as was performed in Chapter 7 (see Section 11.3).

Before doing so we define, as in the hard spheres case, the notion of admissible initial data in Section 11.1. We give the precise version of Theorem 5 in Section 11.2.

## 11.1. Admissible initial data

The characterization of admissible initial data is very similar to the hard spheres case studied in Paragraph 6.1.1. The only new aspect concerns the fact that marginals have been truncated, and that feature will be dealt with in this section.

***Definition 11.1.1* (Admissible Boltzmann data).** — *Admissible Boltzmann data are defined as families $F_0 = (f_0^{(s)})_{s\geq 1}$, with each $f_0^{(s)}$ nonnegative, integrable and continuous over $\Omega_s$, such that*

$$\int_{\mathbf{R}^{2d}} f_0^{(s+1)}(Z_s, z_{s+1})\, dz_{s+1} = f_0^{(s)}(Z_s)\,, \tag{11.1.1}$$

*and which are limits of BBGKY initial data $\widetilde F_{0,N} = (\widetilde f_{0,N}^{(s)})_{1\leq s\leq N} \in \mathbf{X}_{\varepsilon,\beta_0,\mu_0}$ in the following sense: it is assumed that*

$$\sup_{N\geq 1} \|\widetilde F_{0,N}\|_{\varepsilon,\beta_0,\mu_0} < \infty\,, \quad \text{for some } \beta_0>0\,,\ \mu_0\in\mathbf{R}\,,\ \text{as } N\varepsilon^{d-1}\equiv 1\,, \tag{11.1.2}$$

*and that for each given $s\in[1,N]$, the truncated marginal of order $s$ defined by*

$$\widetilde f_{0,N}^{(s)}(Z_s) = \int_{\mathbf{R}^{2d(N-s)}} 1\!\!1_{\mathcal{D}_N^s}(X_N) f_{0,N}^{(N)}(Z_N) dZ_{(s+1,N)}\,, \quad 1\leq s<N\,, \tag{11.1.3}$$

*converges in the Boltzmann-Grad limit:*

$$\widetilde f_{0,N}^{(s)} \longrightarrow f_0^{(s)}\,, \qquad \text{as } N\to\infty \text{ with } N\varepsilon^{d-1}\equiv 1\,, \text{ locally uniformly in } \Omega_s\,. \tag{11.1.4}$$

The following result is proved very similarly to Proposition 6.1.1.

***Proposition 11.1.1.*** — *The set of admissible Boltzmann data, in the sense of Definition* 11.1.1*, is the set of families of marginals* $F_0$ *as in* (11.1.1) *satisfying a uniform bound* $\|F_0\|_{0,\beta_0,\mu_0} < \infty$ *for some* $\beta_0 > 0$ *and* $\mu_0 \in \mathbf{R}$.

We shall not give the proof of that result, as the only difference with Proposition 6.1.1 lies in the presence of a truncation in the marginals, whose effect disappears asymptotically as stated in the following lemma.

***Lemma 11.1.2.*** — *Given* $\widetilde{F}_{0,N} = (\widetilde{f}^{(s)}_{0,N})_{1\leq s\leq N}$ *satisfying* (11.1.2) *and* (11.1.3) *from Definition* 11.1.1*, with associated family* $F_{0,N} = (f^{(s)}_{0,N})_{1\leq s\leq N}$ *of untruncated marginals:*

$$
f^{(s)}_{0,N}(Z_s) = \int_{\mathbf{R}^{2d(N-s)}} f^{(N)}_{0,N}(Z_N)\, dZ_{(s+1,N)}\,, \quad 1\leq s< N\,, \quad Z_s\in\Omega_s\,, \qquad \widetilde{f}^{(N)}_{0,N} = f^{(N)}_{0,N}\,, \tag{11.1.5}
$$

*there holds the convergence*

$$
f^{(s)}_{0,N} - \widetilde{f}^{(s)}_{0,N} \longrightarrow 0\,, \qquad \textit{for fixed } s\geq 1\,,\ \textit{as } N\to\infty \textit{ with } N\varepsilon^{d-1}\equiv 1\,,\ \textit{uniformly in } \Omega_s\,.
$$

*Proof.* — We apply identity (10.1.1) from Lemma 10.1.1 to $f^{(N)}_{0,N}$, and obtain after integration in the velocity variables

$$
f^{(s)}_{0,N}(Z_s) - \widetilde{f}^{(s)}_{0,N}(Z_s) = \sum_{m=1}^{N-s} C^m_{N-s} \int_{\Delta_m(X_s)\times\mathbf{R}^{dm}} \widetilde{f}^{(s+m)}_{0,N}(Z_{s+m}) dZ_{(s+1,s+m)}\,. \tag{11.1.6}
$$

Then, denoting $C_0 = \sup\limits_{M\geq 1} \|F_{0,M}\|_{\varepsilon,\beta_0,\mu_0}$, a finite number by assumption, from

$$
\begin{aligned}
f^{(s+m)}_{0,N}(Z_{s+m}) &\leq \exp\big(-\mu_0(s+m) - \beta_0 E_\varepsilon(Z_{s+m})\big) C_0\\
&\leq \exp\Big(-\mu_0(s+m) - (\beta_0/2)\sum_{1\leq i\leq s}|v_i|^2\Big) C_0\,,
\end{aligned}
$$

we deduce, first by integrating the velocity gaussians and then by using the cluster bound (10.1.2) in Lemma 10.1.1 with $\zeta = \varepsilon^{-d}$, the bound

$$
\begin{aligned}
\int_{\Delta_m(X_s)\times\mathbf{R}^{dm}} f^{(s+m)}_{0,N}(Z_{s+m}) dZ_{(s+1,s+m)} &\leq (C_d/\beta_0)^{md/2} e^{-\mu_0(s+m)} C_0 \int_{\Delta_m(X_s)} dX_{(s+1,s+m)}\\
&\leq m!(C_d/\beta_0)^{md/2}\varepsilon^{md} e^{(\kappa_d-\mu_0)(s+m)} C_0\,.
\end{aligned}
$$

If $2\varepsilon e^{\kappa_d-\mu_0}(C_d/\beta_0)^{d/2} < 1$, then

$$
\sum_{m=1}^{N-s} C^m_{N-s} m! (C_d/\beta_0)^{md/2}\varepsilon^{md} e^{(\kappa_d-\mu_0)(s+m)} \leq e^{(\kappa_d-\mu_0)s} \sum_{m\geq 1} \big(2\varepsilon e^{\kappa_d-\mu_0}(C_d/\beta_0)^{d/2}\big)^m \longrightarrow 0
$$

as $\varepsilon\to 0$, implying $f^{(s)}_{0,N} - \widetilde{f}^{(s)}_{0,N} \longrightarrow 0$ for fixed $s$, uniformly in $\Omega_s$. □

***Remark 11.1.3.*** — *We can reproduce the above proof in the case of a time-dependent family of bounded marginals, i.e.,* $F_N \in \mathbf{X}_{\varepsilon,\boldsymbol{\beta},\boldsymbol{\mu}}$*, with* $\sup\limits_{N\geq 1} |\!|\!| F_N |\!|\!|_{\varepsilon,\boldsymbol{\beta},\boldsymbol{\mu}} < \infty$*, with the notation of Definition* 10.2.1*. This gives uniform convergence to zero, in time* $t\in[0,T]$ *and in space* $X_s\in\Omega_s$*, of the difference between truncated and untruncated marginals:* $\widetilde{f}^{(s)}_N - f^{(s)}_N \to 0$.

We consider therefore families of initial data: Boltzmann initial data $F_0 = (f^{(s)}_0)_{s\in\mathbf{N}}$ such that

$$
\|F_0\|_{0,\beta_0,\mu_0} = \sup_{s\in\mathbf{N}}\sup_{Z_s}\big(\exp(\beta_0 E_0(Z_s) + \mu_0 s) f^{(s)}_0(Z_s)\big) < +\infty
$$

and for each $N$, BBGKY initial data $\widetilde{F}_{N,0} = (\widetilde{f}^{(s)}_{N,0})_{1\leq s\leq N}$ such that

$$\sup_N \|\widetilde{F}_{N,0}\|_{\varepsilon,\beta_0,\mu_0} = \sup_N \sup_{s\leq N} \sup_{Z_s} \big( \exp(\beta_0 E_\varepsilon(Z_s) + \mu_0 s)\widetilde{f}^{(s)}_{N,0}(Z_s)\big) < +\infty\,,$$

satisfying (11.1.3) and (11.1.4). These give rise to a unique, uniformly bounded solution $\widetilde{F}_N$ to the BBGKY hierarchy thanks to Theorem 9 page 87, and to a unique solution $F$ to the Boltzmann hierarchy thanks to Theorem 10 page 88.

## 11.2. Convergence to the Boltzmann hierarchy

Our main result is the following.

***Theorem 11* (Convergence).** — *Assume the potential satisfies Assumption* 1.2.1 *as well as* (8.3.1)*. Let $\beta_0 > 0$ and $\mu_0 \in \mathbf{R}$ be given. There is a time $T > 0$ such that the following holds. For any admissible Boltzmann datum $F_0$ in $\mathbf{X}_{0,\beta_0,\mu_0}$ associated with a family $(\widetilde{F}_{0,N})_{N\geq 1}$ of BBGKY data in $\mathbf{X}_{\varepsilon,\beta_0,\mu_0}$, the solution $\widetilde{F}_N$ to the BBGKY hierarchy satisfies, in the sense of Definition* 6.2.1*,*

$$\widetilde{F}_N \xrightarrow{\sim} F$$

*uniformly on $[0,T]$, where $F$ is the solution to the Boltzmann hierarchy with data $F_0$.*

***Corollary 11.2.1.*** — *Assume the potential satisfies Assumption* 1.2.1 *as well as* (8.3.1)*. Let $\beta_0 > 0$ and $\mu_0 \in \mathbf{R}$ be given. There is a time $T > 0$ such that the following holds. For any admissible Boltzmann datum $F_0$ in $\mathbf{X}_{0,\beta_0,\mu_0}$ associated with a family $(\widetilde{F}_{0,N})_{N\geq 1}$ of BBGKY data in $\mathbf{X}_{\varepsilon,\beta_0,\mu_0}$, the associate family of untruncated marginals $F_N$ satisfies*

$$F_N \xrightarrow{\sim} F\,,$$

*uniformly on $[0,T]$, where $F$ is the solution to the Boltzmann hierarchy with data $F_0$.*

*Proof.* — By Proposition 11.1.1, the family $F_0$ is an admissible Boltzmann datum. Denoting $\widetilde{F}_{0,N}$ an associated BBGKY datum, let $T > 0$ be a time for which the solution the BBGKY hierarchy $\widetilde{F}_N$ with datum $\widetilde{F}_{0,N}$ has a uniform bound, as given by Theorem 9.

By Theorem 11, the convergence $I_{\varphi_s}\big(\widetilde{f}^{(s)}_N - f^{(s)}\big) \to 0$ holds uniformly in $[0,T]$ and locally uniformly in $\Omega_s$. Then, by Lemma 11.1.2 and Remark 11.1.3, there holds $f^{(s)}_N - \widetilde{f}^{(s)}_N \to 0$, for fixed $s$, uniformly in $[0,T]\times\Omega_s$. By Lemma 6.2.2, this implies $I_{\varphi_s}\big(f^{(s)}_N - \widetilde{f}^{(s)}_N\big) \to 0$, uniformly in $[0,T]$ and locally uniformly in $\Omega_s$. We conclude that $f^{(s)}_N \xrightarrow{\sim} f^{(s)}$, uniformly in $[0,T]$. □

In the next paragraph we shall prove that in the sum defining $\widetilde{f}^{(s)}_N(t)$ one can neglect all higher-order interactions and restrict our attention to the case when $m_i = 1$ for each $i \in [1,n]$ and each $n \in \mathbf{N}$. Then we can, exactly as in the hard spheres case discussed in Chapter 7, consider only a finite number of collisions, and reduce the study to bounded energies and well separated collision times.

### 11.3. Reductions of the BBGKY hierarchy, and pseudotrajectories

In this paragraph, we first prove that the estimates obtained in Chapter 10 enable us to reduce the study of the BBGKY hierarchy to the equation

$$\widetilde{g}_N^{(s)}(t,Z_s) = \mathbf{H}_s(t)\widetilde{f}_N^{(s)}(0,Z_s) + \int_0^t \mathbf{H}_s(t-\tau)\mathcal{C}_{s,s+1}\widetilde{g}_N^{(s+1)}(\tau,Z_s)\,d\tau\,, \qquad 1\leq s\leq N-1\,. \tag{11.3.1}$$

Estimate (10.3.2) in Proposition 10.3.1 shows indeed that higher-order collisions are negligible in the Boltzmann-Grad limit. For the solution to the BBGKY hierarchy, this translates as follows.

***Proposition 11.3.1***. — *Let $\beta_0 > 0$ and $\mu_0$ be given. Then with the same notation as Theorem 9, in the Boltzmann-Grad scaling $N\varepsilon^{d-1}\equiv 1$, any family of initial marginals $\widetilde{F}_N(0) = \big(\widetilde{f}_N^{(s)}(0)\big)_{1\leq s\leq N}$ in $\mathbf{X}_{\varepsilon,\beta_0,\mu_0}$ gives rise to a unique solution $\widetilde{G}_N\in\mathbf{X}_{\varepsilon,\boldsymbol{\beta},\boldsymbol{\mu}}$ of* (11.3.1) *and there holds the bound*

$$|\!|\!|\widetilde{G}_N|\!|\!|_{\varepsilon,\boldsymbol{\beta},\boldsymbol{\mu}} \leq 2\|\widetilde{F}_N(0)\|_{\varepsilon,\beta_0,\mu_0}\,.$$

*Besides, the solution $\widetilde{G}_N$ to the modified hierarchy* (11.3.1) *is asymptotically close to the solution $\widetilde{F}_N$ to the BBGKY hierarchy* (9.5.4)*:*

$$|\!|\!|\widetilde{G}_N-\widetilde{F}_N|\!|\!|_{\varepsilon,\boldsymbol{\beta},\boldsymbol{\mu}} \leq 2\varepsilon\|\widetilde{F}_N(0)\|_{\varepsilon,\beta_0,\mu_0}\,. \tag{11.3.2}$$

*Proof.* — We find the bound for $\widetilde{G}_N$, in the same way as for Theorem 9. We turn to the proof of (11.3.2). There holds

$$\begin{aligned}|\!|\!|\widetilde{G}_N-\widetilde{F}_N|\!|\!|_{\varepsilon,\boldsymbol{\beta},\boldsymbol{\mu}} \leq{}& \Big|\!\Big|\!\Big|\int_0^t \Big(\mathbf{H}_s(t-t')\mathcal{C}_{s,s+1}(\widetilde{g}_N^{(s+1)}-\widetilde{f}_N^{(s+1)})(t')\Big)_{1\leq s\leq N}dt'\Big|\!\Big|\!\Big|_{\varepsilon,\boldsymbol{\beta},\boldsymbol{\mu}}\\ &+\Big|\!\Big|\!\Big|\int_0^t \Big(\mathbf{H}_s(t-t')\sum_{2\leq m\leq N-s}\mathcal{C}_{s,s+m}f_N^{(s+m)}(t')\Big)_{1\leq s\leq N}dt'\Big|\!\Big|\!\Big|_{\varepsilon,\boldsymbol{\beta},\boldsymbol{\mu}}\,.\end{aligned}$$

With (10.3.1), this implies

$$|\!|\!|\widetilde{G}_N-\widetilde{F}_N|\!|\!|_{\varepsilon,\boldsymbol{\beta},\boldsymbol{\mu}} \leq c_0\Big|\!\Big|\!\Big|\int_0^t \Big(\mathbf{H}_s(t-t')\sum_{2\leq m\leq N-s}\mathcal{C}_{s,s+m}f_N^{(s+m)}(t')\Big)_{1\leq s\leq N}dt'\Big|\!\Big|\!\Big|_{\varepsilon,\boldsymbol{\beta},\boldsymbol{\mu}}\,,$$

with $c_0 := \big(1-\bar{c}(\beta_0,\mu_0,\lambda,T)\big)^{-1}$, which is indeed strictly positive by assumption. We conclude as in the proof of Proposition 10.3.1 and Lemma 10.3.2. □

One has the following formulation for $\widetilde{g}_N^{(s)}$ in terms of the initial datum:

$$\widetilde{g}_N^{(s)}(t) = \sum_{k=0}^\infty \int_0^t\int_0^{t_1}\dots\int_0^{t_{k-1}} \mathbf{H}_s(t-t_1)\mathcal{C}_{s,s+1}\mathbf{H}_{s+1}(t_1-t_2)\dots\mathbf{H}_{s+k}(t_k)\widetilde{f}_N^{(s+k)}(0)\,dt_k\dots dt_1\,.$$

We define the functional

$$\begin{aligned}I_s(t)(X_s) := \sum_{k=0}^\infty\int\varphi_s(V_s)\int_{\mathcal{T}_k(t)} &\mathbf{H}_s(t-t_1)\mathcal{C}_{s,s+1}\mathbf{H}_{s+1}(t_1-t_2)\mathcal{C}_{s+1,s+2}\\ &\dots\mathcal{C}_{s+k-1,s+k}\mathbf{H}_{s+k}(t_k-t_{k+1})\widetilde{f}_{N,0}^{(s+k)}dT_kdV_s\end{aligned}$$

and following Chapter 7, the reduced elementary functional

$$\begin{aligned}I_{s,k}^{R,\delta}(t)(X_s) := \int\varphi_s(V_s)\int_{\mathcal{T}_{k,\delta}(t)} &\mathbf{H}_s(t-t_1)\mathcal{C}_{s,s+1}\mathbf{H}_{s+1}(t_1-t_2)\mathcal{C}_{s+1,s+2}\\ &\dots\mathcal{C}_{s+k-1,s+k}\mathbf{H}_{s+k}(t_k-t_{k+1})1\!\!1_{E_\varepsilon(Z_{s+k})\leq R^2}\widetilde{f}_{N,0}^{(s+k)}dT_kdV_s\,.\end{aligned} \tag{11.3.3}$$

We can reproduce the proofs of Propositions 7.1.1, 7.2.1 and 7.3.1 to obtain the following result, as in Corollary 7.4.1.

