# WEAK CONVERGENCE RESULTS FOR INHOMOGENEOUS ROTATING FLUID EQUATIONS

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**Abstract.** We consider the equations governing incompressible, viscous fluids in three space dimensions, rotating around an inhomogeneous vector $B(x)$: this is a generalization of the usual rotating fluid model (where $B$ is constant). In the case when $B$ has non degenerate critical points we prove the weak convergence as the rotation rate tends to infinity, of Leray–type solutions towards a vector field which satisfies a heat equation. The method of proof uses weak compactness arguments, which also enable us to recover the usual 2D Navier-Stokes limit in the case when $B$ is constant.

## Résultats de convergence faible pour des équations des fluides tournants non homogènes

**Résumé.** On considère les équations modélisant des fluides incompressibles et visqueux en trois dimensions d'espace, en rotation rapide autour d'un vecteur non homogène $B(x)$: on généralise ainsi le modèle habituel des fluides tournants (où $B$ est constant). Dans le cas où $B$ a des points critiques non dégénérés, on démontre la convergence des solutions de Leray, quand la vitesse de rotation tend vers l'infini, vers un champ de vecteurs qui vérifie une équation de la chaleur. La méthode de démonstration repose sur des arguments de compacité faible, qui nous permettent de retrouver également la limite habituelle Navier-Stokes 2D quand $B$ est constant.

## 1. Introduction

The aim of this article is to study the asymptotics of solutions of rotating fluid equations, in the case when the rotation vector is not homogeneous. We consider a domain $\Omega = \Omega_h \times \Omega_3$, where $\Omega_h$ denotes either the whole space $\mathbf{R}^2$ or any periodic domain of $\mathbf{R}^2$, and similarly $\Omega_3$ denotes $\mathbf{R}$ or $\mathbf{T}$, where $\mathbf{T}$ denotes the one-dimensional torus. We are interested in the following system:

$$(1.1)\qquad \begin{aligned} &\partial_t u + u\cdot\nabla u - \nu\Delta u + \frac{1}{\varepsilon} u \wedge B + \nabla p = 0 \quad \text{ on } \mathbf{R}^+ \times \Omega, \\ &\nabla\cdot u = 0 \quad \text{ on } \mathbf{R}^+ \times \Omega, \\ &u_{|t=0} = u^0 \quad \text{ on } \Omega \end{aligned}$$

where $B = be_3$ is the adimensionalized rotation vector, and $b$ is a smooth function defined in $\Omega_h$. We shall suppose throughout this paper that $b$ does not vanish; more assumptions on $b$ will be made as we go along. In the case when $\Omega$ is unbounded, we suppose that the vector fields vanish at infinity.

Before stating the result we shall prove here, let us recall some well-known facts in the constant case ($b = 1$). The rotating fluid equations, with $b$ constant and homogeneous, modelize the movement of the atmosphere or the oceans at mid-latitudes (see for instance [11] or [18]). The fluid is supposed to be incompressible (which corresponds to the hydrostatic approximation), and its viscosity is $\nu > 0$. The vector field $u$ is the velocity and the scalar $p$ is the pressure, both are unknown. Denoting by $U$ and $L$ the characteristic velocity and length scales of the motion, and by $f$ the local vertical component of the Earth rotation, the parameter $\varepsilon$, known as the Rossby number, is the ratio

$$\varepsilon = \frac{U}{2fL} \cdot$$

Taking the limit $\varepsilon \to 0$ means that the scale of motion of the fluid is much smaller than that of the Earth. In the constant case, those equations have been studied by a number of authors. We refer for instance to the works of A. Babin, A. Mahalov and B. Nicolaenko [2]-[4], P. Embid and A. Majda [7], I. Gallagher [8], E. Grenier [12] for the periodic case, and J.-Y. Chemin, B. Desjardins, I. Gallagher and E. Grenier [5] for the whole space case as well as [6] for the case of horizontal plates with Dirichlet boundary conditions (for such boundary conditions we refer also to the work of E. Grenier and N. Masmoudi [13] as well as N. Masmoudi [17]). We also refer to the survey paper of R. Temam and M. Ziane [20], and the references therein. The results in those papers concern both weak and strong solutions; in this article we shall only be concerned with Leray–type weak solutions ([14]): we will see in Section 2 below that their existence is an easy adaptation of the proof of Leray's existence theorem [14]. In the constant case, it is known that weak solutions converge towards the solution of the two–dimensional Navier-Stokes equations. Such a result in the whole space case is due to Strichartz-type estimates (which are obtained by writing the solution of the linearized problem in Fourier space), whereas in the periodic case it follows from the study of the (discrete) spectrum of the rotating fluid operator (following methods introduced by S. Schochet in [19]).