***Proposition 11.3.2.*** — *With the notation of Theorem 9, given $s \in \mathbf{N}^*$ and $t \in [0,T]$, there are two positive constants $C$ and $C'$ such that for all $n \in \mathbf{N}^*$,*

$$\big\| I_s(t) - \sum_{k=0}^{n} I_{s,k}^{R,\delta}(t) \big\|_{L^\infty(\mathbf{R}^{ds})} \leq C \left( 2^{-n} + e^{-C'\beta_0 R^2} + \frac{n^2}{T}\delta \right) \|\varphi\|_{L^\infty(\mathbf{R}^{ds})} \|\widetilde{F}_{N,0}\|_{\varepsilon,\beta_0,\mu_0} \, .$$

As in the hard-spheres case, in the integrand of the collision operators $\mathcal{C}_{s,s+1}$ defined in (9.4.4), we can distinguish between pre- and post-collisional configurations, as we decompose

$$\mathcal{C}_{s,s+1} = \mathcal{C}^+_{s,s+1} - \mathcal{C}^-_{s,s+1}$$

where

$$\mathcal{C}^{\pm}_{s,s+1} \widetilde{g}^{(s+1)} = \sum_{m=1}^{s} \mathcal{C}^{\pm,m}_{s,s+1} \widetilde{g}^{(s+1)} \tag{11.3.4}$$

the index $m$ referring to the index of the interacting particle among the $s$ "fixed" particles, with the notation

$$\begin{aligned}\big(\mathcal{C}^{\pm,m}_{s,s+1} \widetilde{g}^{(s+1)}\big)(Z_s) := (N-s)\varepsilon^{d-1} \int_{\mathbf{S}_1^{d-1}\times\mathbf{R}^d} (\nu\cdot(v_{s+1}-v_m))_{\pm} \widetilde{g}^{(s+1)}(Z_s, x_m+\varepsilon\nu, v_{s+1}) \\ \times \prod_{\substack{1\leq j\leq s \\ j\neq m}} 1\!\!1_{|x_j - x_{s+1}|\geq\varepsilon} \, d\nu dv_{s+1} \, ,\end{aligned}$$

the index $+$ corresponding to post-collisional configurations and the index $-$ to pre-collisional configurations, according to terminology set out in Chapter 8.

The elementary BBGKY observables we are interested in can therefore be decomposed as

$$I_{s,k}^{R,\delta}(t, X_s) = \sum_{J,M} \Big( \prod_{i=1}^{k} j_i \Big) I_{s,k}^{R,\delta}(t,J,M)(X_s) \tag{11.3.5}$$

where the elementary functionals $I_{s,k}^{R,\delta}(t,J,M)$ are defined by

$$\begin{aligned} I_{s,k}^{R,\delta}(t,J,M)(X_s) := \int \varphi_s(V_s) \int_{\mathcal{T}_{k,\delta}(t)} \mathbf{H}_s(t-t_1) \mathcal{C}^{j_1,m_1}_{s,s+1} \mathbf{H}_{s+1}(t_1-t_2) \mathcal{C}^{j_2,m_2}_{s+1,s+2} \\ \dots \mathbf{H}_{s+k}(t_k - t_{k+1}) 1\!\!1_{E_\varepsilon(Z_{s+k})\leq R^2} \widetilde{f}^{(s+k)}_{N,0} \, dT_k dV_s \, ,\end{aligned}$$

with

$$J := (j_1,\dots,j_k) \in \{+,-\}^k \quad \text{and} \quad M := (m_1,\dots,m_k) \quad \text{with} \quad m_i \in \{1,\dots,s+i-1\} \, .$$

As in the hard spheres case, we still cannot study directly the convergence of $I_{s,n}^{R,\delta}(t,J,M) - I_{s,n}^{0,R,\delta}(t,J,M)$ since the transport operators $\mathbf{H}_k$ do not coincide everywhere with the free transport operators $\mathbf{S}_k$, which means – in terms of pseudo-trajectories – that there are recollisions. Note that, because the interaction potential is compactly supported, recollisions happen only for characteristics such that there exist $i,j \in [1,k]$ with $i \neq j$, and $\tau > 0$ such that

$$|(x_i - \tau v_i) - (x_j - \tau v_j)| \leq \varepsilon \, .$$

We shall thus prove that these recollisions arise only for a few pathological pseudo-trajectories, which can be eliminated by additional truncations of the domains of integration. This is the goal of Part IV, which deals with the hard-spheres and the potential case simultaneously.

# PART IV

# TERM-BY-TERM CONVERGENCE

# CHAPTER 12

# ELIMINATION OF RECOLLISIONS

This last part is the heart of our contribution. We prove the term-by-term convergence of the series giving the observables, both in the case of hard spheres and in the case of smooth hamiltonian systems.

We have indeed seen in Corollary 7.4.1 (for the hard-spheres case) and Proposition 11.3.2 (for the potential case) that the convergence of observables reduces to the convergence to zero of the elementary functionals $I_{s,k}^{R,\delta} - I_{s,k}^{0,R,\delta}$, where $I_{s,k}^{R,\delta}$ is defined in (7.3.1) in the hard-spheres case and in (11.3.3) for the potential case, and $I_{s,k}^{0,R,\delta}$ is defined in (7.3.1). These functionals correspond to dynamics

- involving only a finite number $s+k$ of particles,
- with bounded energies (at most $R^2$),
- such that the $k$ additional particles are adjoined through binary collisions,
- at times separated at least by $\delta$.

What we shall establish is that recollisions can occur only for very pathological pseudo-trajectories, in the sense that the velocities and impact parameters of the additional particles in the collision trees have to be chosen in small measure sets.

We point out the fact that, even in the case of hard spheres, these bad sets are generally not of zero measure because they are built as non countable unions of zero measure sets. The arguments are actually very similar whatever the precise nature of the microscopic interaction.

The only differences we shall see between the case of hard spheres and the case of smooth potentials are the following:

- the parametrization of collisions by the deflection angle is trivial in the case of hard spheres since it coincides exactly with the impact parameter;
- there is no time shift between pre-collisional and post-collisional configurations in the case of hard spheres since the reflection is instantaneous.

These two simplifications will enable us to obtain explicit estimates on the convergence rate in the case of hard spheres. For more general interactions, this convergence rate can be expressed as an implicit function depending on the potential.

### 12.1. Stability of good configurations by adjunction of collisional particles

In this paragraph we momentarily forget the BBGKY and Boltzmann hierarchies, and focus on the study of pseudo-trajectories.

***Definition 12.1.1* (Good configuration).** — *For any constant $c > 0$, we denote by $\mathcal{G}_k(c)$ the set of "good configurations" of $k$ particles, separated by at least $c$ through backwards transport: that is the set of $(X_k, V_k) \in \mathbf{R}^{dk} \times B_R^k$ such that the image of $(X_k, V_k)$ by the backward free transport satisfies the separation condition*

$$\forall \tau \geq 0, \quad \forall i \neq j, \quad |x_i - x_j - \tau(v_i - v_j)| \geq c\,,$$

*in particular it is never collisional.*

We recall that $B_R^k := \Big\{ V_k \in \mathbf{R}^{dk} \,/\, |V_k| \leq R \Big\}$ and in the following we write $B_R := B_R^1$.

Our aim is to show that "good configurations" are stable by adjunction of a collisional particle provided that the deflection angle and the velocity of the additional particle do not belong to a small pathological set. Furthermore the set to be excluded can be chosen in a uniform way with respect to the initial positions of the particles in a small neighborhood of any fixed "good configuration".

***Notation 12.1.2*.** — *In all the sequel, given two positive parameters $\eta_1$ and $\eta_2$, we shall say that*

$$\eta_1 \ll \eta_2 \ \text{ if } \ \eta_1 \leq C\eta_2$$

*for some large constant $C$ which does not depend on any parameter.*

In the following we shall fix three parameters $a, \varepsilon_0, \eta \ll 1$ such that

$$a \ll \varepsilon_0 \ll \eta\delta\,. \tag{12.1.1}$$

We recall that the parameter $\delta$ scales like time while we shall see that $\eta$, like $R$, scales like a velocity. The parameters $a$ and $\varepsilon_0$, just like $\varepsilon$, will have the scaling of a distance.

***Proposition 12.1.1*.** — *Let $a, \varepsilon_0, \eta \ll 1$ satisfy* (12.1.1). *Given $\overline{Z}_k \in \mathcal{G}_k(\varepsilon_0)$, there is a subset $\mathcal{B}_k(\overline{Z}_k)$ of $\mathbf{S}_1^{d-1} \times B_R$ of small measure: for some fixed constant $C > 0$ and some constant $C(\Phi, \eta, R) > 0$,*

$$\begin{aligned} &\big|\mathcal{B}_k(\overline{Z}_k)\big| \leq Ck\Big(R\eta^{d-1} + R^d\Big(\frac{a}{\varepsilon_0}\Big)^{d-1} + R\Big(\frac{\varepsilon_0}{\delta}\Big)^{d-1}\Big) \\ &\qquad \textit{in the case of hard spheres} \\ &\big|\mathcal{B}_k(\overline{Z}_k)\big| \leq Ck\Big(R\eta^{d-1} + C(\Phi, R, \eta)\, R^d\Big(\frac{a}{\varepsilon_0}\Big)^{d-1} + C(\Phi, R, \eta)\, R\Big(\frac{\varepsilon_0}{\delta}\Big)^{d-1}\Big) \\ &\qquad \textit{in the case of a smooth interaction potential } \Phi\,, \end{aligned} \tag{12.1.2}$$

*and such that good configurations close to $\overline{Z}_k$ are stable by adjunction of a collisional particle close to $\bar{x}_k$ and not belonging to $\mathcal{B}_k(\overline{Z}_k)$, in the following sense.*

*Consider $(\nu, v) \in (\mathbf{S}_1^{d-1} \times B_R) \setminus \mathcal{B}_k(\overline{Z}_k)$ and let $Z_k$ be a configuration of $k$ particles such that $V_k = \overline{V}_k$ and $|X_k - \overline{X}_k| \leq a$.*

- *If $\nu \cdot (v - \bar{v}_k) < 0$ then for all $\varepsilon > 0$ sufficiently small,*

$$\forall \tau \geq 0\,, \quad \begin{cases} \forall i \neq j \in [1, k]\,, \quad |(x_i - \tau\bar{v}_i) - (x_j - \tau\bar{v}_j)| \geq \varepsilon\,, \\ \forall j \in [1, k]\,, \quad |(x_k + \varepsilon\nu - \tau v) - (x_j - \tau\bar{v}_j)| \geq \varepsilon\,. \end{cases} \tag{12.1.3}$$

*Moreover after the time $\delta$, the $k+1$ particles are in a good configuration:*

$$(X_k - \delta \overline{V}_k, \overline{V}_k, x_k + \varepsilon\nu - \delta v, v) \in \mathcal{G}_{k+1}(\varepsilon_0/2)\,. \tag{12.1.4}$$

- *If $\nu \cdot (v - \bar v_k) > 0$ then define for $j \in [1, k-1]$*

$$(z_k^{\varepsilon *}, z^{\varepsilon *}) := \sigma^{-1}\big(z_k, (x_k + \varepsilon\nu, v)\big) \quad \text{and} \quad z_j^{\varepsilon *} := (x_j - \bar v_j, \bar v_j)$$

*in the hard-spheres case, where $\sigma$ is defined in* (4.4.2), *and*

$$(z_k^{\varepsilon *}, z^{\varepsilon *}) := \sigma_\varepsilon^{-1}\big(z_k, (x_k + \varepsilon\nu, v)\big) \quad \text{and} \quad z_j^{\varepsilon *} := (x_j - t_\varepsilon \bar v_j, \bar v_j)$$

*in the potential case, where $\sigma_\varepsilon$ is the scattering operator as in Definition* 8.2.1 *and where $t_\varepsilon < \delta$ denotes the scattering time between $z_k$ and $(x_k + \varepsilon\nu, v)$. Then for all $\varepsilon > 0$ sufficiently small,*

$$\forall \tau \geq 0\,, \quad \begin{cases} \forall i \neq j \in [1,k]\,, & |(x_i^{\varepsilon *} - \tau v_i^{\varepsilon *}) - (x_j^{\varepsilon *} - \tau v_j^{\varepsilon *})| \geq \varepsilon\,, \\ \forall j \in [1,k]\,, & |(x^{\varepsilon *} - \tau v^{\varepsilon *}) - (x_j^{\varepsilon *} - \tau v_j^{\varepsilon *})| \geq \varepsilon\,. \end{cases} \tag{12.1.5}$$

*Moreover after the time $\delta$, the $k+1$ particles are in a good configuration:*

$$\Big(X_k^{\varepsilon *} - (\delta - t_\varepsilon) V_k^{\varepsilon *}, V_k^{\varepsilon *}, x^{\varepsilon *} - (\delta - t_\varepsilon) v^{\varepsilon *}, v^{\varepsilon *}\Big) \in \mathcal{G}_{k+1}(\varepsilon_0/2)\,, \tag{12.1.6}$$

*with $t_\varepsilon := 0$ in the hard-spheres case.*

The proof of the proposition may be found in Section 12.3. It relies on some elementary geometrical lemmas, stated and proved in the next section. The first one describes the bad trajectories associated with (free) transport. The other ones explain how they are modified by collisions, both in the case of hard spheres and in the case of smooth interactions.

***Remark 12.1.3***. — *For the sake of simplicity, we have assumed in the statement of Proposition* 12.1.1 *that the additional particle collides with the particle numbered $k$. Of course, a simple symmetry argument shows that an analogous statement holds if the new particle is added close to any of the particles in $Z_k$.*

*The proof of Proposition* 12.1.1 *shows that if $Z_k = \overline{Z}_k$ then the factor $\varepsilon_0/2$ in* (12.1.4) *and* (12.1.6) *may be replaced by $\varepsilon_0$. The loss if $Z_k \neq \overline{Z}_k$ comes from the fact that the set to be excluded has to be chosen in a uniform way with respect to the initial positions of the particles in a small neighborhood of $\overline{X}_k$.*

## 12.2. Geometrical lemmas

We first consider the case of two particles moving freely, and describe the set of velocities $v_2$ leading possibly to collisions (or recollisions).

Here and in the sequel, we denote by $K(w, y, \rho)$ the cylinder of origin $w \in \mathbf{R}^d$, of axis $y \in \mathbf{R}^d$ and radius $\rho > 0$ and by $B_\rho(y)$ the ball centered at $y$ of radius $\rho$.

### 12.2.1. Bad trajectories associated to free transport. —

***Lemma 12.2.1.*** — *Given $\bar a > 0$ satisfying $\varepsilon \ll \bar a \ll \varepsilon_0$, consider $\bar x_1, \bar x_2$ in $\mathbf{R}^d$ such that $|\bar x_1 - \bar x_2| \geq \varepsilon_0$, and $v_1 \in B_R$. Then for any $x_1 \in B_{\bar a}(\bar x_1)$, any $x_2 \in B_{\bar a}(\bar x_2)$ and any $v_2 \in B_R$, the following results hold.*

- *If $v_2 \notin K(v_1, \bar x_1 - \bar x_2, 6R\bar a/\varepsilon_0)$, then*
$$\forall \tau \geq 0\,, \quad |(x_1 - v_1\tau) - (x_2 - v_2\tau)| > \varepsilon\,;$$
- *If $v_2 \notin K(v_1, \bar x_1 - \bar x_2, 6\varepsilon_0/\delta)$*
$$\forall \tau \geq \delta\,, \quad |(x_1 - v_1\tau) - (x_2 - v_2\tau)| > \varepsilon_0\,.$$

*Proof.* — • Assume that there exists $\tau_*$ such that
$$|(x_1 - v_1\tau_*) - (x_2 - v_2\tau_*)| \leq \varepsilon\,.$$
Then, by the triangular inequality and provided that $\varepsilon$ is sufficiently small,
$$|(\bar x_1 - \bar x_2) - \tau_*(v_1 - v_2)| \leq \varepsilon + 2\bar a \leq 3\bar a\,.$$
This means that $(v_1 - v_2)$ belongs to the cone of vertex 0 based on the ball centered at $\bar x_1 - \bar x_2$ and of radius $3\bar a$, which is a cone of solid angle $(3\bar a/|\bar x_1 - \bar x_2|)^{d-1}$ (since $\bar a \ll \varepsilon_0$).

The intersection of this cone and of the sphere of radius $2R$ is obviously embedded in the cylinder of axis $\bar x_1 - \bar x_2$ and radius $6R\bar a/\varepsilon_0$, which proves the first result.

• Similarly assume that there exists $\tau^* \geq \delta$ such that
$$|(x_1 - v_1\tau_*) - (x_2 - v_2\tau_*)| \leq \varepsilon_0\,.$$
Then, by the triangular inequality again,
$$|(\bar x_1 - \bar x_2) - \tau_*(v_1 - v_2)| \leq \varepsilon_0 + 2\bar a \leq 3\varepsilon_0\,.$$
In particular, for any unit vector $n$ orthogonal to $\bar x_1 - \bar x_2$,
$$\tau^*|n \cdot (v_1 - v_2)| = |n \cdot ((\bar x_1 - \bar x_2) - \tau_*(v_1 - v_2))\,| \leq 3\varepsilon_0\,.$$
This tells us exactly that $v_1 - v_2$ belongs to the cylinder of axis $\bar x_1 - \bar x_2$ and radius $3\varepsilon_0/\delta$.

The lemma is proved. $\square$

### 12.2.2. Modification of bad trajectories by hard sphere reflection. —

We now consider the case when particles 1 and 2 undergo a hard sphere collision before being transported, and look at impact parameters $\nu$ and velocities $v_2$ leading possibly to collisions (or recollisions).

***Lemma 12.2.2.*** — *Consider $\rho \ll R$, and $(y, w) \in \mathbf{R}^d \times B_R$. For any $v_1$ in $B_R$, define*
$$\begin{aligned}\mathcal{N}^*(w, y, \rho)(v_1) := \big\{(\nu, v_2) \in \mathbf{S}_1^{d-1} \times B_R \,/\, (v_2 - v_1)\cdot \nu > 0\,,\\ v_1^* \in K(w, y, \rho) \text{ or } v_2^* \in K(w, y, \rho)\big\}\,,\end{aligned}$$
*where*
$$v_1^* := v_1 - \nu \cdot (v_1 - v_2)\,\nu \quad \text{and} \quad v_2^* := v_2 + \ \nu \cdot (v_1 - v_2)\,\nu\,.$$
*Then*
$$|\mathcal{N}^*(w, y, \rho)(v_1)| \leq C_d R \rho^{d-1}\,,$$
*where the constant $C_d$ depends only on the dimension $d$.*

*Proof.* — Denote by $r = |v_1 - v_2| = |v_1^* - v_2^*|$. The reflection condition shows that, as $\nu$ varies in $\mathbf{S}_1^{d-1}$, the velocities $v_1^*$ and $v_2^*$ range over a sphere of diameter $r$.

The solid angle of the intersection of such a sphere with the cylinder $K(w,y,\rho)$ is less than

$$C_d \min\left(1, \left(\frac{\rho}{r}\right)^{d-1}\right)$$

which implies that

$$\begin{aligned}\Big|\big\{(\nu,v_2)\,/\,v_1^* \in K(w,y,\rho) \text{ or } v_2^* \in K(w,y,\rho)\big\}\Big| &\leq C_d \int r^{d-1} \min\left(1, \left(\frac{\rho}{r}\right)^{d-1}\right) dr\\ &\leq C_d R \rho^{d-1}\,.\end{aligned}$$

This proves Lemma 12.2.2. □

### 12.2.3. Modification of bad trajectories by the scattering associated to $\Phi$. —

The last geometrical lemma requires the use of notation coming from scattering theory, introduced in Chapter 8: it states that if two particles $z_1, z_2$ in $\mathbf{R}^{2d}$ are in a post-collisional configuration and if $v_1$ or $v_2$ belong to a cylinder as in Lemma 12.2.1, then the pre-image $z_2^*$ of $z_2$ through the scattering operator belongs to a small set of $\mathbf{R}^{2d}$.

***Lemma 12.2.3.*** — *Consider two parameters $\rho \ll R$ and $\eta \ll 1$, and $(y,w) \in \mathbf{R}^d \times B_R$. For any $v_1$ in $B_R$, define*

$$\begin{aligned}\mathcal{N}^*(w,y,\rho)(v_1) := \big\{(\nu,v_2) \in \mathbf{S}_1^{d-1} \times B_R\,/\,(v_2 - v_1)\cdot\nu > \eta\,,&\\ v_1^* \in K(w,y,\rho) \text{ or } v_2^* \in K(w,y,\rho)\big\}\,,&\end{aligned}$$

*where $(\nu^*, v_1^*, v_2^*) = \sigma_0^{-1}(\nu, v_1, v_2)$ with the notations of Chapter 8. Then*

$$|\mathcal{N}^*(w,y,\rho)(v_1)| \leq C(\Phi,R,\eta) R \rho^{d-1}$$

*where the constant depends on the potential $\Phi$ through the $L^\infty$ norm of the cross-section $b$ on the compact set $B_{2R} \times [\eta/2R, \pi/2]$ defined in Chapter 8.*

*Proof.* — Denote by $r = |v_1 - v_2| = |v_1^* - v_2^*|$, and by $\omega$ the deflection angle.

From the proof of the previous lemma, we deduce that

$$\begin{aligned}\Big|\big\{(\omega,v_2)\,/\,v_1^* \in K(w,y,\rho) \text{ or } v_2^* \in K(w,y,\rho)\big\}\Big| &\leq C_d \int r^{d-1} \min\left(1, \left(\frac{\rho}{r}\right)^{d-1}\right) dr\\ &\leq C_d R \rho^{d-1}\,.\end{aligned}$$

According to Chapter 8, the change of variables $(\nu, v_1 - v_2) \mapsto (\omega, v_1 - v_2)$ is a Lipschitz diffeomorphism away from $\nu \cdot (v_1 - v_2) = 0$. We therefore get the expected estimate. □

***Remark 12.2.4.*** — *Note that the geometrical Lemmas 12.2.1 to 12.2.3 consist in eliminating sets in the velocity variables and deflection angles only, and do not concern the position variables.*

## 12.3. Proof of the geometric proposition

In this section we prove Proposition 12.1.1. We fix a good configuration $\overline{Z}_k \in \mathcal{G}_k(\varepsilon_0)$, and we consider a configuration $Z_k \in \mathbf{R}^{2dk}$, with the same velocities as $\overline{Z}_k$, and neighboring positions: $|X_k - \overline{X}_k| \leq a$. In particular we notice that for all $\tau \geq 0$ and all $i \neq j$,

$$|x_i - x_j - \tau(\bar{v}_i - \bar{v}_j)| \geq |\bar{x}_i - \bar{x}_j - \tau(\bar{v}_i - \bar{v}_j)| - 2a \geq \varepsilon_0/2 \tag{12.3.7}$$

since $a \ll \varepsilon_0$. This implies that $Z_k \in \mathcal{G}_k(\varepsilon_0/2)$. Next we consider an additional particle $(x_k + \varepsilon\nu, v_{k+1})$ and we shall separate the analysis into two parts, depending on whether the situation is pre-collisional (meaning $\nu \cdot (v_{k+1} - \bar{v}_k) < 0$) or post-collisional (meaning $\nu \cdot (v_{k+1} - \bar{v}_k) > 0$).

### 12.3.1. The pre-collisional case. — We assume that

$$\nu \cdot (v_{k+1} - \bar{v}_k) < 0\,,$$

meaning that $(x_k + \varepsilon\nu, v_{k+1})$ and $z_k$ form a pre-collisional pair. In particular we have for all times $\tau \geq 0$ and all $\varepsilon > 0$

$$\Big|\big(x_k + \varepsilon\nu - v_{k+1}\tau\big) - \big(x_k - \bar{v}_k\tau\big)\Big| \geq \varepsilon\,.$$

Furthermore up to excluding the ball $B_\eta(\bar{v}_k)$ in the set of admissible $v_{k+1}$, we may assume that

$$|v_{k+1} - \bar{v}_k| > \eta\,.$$

Under that assumption we have for all $\tau \geq \delta$ and all $\varepsilon > 0$ sufficiently small,

$$\begin{aligned}\Big|\big(x_k + \varepsilon\nu - v_{k+1}\tau\big) - \big(x_k - \bar{v}_k\tau\big)\Big| &\geq \tau|v_{k+1} - \bar{v}_k| - \varepsilon \\ &\geq \delta\eta - \varepsilon > \varepsilon_0/2\,.\end{aligned}$$

Furthermore we know that $Z_k$ belongs to $\mathcal{G}_k(\varepsilon_0/2)$ thanks to (12.3.7).

Now let $j \in [1, k-1]$ be given. According to Lemma 12.2.1, we find that for any $v_{k+1}$ belonging to the set $B_R \setminus K(\bar{v}_j, \bar{x}_j - \bar{x}_k, 6Ra/\varepsilon_0 + 6\varepsilon_0/\delta)$, we have

$$\forall \tau \geq 0\,, \quad |(x_k + \varepsilon\nu - v_{k+1}\tau) - (x_j - \bar{v}_j\tau)| > \varepsilon\,,$$

and

$$\forall \tau \geq \delta\,, \quad |(x_k + \varepsilon\nu - v_{k+1}\tau) - (x_j - \bar{v}_j\tau)| > \varepsilon_0\,.$$

Notice that

$$\Big|B_R \cap K(\bar{v}_j, \bar{x}_j - \bar{x}_k, 6Ra/\varepsilon_0 + 6\varepsilon_0/\delta)\Big| \leq C\Big(R^d\Big(\frac{a}{\varepsilon_0}\Big)^{d-1} + R\Big(\frac{\varepsilon_0}{\delta}\Big)^{d-1}\Big)\,.$$

Defining $\mathcal{M}^-(\overline{Z}_k) := \displaystyle\bigcup_{j \leq k-1} K(\bar{v}_j, \bar{x}_j - \bar{x}_k, 6Ra/\varepsilon_0 + 6\varepsilon_0/\delta)$ and

$$\mathcal{B}_k^-(\overline{Z}_k) := \mathbf{S}_1^{d-1} \times \Big(B_\eta(\bar{v}_k) \cup \mathcal{M}^-(\overline{Z}_k)\Big)$$

we find that

$$\Big|\mathcal{B}_k^-(\overline{Z}_k)\Big| \leq Ck\Big(\eta^d + R^d\big(\frac{a}{\varepsilon_0}\big)^{d-1} + R\big(\frac{\varepsilon_0}{\delta}\big)^{d-1}\Big)$$

and (12.1.3) and (12.1.4) hold as soon as $(\nu, v_{k+1}) \notin \mathcal{B}_k^-(\overline{Z}_k)$.

**12.3.2. The post-collisional case with hard sphere reflection. —**

We now assume that
$$\nu \cdot (v_{k+1} - \bar v_k) > 0\,,$$
meaning that $(x_k + \varepsilon\nu, v_{k+1})$ and $z_k$ form a post-collisional pair. In particular, at time $\tau = 0+$, the configuration is changed and we have the pre-collisional pair $(x_k + \varepsilon\nu, v^*_{k+1})$ and $(x_k, v^*_k)$ where $v^*_k$ and $v^*_{k+1}$ are defined by the usual reflection condition. Furthermore, we have for all times $\tau \geq 0$ and all $\varepsilon > 0$
$$\big|\big(x_k + \varepsilon\nu - v^*_{k+1}\tau\big) - \big(x_k - v^*_k\tau\big)\big| \geq \varepsilon\,.$$

We can then repeat the same arguments as inthe pre-collisional case replacing $\bar v_k, v_{k+1}$ by $v^*_k, v^*_{k+1}$.

Excluding the ball $B_\eta(\bar v_k)$ in the set of admissible $v_{k+1}$, we find that
$$\begin{aligned}\big|\big(x_k + \varepsilon\nu - v^*_{k+1}\tau\big) - \big(x_k - v^*_k\tau\big)\big| &\geq \tau|v_{k+1} - \bar v_k| - \varepsilon\\ &\geq \delta\eta - \varepsilon > \varepsilon_0/2\,.\end{aligned}$$

According to Lemma 12.2.1, if $v^*_k, v^*_{k+1}$ belong to the set $B_R \setminus K(\bar v_j, \bar x_j - \bar x_k, 6Ra/\varepsilon_0 + 6\varepsilon_0/\delta)$, we have
$$\begin{aligned}\forall \tau \geq 0\,,\quad &|(x_k + \varepsilon\nu - v^*_{k+1}\tau) - (x_j - \bar v_j\tau)| > \varepsilon\,,\\ &|(x_k - v^*_k\tau) - (x_j - \bar v_j\tau)| > \varepsilon\,,\end{aligned}$$
and
$$\begin{aligned}\forall \tau \geq \delta\,,\quad &|(x_k + \varepsilon\nu - v^*_{k+1}\tau) - (x_j - \bar v_j\tau)| > \varepsilon_0\\ &|(x_k - v^*_k\tau) - (x_j - \bar v_j\tau)| > \varepsilon_0\,.\end{aligned}$$

Combining Lemmas 12.2.1and 12.2.2, we therefore obtain that (12.1.3) and (12.1.4) hold as soon as $(\nu, v_{k+1}) \notin \mathcal{B}^+_k(\overline Z_k)$ where
$$\mathcal{B}^+_k(\overline Z_k) := \mathbf{S}_1^{d-1} \times B_\eta(\bar v_k) \cup \bigcup_{j \leq k-1} \mathcal{N}^*(\bar v_j, \bar x_j - \bar x_k, 6Ra/\varepsilon_0 + 6\varepsilon_0/\delta)(\bar v_k)\,.$$

In particular,
$$\Big|\mathcal{B}^+_k(\overline Z_k)\Big| \leq Ck\Big(\eta^d + R^d\big(\frac{a}{\varepsilon_0}\big)^{d-1} + R\big(\frac{\varepsilon_0}{\delta}\big)^{d-1}\Big)\,.$$

**12.3.3. The post-collisional case with smooth scattering. —**

In the case of a smooth interaction potential, dealing with the post-collisional case is a little bit more intricate because of the time shift. Furthermore, using Lemma 12.2.3 instead of Lemma 12.2.2, we lose the explicit estimate for the bad set $\mathcal{B}^+_k(\overline Z_k)$.

Let us first define
$$C(\overline Z_k) := \Big\{(\nu, v_{k+1}) \in \mathbf{S}_1^d \times B_R\,,\, \nu \cdot (v_{k+1} - \bar v_k) \leq \eta\Big\}\,, \tag{12.3.8}$$
which satisfies
$$|C(\overline Z_k)| \leq CR\eta^{d-1}\,.$$

Choosing $(\nu, v_{k+1}) \in (\mathbf{S}_1^d \times B_R) \setminus C(\overline Z_k)$ ensures that the cross-section is well defined (see Definition 8.3.3), and that the scattering time $t_\varepsilon$ is of order $C(\Phi, R, \eta)\varepsilon$ by Proposition 8.2.1.

Considering the formulas (8.2.2) expressing $(z_k^{\varepsilon*}, z_{k+1}^{\varepsilon*})$ in terms of $\big(z_k, (x_k+\varepsilon\nu, v_{k+1})\big)$, we know that

$$
\begin{aligned}
|x_k^{\varepsilon*} - x_k| &\leq \frac{1}{2}|x_k^{\varepsilon*} - x_{k+1}^{\varepsilon^*}| + \frac{1}{2}|(x_k^{\varepsilon*} + x_{k+1}^{\varepsilon^*}) - (x_k + x_{k+1})| + \frac{1}{2}|(x_k - x_{k+1})| \\
&\leq Rt_\varepsilon + \varepsilon \leq C(\Phi, R, \eta)\varepsilon\,, \\
|x_{k+1}^{\varepsilon*} - (x_k + \varepsilon\nu)| &\leq \frac{1}{2}|x_k^{\varepsilon*} - x_{k+1}^{\varepsilon^*}| + \frac{1}{2}|(x_k^{\varepsilon*} + x_{k+1}^{\varepsilon^*}) - (x_k + x_{k+1})| + \frac{1}{2}|(x_k - x_{k+1})| \\
&\leq Rt_\varepsilon + \varepsilon \leq C(\Phi, R, \eta)\varepsilon\,.
\end{aligned}
\tag{12.3.9}
$$

Note that due to (12.3.7), all particles $x_j$ with $j \leq k-1$ are at a distance at least $\varepsilon_0/2 - \varepsilon \geq \varepsilon_0/3$ of the particles $x_k$ and $x_k + \varepsilon\nu$. Since they have bounded velocities, they cannot enter the protection spheres of these post-collisional particles during the interaction time $t_\varepsilon$, provided that $\varepsilon$ is small enough:

$$Rt_\varepsilon \ll \varepsilon_0/3\,.$$

Since the dynamics of the particles $j \leq k-1$ is not affected by the scattering, we get that $Z_{k-1}^{\varepsilon*}$ belongs to $\mathcal{G}_{k-1}(\varepsilon_0/2)$:

$$\forall \tau \geq 0\,,\ \forall (i,j) \in [1, k-1]^2 \text{ with } i \neq j\,, \quad |x_i^{\varepsilon*} - x_j^{\varepsilon*} - \tau(v_i^{\varepsilon*} - v_j^{\varepsilon*})| \geq \varepsilon_0/2\,. \tag{12.3.10}$$

The pair $(z_k^{\varepsilon*}, z_{k+1}^{\varepsilon*})$ is a pre-collisional pair by definition, so we know that for all $\tau \geq 0$,

$$|(x_k^{\varepsilon*} - \tau v_k^{\varepsilon*}) - (x_{k+1}^{\varepsilon*} - \tau v_{k+1}^{\varepsilon*})| \geq \varepsilon\,.$$

Excluding the ball $B_\eta(\bar v_k)$ in the set of admissible $v_{k+1}$, we find as above that

$$\forall \tau \geq \delta\,, \quad |x_k^{\varepsilon*} - x^{\varepsilon*} - \tau(v_k^{\varepsilon*} - v_{k+1}^{\varepsilon*})| \geq \eta\delta - \varepsilon \geq \varepsilon_0\,,$$

for $\varepsilon$ sufficiently small, since $\varepsilon_0 \ll \eta\delta$.