In this paper we are interested in the case when the rotation operator is not homoegenous. This is physically motivated by the fact that the vertical component of the rotation depends in fact on the latitude, hence larger geographical zones can be considered with such a model. Moreover new physical phenomena appear when the variation of the rotation is taken into account, namely the presence of Rossby waves (see [9] for a presentation of the various waves present in geophysical flows). Our goal here is not to describe those waves precisely, but rather to show that their presence does not disturb the mean flow. One would like to follow the same methods as in the constant case, described above, but the problem is that it does not seem a good idea to take the Fourier transform of the Coriolis operator

$$Lu \stackrel{\text{def}}{=} P(u \wedge B), \quad \nabla \cdot u = 0,$$

when $B$ is not homogeneous (here $P$ denotes the projector onto divergence free vector fields); moreover the study of the spectrum of $L$ is not an easy matter. So our strategy to study this problem is first to try and recover the well-known results of the constant case without using any information on the spectrum of $L$ (other than the determination of its kernel), and without using the Fourier transform. This will be achieved in Section 3. Then the study of the variable case will be an adaptation of the constant case, in Section 4.

Before stating the results we shall prove in this paper, let us comment on the difficulties compared with the constant case: as stated above, it is easy to construct a bounded family of weak solutions to our problem, whether $b$ is constant or not. Hence one can construct a weak limit point $u$, and the question we want to address is to find the equation satisfied by $u$. Of course the problem consists in taking the limit in the non linear part of the equation. As noted above, we do not wish to study the spectrum of the operator $L$ since that seems to be a difficult issue. So we cannot apply the usual, constant $b$ methods, as to our knowledge they all involve spectral properties of $L$. The idea therefore is to turn to what is known as "weak compactness methods", in the spirit of P.-L. Lions and N. Masmoudi [15]-[16] (for the incompressible limit). We shall recall briefly below what those methods are, and then we shall state the main results of this paper.

1.1. **Weak compactness methods.** Let us explain what weak compactness methods are all about. The idea is as follows: as usual the trouble to find the limit of the equation comes from the bilinear terms. They can be separated into three categories:

- products involving only elements of the kernel of the penalization $L$, which can be shown to be compact;
- products of elements of the kernel against elements of $(\mathrm{Ker}L)^\perp$, for which one can take the limit since elements of $(\mathrm{Ker}L)^\perp$ converge weakly to zero;
- products involving only elements of $(\mathrm{Ker}L)^\perp$, which are the problem.

The idea now is to prove that in the last situation, the limit is in fact zero for algebraic reasons: in previous works on rotating fluids, that result was proved essentially by writing the product of two elements of $(\mathrm{Ker}L)^\perp$ by projection onto eigenvectors of $L$. In the periodic case, a "miracle" in the formulation yielded the result (see [2]-[4] or [8]), whereas in the whole space case, Strichartz estimates did the job (and the convergence was strong), see [5]. In this paper we will show that the result has in fact not much to do with spectral properties of $L$, but is due to simple algebraic properties. Let us recall the result in the case of the incompressible limit, where such properties were first used (see [15]).