Next for $j \leq k-1$ we have for $\varepsilon$ sufficiently small, recalling that the uniform, rectilinear motion of the center of mass as described in (8.1.3),

$$
\begin{aligned}
|x_j^{\varepsilon*} - \bar x_j| &\leq |x_j^{\varepsilon*} - x_j| + |x_j - \bar x_j| \leq Rt_\varepsilon + a \leq 2a \\
|x_k^{\varepsilon*} - \bar x_k| &\leq |x_k^{\varepsilon*} - x_k| + |x_k - \bar x_k| \leq Rt_\varepsilon + \varepsilon + a \leq 2a \\
|x_{k+1}^{\varepsilon*} - \bar x_k| &\leq |x_{k+1}^{\varepsilon*} - x_{k+1}| + |x_k + \varepsilon\nu - \bar x_k| \leq Rt_\varepsilon + 2\varepsilon + a \leq 2a\,.
\end{aligned}
$$

By Lemma 12.2.1, provided $v_k^{\varepsilon*}$ and $v_{k+1}^{\varepsilon*}$ do not belong to

$$K(\bar v_j, \bar x_j - \bar x_k, 12Ra/\varepsilon_0 + 12\varepsilon_0/\delta) \cap B_R\,,$$

we get since $v_j^{\varepsilon*} = \bar v_j$,

$$
\begin{aligned}
\forall \tau \geq 0\,, \quad &|x_k^{\varepsilon*} - x_j^{\varepsilon*} - \tau(v_k^{\varepsilon*} - v_j^{\varepsilon*})| \geq \varepsilon\,, \\
\text{and} \quad &|x_{k+1}^{\varepsilon*} - x_j^{\varepsilon*} - \tau(v_{k+1}^{\varepsilon*} - v_j^{\varepsilon*})| \geq \varepsilon
\end{aligned}
$$

as well as

$$
\begin{aligned}
\forall \tau \geq \delta/2\,, \quad &|x_k^{\varepsilon*} - x_j^{\varepsilon*} - \tau(v_k^{\varepsilon*} - v_j^{\varepsilon*})| \geq \varepsilon_0/2\,, \\
\text{and} \quad &|x_{k+1}^{\varepsilon*} - x_j^{\varepsilon*} - \tau(v_{k+1}^{\varepsilon*} - v_j^{\varepsilon*})| \geq \varepsilon_0/2\,.
\end{aligned}
$$

Lemma 12.2.3 bounds from the above the size of the set $\mathcal{N}^*(\bar v_j, \bar x_j - \bar x_k, \rho)$ of all $(\nu, v_{k+1})$ belonging to $(\mathbf{S}_1^d \times B_R) \setminus C(\overline{Z}_k)$ such that $v_k^{\varepsilon*}$ or $v_{k+1}^{\varepsilon*}$ belongs to $K(\bar v_j, \bar x_j - \bar x_k, \rho)$. We let $\rho = 12Ra/\varepsilon_0 + 12\varepsilon_0/\delta$, and define

$$\mathcal{B}_k^+(\overline{Z}_k) := C(\overline{Z}_k) \cup \big(\mathbf{S}_1^{d-1} \times B_\eta(\bar v_k)\big) \bigcup_{j \leq k-1} \mathcal{N}^*(\bar v_j, \bar x_j - \bar x_k, 12Ra/\varepsilon_0 + 12\varepsilon_0/\delta)(\bar v_k)\,.$$

By Lemma 12.2.3,

$$\Big|\mathcal{B}_k^+(\overline{Z}_k)\Big| \leq CkR\eta^{d-1} + C(\Phi, R, \eta)R\big(R\frac{a}{\varepsilon_0} + \frac{\varepsilon_0}{\delta}\big)^{d-1}$$

and (12.1.5) and (12.1.6) hold as soon as $(\nu, v) \notin \mathcal{B}_k^+(\overline{Z}_k)$. Proposition 12.1.1 is proved. □

Note that, in order to prove that pathological sets have vanishing measure as $\varepsilon \to 0$, we have to choose $\eta$ small enough, and then $a$ and $\varepsilon_0$ even smaller in order that (12.1.1) is satisfied and that (12.1.2) is small. Moreover, if we want to get a rate of convergence, we need to have more precise bounds on the cross-section $b$ in terms of the truncation parameters $R$ and $\eta$.

# CHAPTER 13

# TRUNCATED COLLISION INTEGRALS

Our goal in the present chapter is to slightly modify (in a uniform way) the functionals $I^{R,\delta}_{s,k}$ (defined in (7.3.1) in the hard-spheres case and in (11.3.3) for the potential case) and $I^{0,R,\delta}_{s,k}$, defined in (7.3.1), in order for the corresponding pseudo-trajectories to be decomposed as a succession of free transport and binary collisions, without any recollision. This will be possible thanks to Proposition 12.1.1. We then expect to be able to compare these approximate observables, which will be done in the next chapter.

## 13.1. Initialization

The first step consists in preparing the initial configuration $Z_s$ so that it is a good configuration. We define

$$\Delta_s(\varepsilon_0) := \Big\{ Z_s \in \mathbf{R}^{ds} \times B_R^s \,/\, \inf_{1\leq \ell<j\leq s} |x_\ell - x_j| \geq \varepsilon_0 \Big\}\,,$$

and we shall assume from now on that $Z_s$ belongs to $\Delta_s(\varepsilon_0)$. We also define for convenience

$$\Delta_s^X(\varepsilon_0) := \Big\{ X_s \in \mathbf{R}^{ds} \,/\, \inf_{1\leq \ell<j\leq s} |x_\ell - x_j| \geq \varepsilon_0 \Big\}\,.$$

***Proposition 13.1.1***. — *For all $X_s \in \Delta_s^X(\varepsilon_0)$, there is a subset $\mathcal{M}_s(X_s)$ of $\mathbf{R}^{ds}$ such that*

$$\big|\mathcal{M}_s(X_s)\big| \leq CRs^2 \left( \Big(R\frac{\varepsilon}{\varepsilon_0}\Big)^{d-1} + \Big(\frac{\varepsilon_0}{\delta}\Big)^{d-1} \right)\,,$$

*and defining $\mathcal{P}_s := \Big\{ Z_s \in \Delta_s(\varepsilon_0) \,/\, V_s \notin \mathcal{M}_s(X_s) \Big\}$, then*

$$\begin{aligned}
&\forall \tau \geq 0\,, \quad \mathbb{1}_{\mathcal{P}_s} \circ \mathbf{T}_s(\tau) \equiv \mathbb{1}_{\mathcal{P}_s} \circ \mathbf{S}_s(\tau) \\
&\qquad\qquad\qquad\qquad \textit{in the hard-spheres case}\,, \\
&\forall \tau \geq 0\,, \quad \mathbb{1}_{\mathcal{P}_s} \circ \mathbf{H}_s(\tau) \equiv \mathbb{1}_{\mathcal{P}_s} \circ \mathbf{S}_s(\tau) \qquad\qquad (13.1.1)\\
&\qquad\qquad\qquad\qquad \textit{in the potential case, and} \\
&\forall \tau \geq \delta\,, \quad \mathbb{1}_{\mathcal{P}_s} \circ \mathbf{S}_s(\tau) \equiv \mathbb{1}_{\mathcal{P}_s} \circ \mathbf{S}_s(\tau) \circ \mathbb{1}_{\mathcal{G}_s(\varepsilon_0)}\,.
\end{aligned}$$

*denoting abusively by $\mathbb{1}_A$ the operator of multiplication by the indicator of $A$.*

*Proof.* — The proof is very similar to the arguments of the previous chapter. For any $Z_s$ in $\Delta_s(\varepsilon_0)$, we apply Lemma 12.2.1 which shows that outside a small measure set $\mathcal{M}_s(X_s) \subset \mathbf{R}^{ds}$ of velocities $(v_1, \dots, v_s)$, with

$$|\mathcal{M}_s(X_s)| \leq CRs^2 \left( \left(R\frac{\varepsilon}{\varepsilon_0}\right)^{d-1} + \left(\frac{\varepsilon_0}{\delta}\right)^{d-1} \right),$$

the backward nonlinear flow is actually the free flow and the particles remain at a distance larger than $\varepsilon$ to one another for all times:

$$\forall \tau > 0, \quad \forall \ell \neq \ell' \in \{1, \dots, s\}, \quad |(x_\ell - v_\ell \tau) - (x_{\ell'} - v_{\ell'}\tau)| > \varepsilon ,$$

and that

$$\forall \tau \geq \delta, \quad \forall \ell \neq \ell' \in \{1, \dots, s\}, \quad |(x_\ell - v_\ell \tau) - (x_{\ell'} - v_{\ell'}\tau)| \geq \varepsilon_0 .$$

By construction, $\mathcal{M}_s(X_s)$ depends continuously on $X_s$; the result follows by definition of $\mathcal{P}_s$. □

## 13.2. Approximation of the Boltzmann functional

We recall that we consider a family of initial data $F_0 = (f_0^{(s)})$ satisfying

$$\|F_0\|_{0,\beta_0,\mu_0} := \sup_{s\in\mathbf{N}} \sup_{Z_s} \left( \exp(\beta_0 E(Z_s) + \mu_0 s) f_0^{(s)}(Z_s) \right) < +\infty$$

and after the reductions of Chapters 7 and 11, the observable we are interested in is the following:

$$\begin{aligned} I_{s,k}^{0,R,\delta}(t,J,M)(X_s) := \int \varphi_s(V_s) \int_{\mathcal{T}_{k,\delta}(t)} &\mathbf{S}_s(t-t_1)\mathcal{C}_{s,s+1}^{0,j_1,m_1}\mathbf{S}_{s+1}(t_1-t_2)\mathcal{C}_{s+1,s+2}^{0,j_2,m_2} \\ &\dots \mathbf{S}_{s+k}(t_k - t_{k+1}) \mathbb{1}_{E_0(Z_{s+k})\leq R^2} f_0^{(s+k)} dT_k dV_s , \end{aligned} \tag{13.2.2}$$

By Proposition 13.1.1, up to an error term of order $CRs^2\big((R\dfrac{\varepsilon}{\varepsilon_0}\big)^{d-1} + \big(\dfrac{\varepsilon_0}{\delta}\big)^{d-1}\big)$, we can assume that the initial configuration $Z_s$ is a good configuration, meaning that

$$\begin{aligned} I_{s,k}^{0,R,\delta}(t,J,M)(X_s) = \int_{B_R\setminus\mathcal{M}_s(X_s)} \varphi_s(V_s) \int_{\mathcal{T}_{k,\delta}(t)} &\mathbf{S}_s(t-t_1)\mathcal{C}_{s,s+1}^{0,j_1,m_1}\mathbf{S}_{s+1}(t_1-t_2)\mathcal{C}_{s+1,s+2}^{0,j_2,m_2} \\ &\dots \mathcal{C}_{s+k-1,s+k}^{0,j_k,m_k}\mathbf{S}_{s+k}(t_k - t_{k+1}) \mathbb{1}_{|E_0(Z_{s+k})|\leq R^2} f_0^{(s+k)}\, dT_k dV_s \\ &+ O\left( c_{k,J,M} Rs^2 \left( \left(R\frac{\varepsilon}{\varepsilon_0}\right)^{d-1} + \left(\frac{\varepsilon_0}{\delta}\right)^{d-1} \right) \|F_0\|_{0,\beta_0,\mu_0} \right), \end{aligned}$$

where $\displaystyle\sum_k \sum_{J,M} c_{k,J,M} = 1$ and

$$\big(\mathcal{C}_{s,s+1}^{0,-,m} f^{(s+1)}\big)(Z_s) = \int_{\mathbf{S}_1^{d-1}\times\mathbf{R}^d} ((v_{s+1} - v_m)\cdot\nu_{s+1})_- f^{(s+1)}(Z_s, x_m, v_{s+1})\, d\nu_{s+1} dv_{s+1} \quad \text{and}$$

$$\big(\mathcal{C}_{s,s+1}^{0,+,m} f^{(s+1)}\big)(Z_s) = \int_{\mathbf{S}_1^{d-1}\times\mathbf{R}^d} ((v_{s+1} - v_m)\cdot\nu_{s+1})_+ f^{(s+1)}(z_1, \dots, x_m, v_m^*, \dots, z_s, x_m, v_{s+1}^*)\, d\nu_{s+1} dv_{s+1} .$$

Now let us introduce some notation which we shall be using constantly from now on: given $Z_s \in \Delta_s(\varepsilon_0)$, we call $Z_s^0(\tau)$ the position of the backward free flow initiated from $Z_s$, at time $t_1 \leq \tau \leq t$. Then given $j_1 \in \{+,-\}$, $m_1 \in [1,s]$, a deflection angle $\nu_{s+1}$ and a velocity $v_{s+1}$ we call $Z_{s+1}^0(\tau)$ the position at time $t_2 \leq \tau < t_1$ of the Boltzmann pseudo-trajectory initiated by the adjunction of the particle $(\nu_{s+1}, v_{s+1})$ to the particle $z_{m_1}^0(t_1)$ (which is simply free-flow in the pre-collisional case $j_1 = -$, and free-flow after scattering of particles $z_{m_1}^0(t_1)$ and $(\nu_{s+1}, v_{s+1})$ in the post-collisional case $j_1 = +$).

Similarly by induction given $Z_s \in \Delta_s(\varepsilon_0)$, $T, J$ and $M$ we denote for each $1 \leq k \leq n$ by $Z^0_{s+k}(\tau)$ the position at time $t_{k+1} \leq \tau < t_k$ of the pseudo-trajectory initiated by the adjunction of the particle $(\nu_{s+k}, v_{s+k})$ to the particle $z^0_{m_k}(t_k)$ (which is simply free-flow in the pre-collisional case $j_k = -$, and free-flow after scattering of particles $z^0_{m_k}(t_k)$ and $(\nu_{s+k}, v_{s+k})$ in the post-collisional case $j_k = +$).

Notice that $\tau \mapsto Z^0_{s+k}(\tau)$ is pointwise right-continuous on $[0, t_k]$.

With this notation, the elementary functional $I^{0,R,\delta}_{s,k}$ may be reformulated as

$$\begin{aligned}
I^{0,R,\delta}_{s,k}(t,J,M)(X_s) = \int_{B_R\setminus\mathcal{M}_s(X_s)} dV_s \varphi_s(V_s) \int_{\mathcal{T}_{k,\delta}(t)} dT_k \int_{\mathbf{S}_1^{d-1}\times B_R} d\nu_{s+1} dv_{s+1} ((v_{s+1} - v^0_{m_1}(t_1)\cdot\nu_{s+1})_+ \\
\dots \int_{\mathbf{S}_1^{d-1}\times B_R} d\nu_{s+k} dv_{s+k} ((v_{s+k} - v^0_{m_k}(t_k)\cdot \nu_{s+k})_+ \mathbb{1}_{E_0(Z^0_{s+k}(0))\leq R^2} f_0^{(s+k)}(Z^0_{s+k}(0)) \\
+ O\left( c_{k,J,M} R s^2 \left( \left(R\frac{\varepsilon}{\varepsilon_0}\right)^{d-1} + \left(\frac{\varepsilon_0}{\delta}\right)^{d-1}\right) \|F_0\|_{0,\beta_0,\mu_0}\right),
\end{aligned}$$

where $\displaystyle\sum_k \sum_{J,M} c_{k,J,M} = 1$. Let $a, \varepsilon_0, \eta \ll 1$ be such that

$$a \ll \varepsilon_0 \ll \eta\delta\,.$$

According to Proposition 12.1.1, for any good configuration $\overline{Z}_{s+k-1} \in \mathbf{R}^{2d(s+k-1)}$, we can define a set

$${}^c\mathcal{B}^{m_k}_{s+k-1}(\overline{Z}_{s+k-1}) := \big(\mathbf{S}_1^{d-1}\times B_R\big) \setminus \mathcal{B}^{m_k}_{s+k-1}(\overline{Z}_{s+k-1})\,,$$

such that good configurations $Z_{s+k-1} = (X_{s+k-1}, \overline{V}_{s+k-1})$ with $|X_{s+k-1} - \overline{X}_{s+k-1}| \leq Ca$ are stable by adjunction of a collisional particle $z_{s+k} = (x_{m_k} + \varepsilon\nu_{k+s}, v_{k+s})$ with $(\nu_{k+s}, v_{k+s}) \in {}^c\mathcal{B}^{m_k}_{s+k-1}(\overline{Z}_{s+k-1})$.

We further notice that thanks to Remark 12.1.3, if the adjoined pair $(\nu_{s+k}, v_{s+k})$ belongs to the set ${}^c\mathcal{B}^{m_k}_{s+k-1}(Z^0_{s+k-1}(t_k))$ with $Z^0_{s+k-1}(t_k) \in \mathcal{G}_{s+k-1}(\varepsilon_0)$, then $Z^0_{s+k}(t_{k+1})$ belongs to $\mathcal{G}_{s+k}(\varepsilon_0)$.

As a consequence we may define recursively the approximate Boltzmann functional

$$\begin{aligned}
J^{0,R,\delta}_{s,k}(t,J,M)(X_s) = \int_{B_R\setminus\mathcal{M}_s(X_s)} dV_s \varphi_s(V_s) \int_{\mathcal{T}_{k,\delta}(t)} dT_k \\
\int_{{}^c\mathcal{B}^{m_1}_s(Z^0_s(t_1))} d\nu_{s+1} dv_{s+1} (v_{s+1} - v^0_{m_1}(t_1)\cdot\nu_{s+1})_{j_1} \\
\dots \int_{{}^c\mathcal{B}^{m_k}_{s+k-1}(Z^0_{s+k-1}(t_k))} d\nu_{s+k} dv_{s+k} (v_{s+k} - v^0_{m_k}(t_k)\cdot\nu_{s+k})_{j_k} \\
\times \mathbb{1}_{E_0(Z^0_{s+k}(0))\leq R^2} f_0^{(s+k)}(Z^0_{s+k}(0))\,.
\end{aligned} \tag{13.2.3}$$

The following result is an immediate consequence of Proposition 12.1.1, together with the continuity estimates for the Boltzmann collision operator in Proposition 5.4.2.