**Proposition 1.1.** [15] *Let $(\rho_\varepsilon), (u_\varepsilon), (\theta_\varepsilon)$ be bounded families of $L^2([0,T], H^1(\Omega))$ such that*

$$\rho_\varepsilon \rightharpoonup \rho, \quad u_\varepsilon \rightharpoonup u, \quad \theta_\varepsilon \rightharpoonup \theta \text{ as } \varepsilon \to 0.$$

*Assume that*

$$\begin{aligned}
&\varepsilon\partial_t \rho_\varepsilon + \nabla\cdot u_\varepsilon = 0, \\
&\varepsilon\partial_t u_\varepsilon + \nabla(\rho_\varepsilon + \theta_\varepsilon) = \varepsilon s_\varepsilon, \\
&\varepsilon\partial_t \theta_\varepsilon + \frac{2}{3}\nabla\cdot u_\varepsilon = \varepsilon s'_\varepsilon,
\end{aligned}$$

*where $s_\varepsilon$ and $s'_\varepsilon$ are bounded in $L^1([0,T], H^{-s}(\Omega))$ for some $s > 0$. Then*

$$P\nabla\cdot(u_\varepsilon \otimes u_\varepsilon) \to P\nabla\cdot(u \otimes u) \text{ and } \nabla\cdot(u_\varepsilon\theta_\varepsilon) \to \nabla\cdot(u\theta)$$

*in the sense of distributions, where $P$ is the Leray projector onto divergence free vector fields.*

*Proof.* This result has to be compared with the so-called "compensated compactness" theorems, in the sense that the convergences of some quadratic quantities in $\rho_\varepsilon, u_\varepsilon, \theta_\varepsilon$ are established under the assumption that some combinations of the derivatives of these functions converge strongly in time to 0. The proof consists in checking that the acoustic oscillations do not bring any contribution to the limiting terms. We introduce the following decompositions:

$$u_\varepsilon = Pu_\varepsilon + \nabla\psi_\varepsilon, \quad \text{and} \quad \theta_\varepsilon = \frac{3\theta_\varepsilon - 2\rho_\varepsilon}{5} + \pi_\varepsilon,$$

so that

$$Pu_\varepsilon \text{ and } \frac{3\theta_\varepsilon - 2\rho_\varepsilon}{5} \text{ are bounded in } W^{1,1}([0,T], H^{-s}(\Omega)), \quad \text{and}$$

$$\varepsilon\partial_t \nabla\psi_\varepsilon + \nabla\pi_\varepsilon = \varepsilon(Id - P)s_\varepsilon \to 0, \quad \varepsilon\partial_t\pi_\varepsilon + \frac{2}{3}\Delta\psi_\varepsilon = \frac{2\varepsilon}{5}s'_\varepsilon \to 0 \text{ in } L^1([0,T], H^{-s}(\Omega)).$$

We shall note in the following $S_\varepsilon \stackrel{\text{def}}{=} (Id-P)s_\varepsilon$ and $S'_\varepsilon \stackrel{\text{def}}{=} \frac{2}{5}s'_\varepsilon$. The incompressibility and Boussinesq relations

$$\nabla\cdot u = 0, \quad \nabla(\rho + \theta) = 0$$

allow to identify the limits

$$Pu_\varepsilon \to u, \quad \frac{3\theta_\varepsilon - 2\rho_\varepsilon}{5} \to \theta \text{ in } L^2([0,T]\times\Omega),$$
$$\nabla\psi_\varepsilon \rightharpoonup 0, \quad \pi_\varepsilon \rightharpoonup 0 \text{ in } w - L^2([0,T]\times\Omega),$$

from which we deduce that, in the sense of distributions

$$u_\varepsilon \otimes u_\varepsilon - \nabla\psi_\varepsilon \otimes \nabla\psi_\varepsilon \to u \otimes u,$$
$$\theta_\varepsilon u_\varepsilon - \pi_\varepsilon \nabla\psi_\varepsilon \to \theta u.$$

The key argument is therefore the following formal computation (which can be made rigorous by introducing regularizations with respect to the space variable $x$)