***Proposition 13.2.1***. — *Let $a, \varepsilon_0, \eta \ll 1$ satisfying* (12.1.1). *Then, we have the following error estimates for the observables associated to the Boltzmann dynamics:*

– *with the cross-section associated to hard-spheres,*

$$\begin{aligned}
\Big| \sum_{k=0}^n \sum_{J,M} \mathbb{1}_{\Delta_s(\varepsilon_0)} \big(I^{0,R,\delta}_{s,k} - J^{0,R,\delta}_{s,k}\big)(t,J,M) \Big| \leq C n^2 (s+n) \\
\times \left( R\eta^{d-1} + R^d \Big(\frac{a}{\varepsilon_0}\Big)^{d-1} + R\Big(\frac{\varepsilon_0}{\delta}\Big)^{d-1}\right) \|F_0\|_{0,\beta_0,\mu_0}\,;
\end{aligned}$$

– *with the cross-section associated with a smooth short-range potential* $\Phi$,

$$\Big|\sum_{k=0}^{n}\sum_{J,M}\mathbb{1}_{\Delta_s(\varepsilon_0)}\big(I^{0,R,\delta}_{s,k}-J^{0,R,\delta}_{s,k}\big)(t,J,M)\Big|\leq Cn^2(s+n)$$
$$\times\Big(R\eta^{d-1}+C(\Phi,\eta,R)R^d\Big(\frac{a}{\varepsilon_0}\Big)^{d-1}+C(\Phi,\eta,R)R\Big(\frac{\varepsilon_0}{\delta}\Big)^{d-1}\Big)\|F_0\|_{0,\beta_0,\mu_0}\,.$$

## 13.3. Approximation of the BBGKY functional

We recall that after the reductions of Chapters 7 and 11, the elementary functionals we are interested in are

– in the case of hard spheres:

$$I^{R,\delta}_{s,k}(t,J,M)(X_s):=\int\varphi_s(V_s)\int_{\mathcal{T}_{k,\delta}(t)}\mathbf{T}_s(t-t_1)\mathcal{C}^{j_1,m_1}_{s,s+1}\mathbf{T}_{s+1}(t_1-t_2)\mathcal{C}^{j_2,m_2}_{s+1,s+2}$$
$$\dots\mathcal{C}^{j_k,m_k}_{s+k-1,s+k}\mathbf{T}_{s+k}(t_k-t_{k+1})\mathbb{1}_{E_\varepsilon(Z_{s+k})\leq R^2}f^{(s+k)}_{N,0}\,dT_k dV_s\,,$$

where $F_{N,0}=(f^{(s)}_{N,0})_{1\leq s\leq N}$ satisfies

$$\|F_{N,0}\|_{\varepsilon,\beta_0,\mu_0}:=\sup_{s\in\mathbf{N}}\sup_{Z_s\in\mathcal{D}_s}\big(\exp(\beta_0E_0(Z_s)+\mu_0 s)f^{(s)}_{N,0}(Z_s)\big)<+\infty\,;$$

– in the case of a smooth interaction potential $\Phi$:

$$I^{R,\delta}_{s,k}(t,J,M)(X_s):=\int\varphi_s(V_s)\int_{\mathcal{T}_{k,\delta}(t)}\mathbf{H}_s(t-t_1)\mathcal{C}^{j_1,m_1}_{s,s+1}\mathbf{H}_{s+1}(t_1-t_2)\mathcal{C}^{j_2,m_2}_{s+1,s+2}$$
$$\dots\mathcal{C}^{j_k,m_k}_{s+k-1,s+k}\mathbf{H}_{s+k}(t_k-t_{k+1})\mathbb{1}_{E_\varepsilon(Z_{s+k})\leq R^2}\widetilde{f}^{(s+k)}_{N,0}\,dT_k dV_s\,,$$

where $\widetilde{F}_{N,0}=(\widetilde{f}^{(s)}_{N,0})_{1\leq s\leq N}$ satisfies

$$\|\widetilde{F}_{N,0}\|_{\varepsilon,\beta_0,\mu_0}:=\sup_{s\in\mathbf{N}}\sup_{Z_s}\big(\exp(\beta_0E_\varepsilon(Z_s)+\mu_0 s)\widetilde{f}^{(s)}_{N,0}(Z_s)\big)<+\infty\,.$$

Since both formulas are quite similar, we shall deal with the case of smooth potentials and will indicate – if need be – simplifications arising in the case of hard spheres.

Thanks to Proposition 13.1.1, we have

$$I^{R,\delta}_{s,k}(t,J,M)(X_s)=\int_{B_R\setminus\mathcal{M}_s(X_s)}\varphi_s(V_s)\int_{\mathcal{T}_{k,\delta}(t)}\mathbf{S}_s(t-t_1)\mathbb{1}_{\mathcal{G}_s(\varepsilon_0)}\mathcal{C}^{j_1,m_1}_{s,s+1}\mathbf{H}_{s+1}(t_1-t_2)\mathcal{C}^{j_2,m_2}_{s+1,s+2}$$
$$\dots\mathcal{C}^{j_k,m_k}_{s+k-1,s+k}\mathbf{H}_{s+k}(t_k-t_{k+1})\mathbb{1}_{E_\varepsilon(Z_{s+k}(0))\leq R^2}\widetilde{f}^{(s+k)}_{N,0}\,dT_k dV_s$$
$$+O\left(c_{k,J,M}Rs^2\left(\Big(R\frac{\varepsilon}{\varepsilon_0}\Big)^{d-1}+\Big(\frac{\varepsilon_0}{\delta}\Big)^{d-1}\right)\|\widetilde{F}_{N,0}\|_{\varepsilon,\beta_0,\mu_0}\right)\,,$$

where recall that $c_{k,J,M}$ denotes a sequence of positive real numbers satisfying $\displaystyle\sum_k\sum_{J,M}c_{k,J,M}=1$.

Then using the notation introduced in the previous paragraph for the Boltzmann pseudo-trajectory, let us define the approximate functionals

$$\begin{aligned}J_{s,k}^{R,\delta}(t,J,M)(X_s) := \int_{B_R\setminus\mathcal{M}_s(X_s)} \varphi_s(V_s)\int_{\mathcal{T}_{k,\delta}(t)} \mathbf{S}_s(t-t_1)\mathbb{1}_{\mathcal{G}_s(\varepsilon_0)}\widetilde{\mathcal{C}}_{s,s+1}^{j_1,m_1}\mathbf{H}_{s+1}(t_1-t_2)\\ \dots\widetilde{\mathcal{C}}_{s+k-1,s+k}^{j_k,m_k}\mathbf{H}_{s+k}(t_k-t_{k+1})\mathbb{1}_{E_\varepsilon(Z_{s+k}(0))\leq R^2}\widetilde{f}_0^{(s+k)}dT_kdV_s\,,\end{aligned}$$

where the modified collision operators are obtained by elimination of the pathological set of impact parameters and velocities

$$\begin{aligned}\big(\widetilde{\mathcal{C}}_{s+k-1,s+k}^{\pm,m_k}g^{(s+k)}\big)(Z_{s+k-1}) := (N-s-k+1)\varepsilon^{d-1}\int_{^c\mathcal{B}^{m_k}_{s+k-1}(Z^0_{s+k-1}(t_k))}(\nu_{s+k}\cdot(v_{s+k}-v_{m_k}(t_k)))_\pm\\ \times g^{(s+k)}(\cdot,x_{m_k}(t_k)+\varepsilon\nu_{s+k},v_{s+k}(t_k))\prod_{\substack{1\leq j\leq s+k-1\\ j\neq m_k}}\mathbb{1}_{|(x_j-x_{m_k})(t_k)-\varepsilon\nu_{s+k}|\geq\varepsilon}\,d\nu_{s+k}dv_{s+k}\,.\end{aligned}$$

By construction, we know that the remaining collision trees are nice, in the sense that collisions involve only two particles and are well-separated in time. Using the pre/post-collisional change of variables, we can rewrite the gain terms as follows

$$\begin{aligned}&\mathbb{1}_{\mathcal{G}_{s+k-1}(\varepsilon_0/2)}\big(\widetilde{\mathcal{C}}_{s+k-1,s+k}^{+,m_k}\mathbf{H}_{s+k}(t_k-t_{k+1})g^{(s+k)}\big)(Z_{s+k-1})\\ &:= (N-s-k+1)\varepsilon^{d-1}\mathbb{1}_{\mathcal{G}_{s+k-1}(\varepsilon_0/2)}\int_{^c\mathcal{B}^{m_k}_{s+k-1}(Z^0_{s+k-1}(t_k))}(\nu_{s+k}\cdot(v_{s+k}-v_{m_k}(t_k)))_+\\ &\quad\times\mathbf{H}_{s+k}(t_k-t_{k+1}-t_\varepsilon(Z_{s+k}))g^{(s+k)}(\cdot,x^*_{m_k},v^*_{m_k},\dots,x^*_{s+k},v^*_{s+k})\\ &\quad\times\prod_{\substack{1\leq j\leq s+k-1\\ j\neq m_k}}\mathbb{1}_{|(x_j-x_{m_k})(t_k)-\varepsilon\nu_{s+k}|\geq\varepsilon}\,d\nu_{s+k}dv_{s+k}.\end{aligned}$$

denoting as previously by $(x^*_{m_k},v^*_{m_k},x^*_{s+k},v^*_{s+k})$ the pre-image by the scattering operator $\sigma_\varepsilon$ of the point $(x_{m_k},v_{m_k}(t_k),x_{m_k}(t_k)+\varepsilon\nu_{s+k},v_{s+k}(t_k))$.

Note that this last step is obvious in the case of hard spheres since there is no time shift : $t_\varepsilon\equiv 0$.

As in the Boltzmann case described above, the following result is an immediate consequence of Proposition 12.1.1 together with the continuity estimates for the BBGKY collision operator in Propositions 5.4.1 and 10.3.1.

***Proposition 13.3.1***. — *Let $a,\varepsilon_0,\eta\ll 1$ satisfying* (12.1.1). *Then, for $\varepsilon$ sufficiently small, we have the following error estimates for the observables associated to the BBGKY dynamics:*

– *in the case of hard-spheres*

$$\Big|\sum_{k=0}^n\sum_{J,M}\mathbb{1}_{\Delta_s(\varepsilon_0)}\big(I_{s,k}^{R,\delta}-J_{s,k}^{R,\delta}\big)(t,J,M)\Big|\leq Cn^2(s+n)\Big(R\eta^{d-1}+R^d\Big(\frac{a}{\varepsilon_0}\Big)^{d-1}+R\Big(\frac{\varepsilon_0}{\delta}\Big)^{d-1}\Big)\|F_{N,0}\|_{\varepsilon,\beta_0,\mu_0}\,,$$

– *in the case of some smooth short-range potential $\Phi$*

$$\begin{aligned}\Big|\sum_{k=0}^n\sum_{J,M}\mathbb{1}_{\Delta_s(\varepsilon_0)}\big(I_{s,k}^{0,R,\delta}-J_{s,k}^{0,R,\delta}\big)(t,J,M)\Big|\leq Cn^2(s+n)\\ \times\Big(R\eta^{d-1}+C(\Phi,\eta,R)R^d\Big(\frac{a}{\varepsilon_0}\Big)^{d-1}+C(\Phi,\eta,R)R\Big(\frac{\varepsilon_0}{\delta}\Big)^{d-1}\Big)\|\widetilde{F}_{N,0}\|_{\varepsilon,\beta_0,\mu_0}\,.\end{aligned}$$

The functional $J^{R,\delta}_{s,k}$ can be written in terms of pseudo-trajectories, as in (13.2.3). Let us therefore introduce some notation which we shall be using constantly from now on: given $Z_s \in \Delta_s(\varepsilon_0)$, we call $Z^0_s(\tau)$ the position of the backward free flow initiated from $Z_s$, at time $t_1 \leq \tau \leq t$. Then given $j_1 \in \{+,-\}$, $m_1 \in [1,s]$, an angle $\nu_{s+1}$ (or equivalently a position $x_{s+1} = x^0_{m_1}(t_1) + \varepsilon\nu_{s+1}$) and a velocity $v_{s+1}$ we call $Z^\varepsilon_{s+1}(\tau)$ the position at time $t_2 \leq \tau < t_1$ of the BBGKY pseudo-trajectory initiated by the adjunction of the particle $z_{s+1}$ to the particle $z^0_{m_1}(t_1)$.

Similarly by induction given $Z_s \in \Delta_s(\varepsilon_0)$, $T, J$ and $M$ we denote for each $1 \leq k \leq n$ by $Z^\varepsilon_{s+k}(\tau)$ the position at time $t_{k+1} \leq \tau < t_k$ of the BBGKY pseudo-trajectory initiated by the adjunction of the particle $z_{s+k}$ to the particle $z_{m_k}(t_k)$. We have

$$\begin{aligned}
J^{R,\delta}_{s,k}(t,J,M)(X_s) &= \frac{(N-s)!}{(N-s-k)!}\varepsilon^{k(d-1)} \int_{B_R\setminus\mathcal{M}_s(X_s)} dV_s\varphi_s(V_s)\int_{\mathcal{T}_{k,\delta}(t)} dT_k \\
&\int_{^c\mathcal{B}^{m_1}_s(Z^0_s(t_1))} d\nu_{s+1}dv_{s+1}\,(\nu_{s+1}\cdot(v_{s+1}-v_{m_1}(t_1)))_{j_1} \prod_{\substack{1\leq j\leq s\\ j\neq m_1}} 1\!\!1_{|(x_j-x_{m_1})(t_1)-\varepsilon\nu_{s+1}|\geq\varepsilon} \\
&\dots\int_{^c\mathcal{B}^{j_k}_{s+k-1}(Z^0_{s+k-1}(t_k))} d\nu_{s+k}dv_{s+k}\,(\nu_{s+k}\cdot(v_{s+k}-v_{m_k}(t_k)))_{j_k} \\
&\times\prod_{\substack{1\leq j\leq s+k-1\\ j\neq m_k}} 1\!\!1_{|(x_j-x_{m_k})(t_k)-\varepsilon\nu_{s+k}|\geq\varepsilon} 1\!\!1_{E_\varepsilon(Z_{s+k}(0))\leq R^2}\widetilde{f}^{(s+k)}_{N,0}(Z^\varepsilon_{s+k}(0))\,.
\end{aligned} \tag{13.3.4}$$

Thanks to Propositions 13.2.1 and 13.3.1 the proof of Theorems 8 and 11 reduces to the proof of the convergence to zero of $J^{R,\delta}_{s,k} - J^{0,R,\delta}_{s,k}$. This is the object of the next chapter.

# CHAPTER 14

# CONVERGENCE PROOF

In this chapter we conclude the proof of Theorems 8 and 11 by proving that $J_{s,k}^{R,\delta} - J_{s,k}^{0,R,\delta}$ goes to zero in the Boltzmann-Grad limit, with the notation of the previous chapter, namely (13.2.3) and (13.3.4). The main difficulty lies in the fact that in contrast to the Boltzmann situation, collisions in the BBGKY configuration are not pointwise in space (nor in time in the case of the smooth Hamiltonian system). At each collision time $t_k$ a small error is therefore introduced, which needs to be controlled.

We recall that, as in the previous chapters, we consider dynamics

- involving only a finite number $s+k$ of particles,
- with bounded energies (at most $R^2 \gg 1$),
- such that the $k$ additional particles are adjoined through binary collisions at times separated at least by $\delta \ll 1$.

The additional truncation parameters $a, \varepsilon_0, \eta \ll 1$ satisfy (12.1.1).

## 14.1. Proximity of Boltzmann and BBGKY trajectories

This paragraph is devoted to the proof, by induction, that the BBGKY and Boltzmann pseudo-trajectories remain close for all times, in particular that there is no recollision for the BBGKY dynamics.

We recall that the notation $Z_k^0(t)$ and $Z_k(t)$ were defined in Paragraphs 13.2 and 13.3 respectively.

***Lemma 14.1.1.*** — *Fix $T \in \mathcal{T}_{n,\delta}(t)$, $J$, and $M$ and given $Z_s$ in $\Delta_s(\varepsilon_0)$, consider for all $i \in \{1, \dots n\}$, an impact parameter $\nu_{s+i}$ and a velocity $v_{s+i}$ such that $(\nu_{s+i}, v_{s+i}) \notin \mathcal{B}_{s+i-1}(Z^0_{s+i-1}(t_i))$. Then, for $\varepsilon$ sufficiently small, for all $i \in [1, n]$, and all $k \le s+i$,*

- *for the hard sphere dynamics*

$$|x_k^\varepsilon(t_{i+1}) - x_k^0(t_{i+1})| \le \varepsilon i \quad \textit{and} \quad v_k(t_{i+1}) = v_k^0(t_{i+1}) \,, \tag{14.1.1}$$

- *for the hamiltonian dynamics associated to $\Phi$*

$$|x_k^\varepsilon(t_{i+1}) - x_k^0(t_{i+1})| \le C(\Phi, R, \eta)\varepsilon i \quad \textit{and} \quad v_k(t_{i+1}) = v_k^0(t_{i+1}) \,, \tag{14.1.2}$$

*where the constant $C(\Phi, R, \eta)$ depends only on $\Phi$, $R$, and $\eta$.*

*Proof.* — We proceed by induction on $i$, the index of the time variables $t_{i+1}$ for $0 \leq i \leq n$.