$$\begin{aligned}
P\nabla.(\nabla\psi_\varepsilon \otimes \nabla\psi_\varepsilon) &= \frac{1}{2}P\nabla|\nabla\psi_\varepsilon|^2 + P(\Delta\psi_\varepsilon\nabla\psi_\varepsilon)\\
&= \frac{3}{2}P(-\partial_t(\varepsilon\pi_\varepsilon\nabla\psi_\varepsilon) - \pi_\varepsilon\nabla\pi_\varepsilon + \pi_\varepsilon\varepsilon S_\varepsilon + \varepsilon {S'}_\varepsilon\nabla\psi_\varepsilon)\\
&= \frac{3}{2}P(-\partial_t(\varepsilon\pi_\varepsilon\nabla\psi_\varepsilon) + \pi_\varepsilon\varepsilon S_\varepsilon + \varepsilon {S'}_\varepsilon\nabla\psi_\varepsilon),\\
\nabla.(\pi_\varepsilon\nabla\psi_\varepsilon) &= \pi_\varepsilon\Delta\psi_\varepsilon + \nabla\psi_\varepsilon.\nabla\pi_\varepsilon\\
&= \frac{3}{2}\pi_\varepsilon(\varepsilon {S'}_\varepsilon - \varepsilon\partial_t\pi_\varepsilon) + \nabla\psi_\varepsilon\cdot(\varepsilon S_\varepsilon - \varepsilon\partial_t\nabla\psi_\varepsilon)\\
&= \frac{3}{2}\pi_\varepsilon\varepsilon {S'}_\varepsilon + \nabla\psi_\varepsilon\cdot\varepsilon S_\varepsilon - \frac{3\varepsilon}{4}\partial_t|\pi_\varepsilon|^2 - \frac{\varepsilon}{2}\partial_t|\nabla\psi_\varepsilon|^2,
\end{aligned}$$

which shows that the contribution of the acoustic oscillations is negligible. □

Inspired by the previous computation, we shall in this article try to use a similar method in the case of rotating fluids: we refer to the proofs of Propositions 3.4 and 4.3 for precise computations.

1.2. **Main results.** Since we consider incompressible flows, we introduce the following subspaces of $L^2(\Omega)$ and $H^1(\Omega)$

$$\mathrm{H}(\Omega) = \{u \in L^2(\Omega) \,/\, \nabla\cdot u = 0\}, \quad \mathrm{V}(\Omega) = \{u \in H^1(\Omega) \,/\, \nabla\cdot u = 0\}.$$

Finally $\mathrm{V}'(\Omega)$ will denote the dual space of V. We will omit the mention of the space $\Omega$ in the notation, whenever no confusion is possible. We will also use the following notation for the inhomogeneous Sobolev spaces

$$H^s(\Omega) = \{u \in \mathcal{D}'(\Omega) \,/\, (Id - \Delta)^{s/2}u \in L^2(\Omega)\}.$$

We will also use the homogeneous couterpart

$$\dot{H}^s(\Omega) = \{u \in \mathcal{D}'(\Omega) \,/\, (-\Delta)^{s/2} u \in L^2(\Omega)\}.$$

It will appear clearly in the following that the horizontal variables play a special role in this problem. Consequently we shall use the following notation: if $x$ is a point in $\Omega$, then we shall note its cartesian coordinates by $(x_1, x_2, x_3)$, and the horizontal part of $x$ will be denoted $x_h \stackrel{\text{def}}{=} (x_1, x_2) \in \Omega_h$. Similarly we will denote the horizontal part of any vector field $f$ by $f_h$, the horizontal gradient by $\nabla_h \stackrel{\text{def}}{=} (\partial_1, \partial_2)$ and its orthogonal by $\nabla_h^\perp = (\partial_2, -\partial_1)$, and the horizontal divergence and Laplacian respectively by $\text{div}_h f \stackrel{\text{def}}{=} \partial_1 f_1 + \partial_2 f_2 = \nabla_h \cdot f_h$ and $\Delta_h \stackrel{\text{def}}{=} \partial_1^2 + \partial_2^2$.

Finally as usual, $C$ will denote a constant which can change from line to line, and $\nabla p$ will denote the gradient of a function which can also change from line to line.

Now we are ready to state the main theorems of this paper. The first result, rather standard, shows that there are weak solutions to the system (1.1).

**Theorem 1.** *Let $u^0$ be any vector field in* H. *Then for all $\varepsilon > 0$, Equation (1.1) has at least one weak solution $u_\varepsilon \in L^\infty(\mathbf{R}^+, \mathrm{H}) \cap L^2(\mathbf{R}^+, \dot{H}^1)$. Moreover, for all $t > 0$, the following energy estimate holds:*

$$\|u_\varepsilon(t)\|_{L^2}^2 + 2\nu \int_0^t \|\nabla u_\varepsilon(t')\|_{L^2}^2 dt' \leq \|u^0\|_{L^2}^2. \tag{1.2}$$

The aim of the paper is to describe the limit of $u_\varepsilon$ as $\varepsilon$ goes to zero. We will first concentrate on the constant case.