We first notice that by construction, $Z_s(t_1) - Z_s^0(t_1) = 0$, so (14.1.2) holds for $i = 0$. The initial configuration being a good configuration, we indeed know – by definition – that there is no possible recollision.

Now let $i \in [1, n]$ be fixed, and assume that for all $\ell \leq i$

$$\forall k \leq s + \ell - 1, \qquad |x_k^\varepsilon(t_\ell) - x_k^0(t_\ell)| \leq C\varepsilon(\ell - 1) \quad \text{and} \quad v_k(t_\ell) = v_k^0(t_\ell)\,, \tag{14.1.3}$$

with $C = 1$ for hard spheres.

Let us prove that (14.1.3) holds for $\ell = i + 1$. We shall consider two cases depending on whether the particle adjoined at time $t_i$ is pre-collisional or post-collisional.

• As usual, the case of pre-collisional velocities $(v_{s+i}, v_{m_i}(t_i))$ at time $t_i$ is the most simple to handle. We indeed have $\forall \tau \in [t_{i+1}, t_i]$

$$\begin{aligned} \forall k < s+i\,, \quad x_k^0(\tau) &= x_k^0(t_i) + (\tau - t_i) v_k^0(t_i)\,, \qquad v_k^0(\tau) = v_k^0(t_i)\,, \\ x_{s+i}^0(\tau) &= x_{m_i}^0(t_i) + (\tau - t_i) v_{s+i}\,, \qquad v_{s+i}^0(\tau) = v_{s+i}\,. \end{aligned}$$

Now let us study the BBGKY trajectory. We recall that the particle is adjoined in such a way that $(\nu_{s+i}, v_{s+i})$ belongs to ${}^c\mathcal{B}_{s+i-1}(Z_{s+i-1}^0(t_i))$. Provided that $\varepsilon$ is sufficiently small, by the induction assumption (14.1.3), we have

$$\forall k \leq s + i - 1, \quad |x_k^\varepsilon(t_i) - x_k^0(t_i)| \leq C\varepsilon(i-1) \leq a\,,$$

with $C = 1$ for hard spheres.

Since $Z_{s+i-1}^0(t_i)$ belongs to $\mathcal{G}_{s+i-1}(\varepsilon_0)$ (see Paragraph 13.2), we can apply Proposition 12.1.1 which implies that backwards in time, there is free flow for $Z_{s+i}^\varepsilon$. In particular,

$$\begin{aligned} &\forall k < s+i\,, \quad x_k(\tau) = x_k(t_i) + (\tau - t_i) v_k(t_i)\,, \qquad v_k(\tau) = v_k(t_i)\,, \\ &x_{s+i}(\tau) = x_{m_i}(t_i) + \varepsilon\nu_{s+i} + (\tau - t_i) v_{s+i}\,, \qquad v_{s+i}(\tau) = v_{s+i}\,. \end{aligned}$$

We therefore obtain

$$\forall k \leq s+i\,, \quad \forall \tau \in [t_{i+1}, t_i]\,, \quad v_k(\tau) - v_k^0(\tau) = v_k(t_i) - v_k^0(t_i) = 0\,, \tag{14.1.4}$$

and

$$\forall k \leq s+i\,, \quad \forall \tau \in [t_{i+1}, t_i]\,, \quad |x_k(\tau) - x_k^0(\tau)| \leq C\varepsilon(i-1) + \varepsilon\,, \tag{14.1.5}$$

with $C = 1$ in the case of hard spheres.

• The case of post-collisional velocities $(v_{s+i}, v_{m_i}(t_i))$ at time $t_i$ for the hard sphere dynamics is very similar. We indeed have $\forall \tau \in [t_{i+1}, t_i[$

$$\begin{aligned} &\forall k < s+i, \quad k \neq m_i\,, \quad x_k^0(\tau) = x_k^0(t_i) + (\tau - t_i) v_k^{0*}(t_i)\,, \qquad v_k^0(\tau) = v_k^0(t_i)\,, \\ &x_{m_i}^0(\tau) = x_{m_i}^0(t_i) + (\tau - t_i) v_{m_i}^{0*}(t_i)\,, \qquad v_k^0(\tau) = v_{m_i}^{0*}(t_i)\,, \\ &x_{s+i}^0(\tau) = x_{m_i}^0(t_i) + (\tau - t_i) v_{s+i}^*\,, \qquad v_{s+i}^0(\tau) = v_{s+i}^*\,. \end{aligned}$$

Now let us study the BBGKY trajectory. We recall that the particle is adjoined in such a way that $(\nu_{s+i}, v_{s+i})$ belongs to ${}^c\mathcal{B}_{s+i-1}^{m_i}(Z_{s+i-1}^0(t_i))$. Provided that $\varepsilon$ is sufficiently small, by the induction assumption (14.1.3), we have

$$\forall k \leq s + i - 1, \quad |x_k^\varepsilon(t_i) - x_k^0(t_i)| \leq \varepsilon(i-1)\,.$$

Since $Z^0_{s+i-1}(t_i)$ belongs to $\mathcal{G}_{s+i-1}(\varepsilon_0)$ (see Paragraph 13.2), we can apply Proposition 12.1.1 which implies that backwards in time, there is free flow for $Z^\varepsilon_{s+i}$. In particular,

$$\forall k \leq s+i\,, \quad \forall \tau \in [t_{i+1}, t_i[\,, \quad v_k(\tau) - v^0_k(\tau) = v_k(t_i^-) - v^0_k(t_i^-) = 0\,, \tag{14.1.6}$$

and

$$\forall k \leq s+i\,, \quad \forall \tau \in [t_{i+1}, t_i[\,, \quad |x_k(\tau) - x^0_k(\tau)| \leq \varepsilon(i-1) + \varepsilon \leq i\varepsilon\,. \tag{14.1.7}$$

• The case of post-collisional velocities is a little more complicated since there is a (small) time interval during which interaction occurs.

Let us start by describing the Boltzmann flow. By definition of the post-collisional configuration, we know that the following identities hold:

$$\forall t_{i+1} \leq \tau < t_i\,, \begin{cases} (v^0_{m_i}, v^0_{s+i})(\tau) = (v^{0*}_{m_i}(t_i), v^*_{s+i}) \text{ with } (\nu^*_{s+i}, v^{0*}_{m_i}(t_i), v^*_{s+i}) := \sigma_0^{-1}(\nu_{s+i}, v^0_{m_i}(t_i), v_{s+i}) \\ x^0_{m_i}(\tau) = x^0_{m_i}(t_i) + (\tau - t_i)v^{0*}_{m_i}(t_i)\,,\; x^0_{s+i}(\tau) = x^0_{s+i}(t_i) + (\tau - t_i)v^*_{s+i} \\ \forall j \notin \{m_i, s+1\}\,, \quad v^0_j(\tau) = v^0_j(t_i)\,,\; x^0_j(\tau) = x^0_j(t_i) + (\tau - t_i)v^0_j(t_i)\,, \end{cases}$$

where $\sigma_0$ denotes the scattering operator defined in Definition 8.2.1 in Chapter 8.

First, by Proposition 12.1.1, we know that for $j \notin \{m_i, s+i\}$ and $\forall \tau \in [t_{i+1}, t_i]$,

$$x_j(\tau) = x_j(t_i) + (\tau - t_i)v_j(t_i)\,, \qquad v_j(\tau) = v_j(t_i)\,,$$

so that by the induction assumption (14.1.3) we obtain

$$\begin{aligned} \forall j \notin \{m_i, s+i\}\,, \forall \tau \in [t_{i+1}, t_i]\,, \quad |x_j(\tau) - x^0_j(\tau)| = |x_j(t_i) - x^0_j(t_i)| \leq C\varepsilon(i-1) \\ \text{and} \quad v_j(\tau) = v^0_j(\tau)\,. \end{aligned} \tag{14.1.8}$$

We now have to focus on the pair $(s+i, m_i)$. According to Chapter 8, the relative velocity evolves under the nonlinear dynamics on a time interval $[t_i - t_\varepsilon, t_i]$ with $t_\varepsilon \leq C(\Phi, R, \eta)\varepsilon$ (recalling that by construction, the relative velocity $|v_{s+i} - v_{m_i}(t_i)|$ is bounded from above by $R$ and from below by $\eta$, and that the impact parameter is also bounded from below by $\eta$). Then, for all $\tau \in [t_{i+1}, t_i - t_\varepsilon]$,

$$v_{s+i}(\tau) = v^*_{s+i} = v^0_{s+i}(\tau)\,, \quad v_{m_i}(\tau) = v^*_{m_i}(t_i) = v^{0*}_{m_i}(t_i) = v^0_{m_i}(\tau)\,. \tag{14.1.9}$$

In particular,

$$v_{s+i}(t_{i+1}) = v^0_{s+i}(t_{i+1}) \quad \text{and} \quad v_{m_i}(t_{i+1}) = v^0_{m_i}(t_{i+1})\,. \tag{14.1.10}$$

The conservation of total momentum as in Paragraph 12.3.3 shows that

$$\begin{aligned} &\Big|\frac{1}{2}(x^\varepsilon_{m_i}(t_i - t_\varepsilon) + x^\varepsilon_{s+i}(t_i - t_\varepsilon)) - \frac{1}{2}(x^0_{m_i}(t_i - t_\varepsilon) + x^0_{s+i}(t_i - t_\varepsilon))\Big| \\ &\qquad = \Big|\frac{1}{2}(x^\varepsilon_{m_i}(t_i) + x^\varepsilon_{s+i}(t_i) - \frac{1}{2}(x^0_{m_i}(t_i) + x^0_{s+i}(t_i))\Big| \\ &\qquad = \Big|x^\varepsilon_{s+i}(t_i) - x^0_{s+i}(t_i)\Big| + \frac{\varepsilon}{2} \leq C\varepsilon(i-1) + \frac{\varepsilon}{2}\cdot \end{aligned}$$

On the other hand, by definition of the scattering time $t_\varepsilon$,

$$|x^\varepsilon_{m_i}(t_i - t_\varepsilon) - x^\varepsilon_{s+i}(t_i - t_\varepsilon)| = \varepsilon\,,$$
$$|x^0_{m_i}(t_i - t_\varepsilon) - x^0_{s+i}(t_i - t_\varepsilon)| = t_\varepsilon |v^*_{m_i} - v^*_{s+i}| \leq C(\Phi, R, \eta)\,\varepsilon\,.$$

We obtain finally

$$|x^\varepsilon_{m_i}(t_i - t_\varepsilon) - x^0_{m_i}(t_i - t_\varepsilon)| \leq C\varepsilon i \text{ and } |x^\varepsilon_{s+i}(t_i - t_\varepsilon) - x^0_{s+i}(t_i - t_\varepsilon)| \leq C\varepsilon i \tag{14.1.11}$$

provided that $C$ is chosen sufficiently large (depending on $\Phi$, $R$ and $\eta$).

Now let us apply Proposition 12.1.1, which implies that for all $\tau \in [t_{i+1}, t_i - t_\varepsilon]$ the backward in time evolution of the two particles $x^\varepsilon_{s+i}(t_i - t_\varepsilon)$ and $x^\varepsilon_{m_i}(t_i - t_\varepsilon)$, is that of free flow: we have therefore, using (14.1.9),

$$x^\varepsilon_{m_i}(t_{i+1}) - x^0_{m_i}(t_{i+1}) = x^\varepsilon_{m_i}(t_i - t_\varepsilon) - x^0_{m_i}(t_i - t_\varepsilon)\,,$$
$$x^\varepsilon_{s+i}(t_{i+1}) - x^0_{s+i}(t_{i+1}) = x^\varepsilon_{s+i}(t_i - t_\varepsilon) - x^0_{s+i}(t_i - t_\varepsilon)\,.$$

From (14.1.11) we therefore deduce that the induction assumption is satisfied at time step $t_{i+1}$, and the proposition is proved. □

Note that, by construction,

$$Z^0_{s+k}(0) \in \mathcal{G}_{s+k}(\varepsilon_0)\,,$$

so that an obvious application of the triangular inequality leads to

$$Z^\varepsilon_{s+k}(0) \in \mathcal{G}_{s+k}(\varepsilon_0/2)\,.$$

Note also that the indicator functions are identically equal to 1 for good configurations. We therefore have the following

***Corollary 14.1.2***. — *Under the assumptions of Lemma* 14.1.1*, the functional $J^{R,\delta}_{s,n}(t,J,M)$ defined in* (13.3.4) *may be written as follows:*

$$\begin{aligned}
J^{R,\delta}_{s,k}(t,J,M)(X_s) = \frac{(N-s)!}{(N-s-k)!}\varepsilon^{k(d-1)} \int_{B_R\setminus\mathcal{M}_s(X_s)} dV_s\varphi_s(V_s) \int_{\mathcal{T}_{k,\delta}(t)} dT_k \\
\int_{{}^c\mathcal{B}^{m_1}_s(Z^0_s(t_1))} d\nu_{s+1}dv_{s+1}\left(\nu_{s+1}\cdot(v_{s+1}-v_{m_1}(t_1))\right)_{j_1} \\
\dots\int_{{}^c\mathcal{B}^{m_k}_{s+k-1}(Z^0_{s+k-1}(t_n))} d\nu_{s+k}dv_{s+k}\left(\nu_{s+n}\cdot(v_{s+k}-v_{m_k}(t_k))\right)_{j_k} \\
\times \mathbb{1}_{E_\varepsilon(Z_{s+k}(0))\leq R^2}\mathbb{1}_{Z_{s+k}(0)\in\mathcal{G}_{s+k}(\varepsilon_0/2)}\widetilde{f}^{(s+k)}_{N,0}(Z^\varepsilon_{s+k}(0))\,.
\end{aligned}$$

## 14.2. Proof of convergence for the hard sphere dynamics: proof of Theorem 8

In this section we prove Theorem 8, which concerns the case of hard spheres. The potential case will be treated in the following section.

From Corollary 7.4.1, we know that any observable associated to the BBGKY hierarchy can be approximated by a finite sum : more precisely, given $s$ and $t \in [0,T]$, there are two positive constants $C$ and $C'$ such that

$$\Big\|I_s(t) - \sum_{k=0}^n I^{R,\delta}_{s,k}(t)\Big\|_{L^\infty(\mathbf{R}^{ds})} \leq C\left(2^{-n} + e^{-C'\beta_0R^2} + \frac{n^2}{T}\delta\right)\|\varphi\|_{L^\infty(\mathbf{R}^{ds})}\|F_{N,0}\|_{\varepsilon,\beta_0,\mu_0}\,. \tag{14.2.12}$$

Similarly, for the Boltzmann hierarchy, we get

$$\Big\|I^0_s(t) - \sum_{k=0}^n I^{R,\delta}_{0,s,k}(t)\Big\|_{L^\infty(\mathbf{R}^{ds})} \leq C\left(2^{-n} + e^{-C'\beta_0R^2} + \frac{n^2}{T}\delta\right)\|\varphi\|_{L^\infty(\mathbf{R}^{ds})}\|F_0\|_{0,\beta_0,\mu_0}\,. \tag{14.2.13}$$

Then, from Propositions 13.2.1 and 13.3.1, we obtain the error terms corresponding to the elimination of pathological velocities and impact parameters

$$\begin{aligned}&\Big|1\!\!1_{\Delta_s(\varepsilon_0)}\sum_{k=0}^n\sum_{J,M}\big(I^{0,R,\delta}_{s,k}-J^{0,R,\delta}_{s,k}\big)(t,J,M)\Big|\leq Cn^2(s+n)\\&\qquad\qquad\times\Big(R\eta^{d-1}+R^d\Big(\frac{a}{\varepsilon_0}\Big)^{d-1}+R\Big(\frac{\varepsilon_0}{\delta}\Big)^{d-1}\Big)\|F_0\|_{0,\beta_0,\mu_0}\|\varphi\|_{L^\infty(\mathbf{R}^{ds})}\,,\end{aligned}\tag{14.2.14}$$

and

$$\begin{aligned}&\Big|1\!\!1_{\Delta_s(\varepsilon_0)}\sum_{k=0}^n\sum_{J,M}\big(I^{R,\delta}_{s,k}-J^{R,\delta}_{s,k}\big)(t,J,M)\Big|\leq Cn^2(s+n)\\&\qquad\qquad\times\Big(R\eta^{d-1}+R^d\Big(\frac{a}{\varepsilon_0}\Big)^{d-1}+R\Big(\frac{\varepsilon_0}{\delta}\Big)^{d-1}\Big)\|F_{N,0}\|_{\varepsilon,\beta_0,\mu_0}\|\varphi\|_{L^\infty(\mathbf{R}^{ds})}\,.\end{aligned}\tag{14.2.15}$$

The end of the proof of Theorem 8 consists in estimating the error terms in $J^{R,\delta}_{s,k}-J^{0,R,\delta}_{s,k}$ coming essentially from the micro-translations described in the previous paragraph and from the initial data.

### 14.2.1. Error coming from the initial data. —

Let us replace the initial data in $J^{R,\delta}_{s,k}$ by that of the Boltzmann hierarchy, defining:

$$\begin{aligned}\widetilde J^{R,\delta}_{s,k}(t,J,M)(X_s):=\frac{(N-s)!}{(N-s-k)!}\varepsilon^{k(d-1)}&\int_{B_R\setminus\mathcal{M}_s(X_s)}dV_s\varphi_s(V_s)\int_{\mathcal{T}_{k,\delta}(t)}dT_k\\&\int_{^c\mathcal{B}^{m_1}_s(Z^0_s(t_1))}d\nu_{s+1}dv_{s+1}\,(\nu_{s+1}\cdot(v_{s+1}-v_{m_1}(t_1)))_{j_1}\\&\quad\dots\int_{^c\mathcal{B}^{m_k}_{s+k-1}(Z^0_{s+k-1}(t_k))}d\nu_{s+k}dv_{s+k}\,(\nu_{s+k}\cdot(v_{s+k}-v_{m_k}(t_k)))_{j_k}\\&\quad\times 1\!\!1_{E_0(Z_{s+k}(0))\leq R^2}1\!\!1_{Z^\varepsilon_{s+k}(0)\in\mathcal{G}_{s+k}(\varepsilon_0/2)}f_0^{(s+k)}(Z_{s+k}(0))\,.\end{aligned}$$

Since, by definition of admissible Boltzmann data, we have for any fixed $s$

$$f^{(s)}_{0,N}\longrightarrow f^{(s)}_0\quad\text{as } N\to\infty\text{ with }N\varepsilon^{d-1}\equiv1\,,\text{ locally uniformly in }\Omega_s\,,$$

we expect that

$$J^{R,\delta}_{s,k}(t,J,M)(X_s)-\widetilde J^{R,\delta}_{s,k}(t,J,M)(X_s)\to0$$

as $N\to\infty$ with $N\varepsilon^{d-1}\equiv1$, locally uniformly in $\Omega_s$.

***Lemma 14.2.1***. — *Let $F_0$ be an admissible Boltzmann datum and $F_{0,N}$ an associated BBGKY datum. Then, in the Boltzmann-Grad scaling $N\varepsilon^{d-1}=1$, for all fixed $s,k\in\mathbf{N}$ and $t<T$,*

$$J^{R,\delta}_{s,k}(t,J,M)(X_s)-\widetilde J^{R,\delta}_{s,k}(t,J,M)(X_s)\to0\,,$$

*locally uniformly in $\Omega_s$.*

*For tensorized initial data*

$$f^{(N)}_{0,N}(Z_N)=\mathcal{Z}_N^{-1}1\!\!1_{Z_N\in\mathcal{D}_N}f_0^{\otimes N}(Z_N)\qquad\textit{with}\qquad\big\|f_0\exp(\beta_0|v|^2)\big\|_{L^\infty}<+\infty\,,$$

*we further have the following error estimate :*

$$\Big|1\!\!1_{\Delta_s^X(\varepsilon_0)}\sum_{k=0}^n\sum_{J,M}(J^{R,\delta}_{s,k}-\widetilde J^{R,\delta}_{s,k})(t,J,M)(X_s)\Big|\leq C\varepsilon(s+n)\|F_0\|_{0,\beta_0,\mu_0}\|\varphi\|_{L^\infty(\mathbf{R}^{ds})}\,.$$

*Proof.* — By definition of the good sets $\mathcal{G}_k(c)$, the positions in the argument of $f^{(s+k)}_{N,0}-f_0^{(s+k)}$ satisfy the separation condition $|x_i-x_j|\geq\varepsilon_0/2>\varepsilon$ for $i\neq j$ :

$$1\!\!1_{\mathcal{G}_{s+k}(\varepsilon_0/2)}(f^{(s+k)}_{N,0}-f_0^{(s+k)})=1\!\!1_{\mathcal{G}_{s+k}(\varepsilon_0/2)}1\!\!1_{\Delta^X_{s+k}(\varepsilon_0/2)}(f^{(s+k)}_{N,0}-f_0^{(s+k)})\,.$$

So we can write

$$\begin{aligned}(J^{R,\delta}_{s,k}(t,J,M)-\widetilde J^{R,\delta}_{s,k}(t,J,M))(X_s)=\frac{(N-s)!}{(N-s-k)!}\varepsilon^{k(d-1)}&\int_{B_R\setminus\mathcal{M}_s(X_s)}dV_s\varphi_s(V_s)\int_{\mathcal{T}_{k,\delta}(t)}dT_k\\ &\int_{{}^c\mathcal{B}_s^{m_1}(Z_s^0(t_1))}d\nu_{s+1}dv_{s+1}(\nu_{s+1}\cdot(v_{s+1}-v_{m_1}(t_1)))_{j_1}\\ &\quad\dots\int_{{}^c\mathcal{B}^{m_k}_{s+k-1}(Z^0_{s+k-1}(t_k))}d\nu_{s+k}dv_{s+k}\,(\nu_{s+k}\cdot(v_{s+k}-v_{m_k}(t_k)))_{j_k}\\ &\quad\times 1\!\!1_{E_\varepsilon(Z^\varepsilon_{s+k}(0))\leq R^2}1\!\!1_{\Delta_{s+k}(\varepsilon_0/2)}(f^{(s+k)}_{N,0}-f_0^{(s+k)})\,,\end{aligned}$$

and we find directly that

$$\Big|1\!\!1_{\Delta_s^X(\varepsilon_0)}(J^{R,\delta}_{s,k}(t,J,M)-\widetilde J^{R,\delta}_{s,k}(t,J,M))(X_s)\Big|\leq C\frac{R^{k(d+1)}t^k}{k!}\Big\|1\!\!1_{\Delta_{s+k}(\varepsilon_0/2)}(f^{(s+k)}_{N,0}-f_0^{(s+k)})\Big\|_{L^\infty}\,.$$

Note that, summing all the elementary contributions (i.e. summing over $J$, $M$ and $k$), we get the convergence to 0, but with a very bad dependence with respect to $R$ and $n$.

In the case of tensorized initial data, this estimate can be improved using some explicit control on the convergence of the initial data. Looking at the proof of Proposition 6.1.2, we indeed see that

$$1\!\!1_{Z_s\in\mathcal{D}_s}f_0^{\otimes s}-f^{(s)}_{0,N}=\Big(1-\mathcal{Z}_N^{-1}\mathcal{Z}_{N-s}\Big)1\!\!1_{Z_s\in\mathcal{D}_s}f_0^{\otimes s}+\mathcal{Z}_N^{-1}\mathcal{Z}^\flat_{(s+1,N)}1\!\!1_{Z_s\in\mathcal{D}_s}f_0^{\otimes s}$$

with

$$\Big|1-\mathcal{Z}_N^{-1}\mathcal{Z}_{N-s}\Big|\leq(1-\varepsilon\kappa_d|f_0|_{L^\infty L^1})^{-s}-1\leq\varepsilon s\kappa_d|f_0|_{L^\infty L^1}\big(1-\varepsilon\kappa_d|f_0|_{L^\infty L^1}\big)^{-(s+1)}$$

according to Lemma 6.1.2, and

$$\mathcal{Z}_N^{-1}\mathcal{Z}^\flat_{(s+1,N)}\leq\varepsilon s\kappa_d|f_0|_{L^\infty L^1}\big(1-\varepsilon\kappa_d|f_0|_{L^\infty L^1}\big)^{-(s+1)}\,.$$

Using the continuity estimate in Proposition 5.4.1, we then deduce that

$$\begin{aligned}&\Big|1\!\!1_{\Delta_s^X(\varepsilon_0)}(J^{R,\delta}_{s,k}(t,J,M)-\widetilde J^{R,\delta}_{s,k}(t,J,M))(X_s)\Big|\\ &\qquad\leq\varepsilon(s+k)\kappa_d|f_0|_{L^\infty L^1}\|F_0\|_{0,\beta_0,\mu_0}\|\varphi\|_{L^\infty(\mathbf{R}^{ds})}c_{k,J,M}\,.\end{aligned}$$

denoting by $(c_{k,J,M})$ a sequence of nonnegative real numbers such that $\sum_k\sum_{J,M}c_{k,J,M}=1$. This concludes the proof of Lemma 14.2.1. □

**14.2.2. Error coming from the prefactors in the collision operators.** —

As $\varepsilon \to 0$ in the Boltzmann-Grad scaling, we have

$$\frac{(N-s)!}{(N-s-k)!}\varepsilon^{k(d-1)} \to 1\,.$$

Defining

$$\begin{aligned}\overline{J}^{R,\delta}_{s,k}(t,J,M)(X_s) = &\int_{B_R\setminus\mathcal{M}_s(X_s)} dV_s\varphi_s(V_s)\int_{\mathcal{T}_{k,\delta}(t)} dT_k \\ &\int_{{}^c\mathcal{B}^{m_1}_s(Z^0_s(t_1))} d\nu_{s+1}dv_{s+1}\,(\nu_{s+1}\cdot(v_{s+1}-v_{m_1}(t_1)))_{j_1} \\ &\quad\dots\int_{{}^c\mathcal{B}^{m_k}_{s+k-1}(Z^0_{s+k-1}(t_k))} d\nu_{s+k}dv_{s+k}\,(\nu_{s+k}\cdot(v_{s+k}-v_{m_k}(t_k)))_{j_k} \\ &\quad\times 1\!\!1_{E_0(Z_{s+k}(0))\leq R^2}1\!\!1_{Z^\varepsilon_{s+k}(0)\in\mathcal{G}_{s+k}(\varepsilon_0/2)}f_0^{(s+k)}(Z_{s+k}(0))\,,\end{aligned} \tag{14.2.16}$$

and using again the continuity estimate in Proposition 5.4.1, we have the following obvious convergence.

***Lemma 14.2.2.*** — *In the Boltzmann-Grad scaling $N\varepsilon^{d-1}=1$,*

$$\Big|1\!\!1_{\Delta^X_s(\varepsilon_0)}\sum_{k=0}^n\sum_{J,M}(\widetilde{J}^{R,\delta}_{s,k}-\overline{J}^{R,\delta}_{s,k})(t,J,M)(X_s)\Big| \leq C\frac{(s+n)^2}{N}\|\varphi\|_{L^\infty(\mathbf{R}^{ds})}\|F_0\|_{0,\beta_0,\mu_0}\,.$$

**14.2.3. Error coming from the divergence of trajectories.** —

We can now compare the definition (13.2.3) of $J^{0,R,\delta}_{s,k}(t,J,M)$:

$$\begin{aligned}J^{0,R,\delta}_{s,k}(t,J,M)(X_s) = &\int_{B_R\setminus\mathcal{M}_s(X_s)} dV_s\varphi_s(V_s)\int_{\mathcal{T}_{k,\delta}(t)} dT_k\int_{{}^c\mathcal{B}^{m_1}_s(Z^0_s(t_1))} d\nu_{s+1}dv_{s+1}((v_{s+1}-v^0_{m_1}(t_1)\cdot\nu_{s+1})_{j_1} \\ &\dots\int_{{}^c\mathcal{B}^{m_k}_{s+k-1}(Z^0_{s+k-1}(t_k))} d\nu_{s+k}dv_{s+k}((v_{s+k}-v^0_{m_k}(t_n)\cdot\nu_{s+k})_{j_k} \\ &\times 1\!\!1_{E_0(Z^0_{s+k}(0))\leq R^2}f_0^{(s+k)}(Z^0_{s+k}(0))\,.\end{aligned}$$

and the formulation (14.2.16) for the approximate BBGKY hierarchy.

Lemma 14.1.1 implies that at time 0 we have

$$|X_{s+k}(0)-X^0_{s+k}(0)|\leq Ck\varepsilon\,,\quad\text{and}\quad V_{s+k}(0)=V^0_{s+k}(0)\,.$$

Since $f_0^{(s+k)}$ is continuous, we obtain the expected convergence as stated in the following lemma.

***Lemma 14.2.3.*** — *In the Boltzmann-Grad scaling $N\varepsilon^{d-1}=1$, for all fixed $s,k\in\mathbf{N}$ and $t<T$,*

$$\bar{J}^{R,\delta}_{s,k}(t,J,M)(X_s)-J^{0,R,\delta}_{s,k}(t,J,M)(X_s)\to 0\,.$$

*For tensorized Lipschitz initial data, we further have the following error estimate :*

$$\Big|1\!\!1_{\Delta^X_s(\varepsilon_0)}\sum_{k=0}^n\sum_{J,M}(\bar{J}^{R,\delta}_{s,k}-J^{0,R,\delta}_{s,k})(t,J,M)(X_s)\Big|\leq C\varepsilon n\|\nabla_x f_0\|_\infty\|F_0\|_{0,\beta_0,\mu_0}\|\varphi\|_{L^\infty(\mathbf{R}^{ds})}\,.$$

Notice that putting together Lemmas 14.2.1, 14.2.2 and 14.2.3, along with the estimates (14.2.12)-(14.2.13) and (14.2.14)-(14.2.15), end the proof of Theorem 8 up to the rate of convergence. This is the object of the next paragraph.

**14.2.4. Optimization for tensorized Lipschitz initial data.** — We can now conclude the proof of Theorem 8. Gathering the results of Lemmas 14.2.1, 14.2.2 and 14.2.3, together with the estimates (14.2.12)-(14.2.13) and (14.2.14)-(14.2.15), we get

$$\begin{aligned}\left\|I_s(t) - I_s^0(t)\right\|_{L^\infty(\mathbf{R}^{ds})} \leq & C\left(2^{-n} + e^{-C'\beta_0 R^2} + \frac{n^2}{T}\delta\right)\|\varphi\|_{L^\infty(\mathbf{R}^{ds})}\sup_N \|F_{N,0}\|_{\varepsilon,\beta_0,\mu_0}\\ & + Cn^2(s+n)\Big(R\eta^{d-1} + R^d\Big(\frac{a}{\varepsilon_0}\Big)^{d-1} + R\Big(\frac{\varepsilon_0}{\delta}\Big)^{d-1}\Big)\|F_{N,0}\|_{\varepsilon,\beta_0,\mu_0}\|\varphi\|_{L^\infty(\mathbf{R}^{ds})}\\ & + C\varepsilon(s+n)\|F_0\|_{0,\beta_0,\mu_0}\|\varphi\|_{L^\infty(\mathbf{R}^{ds})}\\ & + C\frac{(s+n)^2}{N}\|\varphi\|_{L^\infty(\mathbf{R}^{ds})}\|F_0\|_{0,\beta_0,\mu_0}\\ & + Cn\varepsilon\|\nabla_x f_0\|_{L^\infty}\|\varphi\|_{L^\infty(\mathbf{R}^{ds})}\|F_0\|_{0,\beta_0,\mu_0}\end{aligned}$$

Therefore, choosing

$$n \sim C_1|\log\varepsilon|, \quad R^2 \sim C_2|\log\varepsilon|$$

for some sufficiently large constants $C_1$ and $C_2$, and

$$\delta = \varepsilon^{(d-1)/(d+1)}, \quad \varepsilon_0 = \varepsilon^{d/(d+1)}$$

we find that the total error is smaller than $C\varepsilon^\alpha$ for any $\alpha < (d-1)/(d+1)$.

This ends the proof of Theorem 8.

### 14.3. Convergence in the case of a smooth interaction potential: proof of Theorem 11

Let us now prove Theorem 11.

The same arguments as in the previous section provide the convergence for any smooth short-range potential satisfying (8.3.1). Let us only sketch the proof and point out how to deal with the following minor differences.

– The elimination of multiple collisions gives an additional error term : from Propositions 11.3.1 and 11.3.2, we indeed deduce the analogue of (14.2.12):

$$\left\|I_s(t) - I_{s,n}^{R,\delta}(t)\right\|_{L^\infty(\mathbf{R}^{ds})} \leq C\left(\varepsilon + 2^{-n} + e^{-C'\beta_0 R^2} + \frac{n^2}{T}\delta\right)\|\varphi\|_{L^\infty(\mathbf{R}^{ds})}\|\widetilde{F}_{N,0}\|_{\varepsilon,\beta_0,\mu_0}\,. \tag{14.3.17}$$

– The error term coming from the elimination of pathological velocities and impact parameters depends (in a non trivial way) on the local $L^\infty$ norm of the cross-section: estimate (14.2.14) becomes

$$\begin{aligned}&\Big|\mathbb{1}_{\Delta_s(\varepsilon_0)}\sum_{k=0}^n\sum_{J,M}\big(I_{s,k}^{0,R,\delta} - J_{s,k}^{0,R,\delta}\big)(t,J,M)\Big|\\ &\leq Cn^2(s+n)\Big(R\eta^{d-1} + C(\Phi,R,\eta)R^d\Big(\frac{a}{\varepsilon_0}\Big)^{d-1} + C(\Phi,R,\eta)R\Big(\frac{\varepsilon_0}{\delta}\Big)^{d-1}\Big)\|F_0\|_{0,\beta_0,\mu_0}\|\varphi\|_{L^\infty(\mathbf{R}^{ds})}\,.\end{aligned}$$

– Additional error terms come from the difference between truncated marginals and true marginals (namely on the initial data) : by Lemma 11.1.2, there holds the convergence

$$f_{0,N}^{(s)} - \widetilde{f}_{0,N}^{(s)} \longrightarrow 0\,, \qquad \text{for fixed } s \geq 1\,, \text{ as } N \to \infty \text{ with } N\varepsilon^{d-1} \equiv 1\,, \text{ uniformly in } \Omega_s\,.$$

Together with Lemma 14.2.1, this implies that

$$J_{s,k}^{R,\delta}(t,J,M)(X_s) - \widetilde{J}_{s,k}^{R,\delta}(t,J,M)(X_s) \to 0\,.$$

– The micro-translations between the "good" Boltzmann and BBGKY pseudo-trajectories depend on the maximal duration of the interactions to be considered

$$|X_{s+k}(0) - X_{s+k}^0(0)| \leq C(\Phi,R,\eta)k\varepsilon\,, \quad \text{and} \quad V_{s+k}(0) = V_{s+k}^0(0)\,,$$

so that the convergence

$$\bar{J}_{s,k}^{R,\delta}(t,J,M)(X_s) - J_{s,k}^{0,R,\delta}(t,J,M)(X_s) \to 0$$

may be very slow.

Combining all estimates shows that for any fixed $s \in \mathbf{N}$ and any $t < T$

$$I_s(t)(X_s) - I_s^0(t)(X_s) \to 0$$

locally uniformly in $\Omega_s$, which concludes the proof of Theorem 11.

# CHAPTER 15

# CONCLUDING REMARKS

## 15.1. On the time of validity of Theorems 9 and 8

Let us first note that, for any fixed $N$, the BBGKY hierarchy has a global solution since it is formally equivalent to the Liouville equation in the phase space of dimension $2Nd$, which is nothing else than a linear transport equation. The fact that we obtain a uniform bound on a finite life span only, is therefore due to the analytical-type functional spaces $\mathbf{X}_{\varepsilon,\beta,\mu}$ we consider. Belonging to such a functional space requires indeed a strong control on the growth of marginals.

An important point is that the time of convergence is exactly the time for which these uniform a priori estimates hold. By definition of the functional spaces, we are indeed in a situation where the high order correlations can be neglected (see (14.2.12) and (14.3.17)), so that we only have to study the dynamics of a finite system of particles. The term-by-term convergence relies then on geometrical properties of the transport in the whole space, which do not introduce any restriction on the time of convergence.

A natural question is therefore to know whether or not it is possible to get better uniform a priori estimates and thus to improve the time of convergence. Let us first remark that such a priori estimates would hold for the Boltzmann hierarchy and thus for the nonlinear non homogeneous Boltzmann equation. As mentioned in Chapter 2, Remark 2.3.2, the main difficulty is to control the possible spatial concentrations of particles, which would contradict the rarefaction assumption and lead to an uncontrolled collision process.

## 15.2. More general potentials

A first natural extension to this work concerns the case of a compactly supported, repulsive potential, but no longer satisfying (8.3.1). As explained in Chapter 8, that assumption guarantees that the cross section is well defined everywhere, since the deflection angle is a one-to-one function of the impact parameter. If that is no longer satisfied, additional decompositions are necessary to split the integration domain in subdomains where the cross-section is well-defined : as mentioned in Remark 3.1.3, we then expect to be able to extend the convergence proof, up to some technical complications due to the resummation procedures (see [**39**] for an alternative method). Note that, if the deflection angle can be locally constant as a function of the impact parameter, the method does not apply, which is consistent with the fact that we do not expect the Boltzmann equation to be a good approximation of the dynamics (see the by now classical counterexample by Uchiyama [**15**]).

From a physical point of view it would be more interesting to study the case of long-range potentials. Then the cross section actually becomes singular, so a different notion of limit must be considered, possibly in the spirit of Alexandre and Villani [**3**]. One intermediate step, as in [**16**], would be to extend this work to the case when the support of the potential goes to infinity with the number of particles. Then one could try truncating the long-range potential and showing that the tail of the potential has very little effect in the convergence.

Note that in the case when grazing collisions become predominant, then the Boltzmann equation should be replaced by the Landau equation, whose derivation is essentially open; a first result in that direction was obtained very recently by A. Bobylev, M. Pulvirenti and C. Saffirio in [**4**], where a time zero convergence result is established.

## 15.3. Other boundary conditions

As it stands, our analysis is restricted to the whole space (namely $X_N \in \mathbf{R}^{dN}$). It is indeed important that free flow corresponds to straight lines (see in particular Lemmas 12.2.1 and 12.2.3 as well more generally as the analysis of pathological trajectories in Chapter 12).

It would be very interesting to generalize this work to more general geometries. A first step in that direction is to study the case of periodic flows in $X_N$. The geometric lemmas must be adapted to that framework, and in particular it appears that a finite life span must a priori be given before the surgery of the collision integrals may be performed (see [**5**]).

The case of a general domain is again much more complicated, and results from the theory of billiards would probably need to be used.

# BIBLIOGRAPHY


[1] R. Alexander, The Infinite Hard Sphere System, *Ph.D. dissertation,* Dept. Mathematics, Univ. California, Berkeley, 1975.

[2] R. Alexander, Time Evolution for Infinitely Many Hard Spheres, *Commun. in Math. Phys.* **49** (1976), p.217-232.

[3] R. Alexandre and C. Villani, On the Boltzmann equation for long-range interactions, *Comm. Pure Appl. Math.* **55** (2002), no. 1, p. 30-70.

[4] A. Bobylev, M. Pulvirenti and C. Saffirio, From particle systems to the Landau equation: a consistency result, *preprint.*

[5] T. Bodineau, I. Gallagher and L. Saint-Raymond. The linear Boltzmann equation as the low density limit of short-range interacting system of particles, *preprint.*

[6] N. Bogoliubov, *Problems of Dynamical Theory in Statistical Physics*, in Studies in Statistical Mechanics, J. de Boer and G.E. Uhlenbeck, Ed. Interscience, New York, 1962.

[7] B. Bollobás, *Random graphs*, Second edition. Cambridge Studies in Advanced Mathematics, **73**, Cambridge University Press, Cambridge, 2001. xviii+498 pp.

[8] L. Boltzmann, Weitere Studien uber das Warme gleichgenicht unfer Gasmolakular. Sitzungsberichte der Akademie der Wissenschaften 66 (1872), 275-370. Translation : Further studies on the thermal equilibrium of gas molecules, in Kinetic Theory 2, 88-174, Ed. S.G. Brush, Pergamon, Oxford (1966).

[9] L. Boltzmann, *Leçons sur la théorie des gaz*, Gauthier-Villars (Paris, 1902-1905). Ré-édition Jacques Gabay, 1987.

[10] M. Born, H. S. Green, A general kinetic theory of liquids. I. The molecular distribution functions, *Proc. Roy. Soc. London.* Ser. A. **188** (1946), p. 10-18.

[11] W. Braun, K. Hepp, The Vlasov dynamics and its fluctuations in the 1/N limit of interacting classical particles, *Comm. Math. Phys.,* **56** (1977), p. 101-113.

[12] C. Cercignani, On the Boltzmann equation for rigid spheres, *Transport Theory Statist. Phys.* **2** (1972), no. 3, p. 211-225.

[13] C. Cercignani, *The Boltzmann equation and its application*, Springer-Verlag, 1988.

[14] C. Cercignani, V. I. Gerasimenko, D. I. Petrina, *Many-Particle Dynamics and Kinetic Equations*, Kluwer Academic Publishers, Netherlands, 1997.

[15] C. Cercignani, R. Illner, M. Pulvirenti, *The Mathematical Theory of Dilute Gases*, Springer Verlag, New York NY, 1994.

[16] L. Desvillettes and M. Pulvirenti, The linear Boltzmann equation for long-range forces: a derivation from particle systems, *Math. Models Methods Appl. Sci.* **9** (1999), no. 8, p. 1123-1145.

[17] L. Desvillettes, V. Ricci, A rigorous derivation of a linear kinetic equation of Fokker-Planck type in the limit of grazing collisions. *J. Statist. Phys.* **104** (2001), no. 5-6, p. 1173-1189.

[18] R. DiPerna and P.-L. Lions, On the Cauchy problem for Boltzmann equations: global existence and weak stability, *Ann. of Math.*(2) **130** (1989), no. 2, 321366.

[19] R. Esposito, M. Pulvirenti, From particles to fluids, *Handbook of mathematical fluid dynamics.* Vol. III, p. 1-82, North-Holland, Amsterdam, 2004.

[20] H.-O. Georgii, The Equivalence of Ensembles for Classical Systems of Particles, *Journal of Statistical Physics* **80** (1995), p. 1341-1378.

[21] F. Golse, *The Mean-Field Limit for the Dynamics of Large Particle Systems.* Journées"Équations aux Dérivées Partielles", Exp. No. IX, 47 pp., Univ. Nantes, Nantes, 2003.

[22] F. Golse, P.-L. Lions, B. Perthame, R. Sentis, Regularity of the moments of the solution of a transport equation, *J. Funct. Anal.* **76** (1988), no. 1, 110125.

[23] H. Grad, Principles of the kinetic theory of gases, *Handbuch der Physik* **12**, Thermodynamik der Gase p. 205-294 Springer-Verlag, Berlin-Gottingen-Heidelberg, 1958.

[24] H. Grad, On the kinetic theory of rarefied gases, *Comm. Pure Appl. Math.* **2** (1949), p. 331-407.

[25] M. Hauray, P.-E. Jabin, N-particles approximation of the Vlasov equations with singular potential, *Arch. Ration. Mech. Anal.***183** (2007), no. 3, p. 489524.

[26] E. Hewitt, L. Savage, Symmetric measures on Cartesian products. *Trans. Amer. Math. Soc.* **80** (1955), p. 470-501.

[27] D. Hilbert, *Sur les problèmes futurs des mathématiques*, Compte-Rendu du 2ème Congrés International de Mathématiques, tenu à Paris en 1900 : Gauthier-Villars (Paris, 1902), p. 58-114.

[28] R. Illner, M. Pulvirenti, Global validity of the Boltzmann equation for a two-dimensional rare gas in vacuum. *Comm. Math. Phys.* **105** (1986), no. 2, p. 189-203.

[29] R. Illner, M. Pulvirenti, Global validity of the Boltzmann equation for two- and three dimensional rare gas in vacuum. Erratum and improved result, *Comm. Math. Phys.* **121** (1989), p. 143-146.

[30] F. King, BBGKY hierarchy for positive potentials, *Ph.D. dissertation,* Dept. Mathematics, Univ. California, Berkeley, 1975.

[31] J. G. Kirkwood, The Statistical Mechanical Theory of Transport Processes I. General Theory, *Journal of Chemical Physics* **14** (1946) p.180-202.

[32] L.D. Landau, E.M. Lifshitz, *Statistical Physics*, Part 1. Vol. 5 (3rd ed.) (1980), Butterworth-Heinemann.

[33] L.D. Landau, E.M. Lifshitz, *Statistical Physics*, Part 2. Vol. 9 (1rst ed.) (1980), Butterworth-Heinemann.

[34] O.E. Lanford, *Time evolution of large classical systems*, Lect. Notes in Physics **38**, J. Moser ed., 1–111, Springer Verlag (1975).

[35] L. Nirenberg, An abstract form of the nonlinear Cauchy-Kowalewski theorem, *Journal of Differential Geometry*, **6** (1972), p. 561-576.

[36] T. Nishida, A note on a theorem by Nirenberg, *Journal of Differential Geometry*, **12** (1977), p. 629-633.

[37] H. Poincaré, *Works. Vol. VII : Rotating fluid masses. Principles of analytic mechanics. Three-body problem. With a preface by Jacques Lévy.* Reprint of the 1952 edition. Gauthier-Villars Great Classics, 1996.

[38] M. Pulvirenti, *A short introduction to the kinetic theory*, available on the webpage `http://www.mat.uniroma1.it/people/pulvirenti/didattica/index.html`

[39] M. Pulvirenti, C. Saffirio and S. Simonella, *personal communication.*

[40] H. Spohn, Boltzmann hierarchy and Boltzmann equation, in *Kinetic theories and the Boltzmann equation* (Montecatini, 1981), p. 207-220.

[41] H. Spohn, *Large Scale Dynamics of Interacting Particles*, Texts and Monographs in Physics, Springer Verlag, Heidelberg, 1991.

[42] K. Uchiyama, Derivation of the Boltzmann equation from particle dynamics, *Hiroshima Math. J.* **18** (1988), no. 2, p. 245-297.

[43] S. Ukai, On the existence of global solutions of mixed problem for non-linear Boltzmann equation, *Proc. Japan Acad.,* **50** (1974), p. 179-184.

[44] S. Ukai, Les solutions globales de l'équation de Boltzmann dans l'espace tout entier et dans le demi-espace, *C.R. Acad. Sci. Paris,* **282**(6) (1976), p. 317-320.

[45] S. Ukai, The Boltzmann-Grad Limit and Cauchy-Kovalevskaya Theorem, *Japan J. Indust. Appl. Math.,* **18** (2001), p. 383-392.

[46] C. Villani, *A review of mathematical topics in collisional kinetic theory*, Handbook of mathematical fluid dynamics, Vol. I, 71-305, North-Holland, Amsterdam, 2002.

[47] J. Yvon, *La théorie statistique des fluides et l'équation d'état*, Actual. Sci. et Indust. **203** (Paris, Hermann), 1935.

# NOTATION INDEX


$B_R$, ball of radius $R$ centered at zero in $\mathbf{R}^d$, page 26

$B_R^s$, ball of radius $R$ centered at zero in $\mathbf{R}^{ds}$, page 26

$B_R(x)$, ball of radius $R$ centered at $x$ in $\mathbf{R}^d$, page 82

$\mathcal{B}_k(\overline{Z}_k)$ a small set of angles and velocities of a particle adjoined to $\overline{Z}_k$ (or a neighboring configuration), leading to pathological trajectories, page 98

$b(w,\omega)/|w|$, cross-section, page 69

$\mathbf{C}_N$, BBGKY hierarchy collision operator, page 29 for the hard-spheres case and page 78 for the potential case

$\mathbf{C}^0$, Boltzmann hierarchy collision operator, page 31 for the hard-spheres case and page 79 for the potential case

$\mathcal{C}_{s,s+1}$, BBGKY collision operator, page 29 for the hard-spheres case and page 77 for the potential case

$\mathcal{C}_{s,s+m}$, BBGKY collision operator involving $m$ additional particles, page 77

$\mathcal{C}^0_{s,s+1}$, Boltzmann collision operator, page 31 for the hard-spheres case and page 70 for the potential case

$\mathcal{D}_N$, domain on which the hard-spheres dynamics take place, page 25

$\mathcal{D}_N^s$, artificial set in $X_N$ variables on which the Hamiltonian dynamics take place, page 72

$\Delta_m(X_s)$, $m$-particle cluster based on $X_s$, page 76

$\Delta_s$, well-separated initial configurations, page 107

$\Delta_s^X$, well-separated initial positions, page 107

$d\sigma_N^{i,j}$, surface measure on $\Sigma_N^s(i,j)$, page 73

$d\sigma$, surface measure on $S_\varepsilon(x_i)$, page 77

$dZ_{(i,j)}$, $2d(j-i+1)$-dimensional Lebesgue measure, page 47

$E(X_s,X_n)$, $\varepsilon$-closure of $X_s$ in $X_N$, page 76

$E_{<i_0,j_0>}(X_s,X_n)$, $\varepsilon$-closure of $X_s$ in $X_N$ with a weak link at $(i_0,j_0)$, page 76

$E_\varepsilon(Z_s)$, $s$-particle Hamiltonian, page 84

$E_0(Z_s)$, $s$-particle free Hamiltonian, page 38

$f_N^{(s)}$, marginal of order $s$ of the $N$-particle distribution function, page 27 for the hard-spheres case, page 71 for the potential case

$\widetilde{f}_N^{(s)}$, truncated marginal of order $s$ of the $N$-particle distribution function, page 72

$f^{(s)}$, marginal of order $s$ associated with the Boltzmann hierarchy, page 31

$\mathbf{\Phi}_\varepsilon$, rescaled potential, page 84

$\mathcal{G}_k$, set of good configurations of $k$ particles, page 98

$\mathbf{H}_s(t)$, $s$-particle flow in the potential case, page 78

$\mathbf{H}(t)$, BBGKY hierarchy flow in the potential case, page 78

$I_\varphi$, observable (average with respect to momentum variables), page 51

$I_s(t)(X_s)$ BBGKY observable, page 55 for the hard-spheres case, page 92 for the potential case

$I_s^0(t)(X_s)$ Boltzmann observable, page 56

$I_{s,k}^{R,\delta}(t)(X_s)$ reduced BBGKY observable, page 59 for the hard-spheres case, page 92 for the potential case

$I_{s,k}^{0,R,\delta}(t)(X_s)$ reduced Boltzmann observable, page 59

$K(w,y,\rho)$, cylinder of origin $w \in \mathbf{R}^d$, of axis $y \in \mathbf{R}^d$ and radius $\rho > 0$, page 99

$\kappa_d$, volume of the unit ball in $\mathbf{R}^d$, page 41

$n^{i,j}$, outward normal to $\Sigma_N(i,j)$, page 28

$\nu^{i,j}$, direction of $x_i - x_j$, page 3

$\mathcal{M}_s(X_s)$, good set of initial velocities associated with well separated positions, page 107

$\mathcal{P}$, the set of continuous densities of probability in $\mathbf{R}^{2d}$, page 49

$\rho_*$, distance of minimal approach, page 64

$\mathbf{S}_s(t)$, $s$-particle free flow, page 31

$\mathbf{S}(t)$, total free flow, page 31

$\mathbf{S}_1^{d-1}$, unit sphere in $\mathbf{R}^d$, page 9

$S_\varepsilon(x_i)$, sphere in $\mathbf{R}^d$ of radius $\varepsilon$, centered at $x_i$, page 77

$\sigma$, scattering operator in the hard-spheres case, page 31

$\sigma_\varepsilon$, scattering operator in the case of a potential, page 66

$\sigma_0$, Boltzmann scattering operator, page 66

$\Sigma_N(i,j)$, boundary of $\mathcal{D}_N$, page 28

$\Sigma_N^s(i,j)$, boundary of the artificial set $\mathcal{D}_N^s$, page 73

$\mathbf{T}_s(t)$, $s$-particle flow for hard spheres, page 30

$\mathbf{T}(t)$, total flow for hard spheres, page 30

$t_\varepsilon = \varepsilon\tau_*$, nonlinear interaction time, page 64

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