**Theorem 2.** *Suppose that $B = be_3$ where $b$ is constant and homogeneous.*

*Let $u^0$ be any vector field in* H, *and let $u_\varepsilon$ be any weak solution of (1.1) in the sense of Theorem 1. Then $u_\varepsilon$ converges weakly in $L^2_{loc}(\mathbf{R}^+ \times \Omega)$ to a limit $u$ which if $\Omega_3 = \mathbf{R}$ is zero, and if $\Omega_3 = \mathbf{T}$ is the solution of the two dimensional Navier–Stokes equations*

$$(NS2D) \qquad \partial_t u - \nu \Delta_h u + u_h \cdot \nabla_h u = (-\nabla_h p, 0), \quad div_h u_h = 0,$$

$$u_{|t=0} = \int_{\mathbf{T}} u^0(x_h, x_3)\, dx_3 - \int_{\Omega_h \times \mathbf{T}} (u_h^0(x), 0)\, dx.$$

**Remark 1.2.** *This theorem is by no means a novelty, it is even rather less precise than other such results one can find in the literature* ([2]-[4], [5], [8], [12])*. As we will see in Section 3, the interest of this result lies in its proof, which contrary to the references above, does not depend on the boundary conditions (which can be the whole space or periodic, in each direction).*

Now let us state the new result of this paper, concerning the case when $b$ is not homogeneous. We will suppose that $B = be_3$ where $b = b(x_h)$ is a smooth function, with non degenerate critical points in the following sense: denoting by $\mu(X)$ the Lebesgue measure of any set $X$ we suppose that

$$\lim_{\delta \to 0} \mu\left(\{x \in \Omega_h \,/\, |\nabla b(x)| \leq \delta\}\right) = 0. \tag{1.3}$$

**Theorem 3.** *Suppose that $B = be_3$ where $b = b(x_h)$ is a smooth function, with non degenerate critical points in sense of (1.3).*

*Let $u^0$ be any vector field in H, and let $u_\varepsilon$ be any weak solution of (1.1) in the sense of Theorem 1. Then $u_\varepsilon$ converges weakly in $L^2_{loc}(\mathbf{R}^+ \times \Omega)$ to a limit $u$ which if $\Omega_3 = \mathbf{R}$ is zero, and if $\Omega_3 = \mathbf{T}$ is defined as follows: $u$ belongs to $\cap \mathrm{Ker}(L)$ and its third component $u_3 \in L^\infty(\mathbf{R}^+; L^2) \cap L^2(\mathbf{R}^+; \dot{H}^1)$ satisfies the transport-diffusion equation*

$$\partial_t u_3 - \nu \Delta_h u_3 + u_h \cdot \nabla_h u_3 = 0, \quad \partial_3 u_3 = 0, \quad u_{3|t=0} = \int_{\mathbf{T}} u_3^0(x_h, x_3)\, dx_3 \quad \text{in} \quad \mathbf{R}^+ \times \Omega,$$

*while the horizontal component $u_h \in C(\mathbf{R}^+; \mathrm{V}'(\Omega_h)) \cap L^2_{loc}(\mathbf{R}^+; \mathrm{V}(\Omega_h))$ satisfies the following property: for any vector field $\Phi \in \mathrm{V}(\Omega_h) \cap \mathrm{Ker}(L)$ and for any time $t > 0$,*

$$(u_h(t)|\Phi_h)_{L^2(\Omega_h)} + \nu \int_0^t (\nabla_h u_h(t')|\nabla_h \Phi_h)_{L^2(\Omega_h)}\, dt' = (u_h^0|\Phi_h)_{L^2(\Omega_h)}. \tag{1.4}$$

**Remark 1.3.** *Formally Equation (1.4) can be written as a heat equation on $\mathrm{Ker}(L)$, as writing $\Pi$ the orthogonal projector in $L^2$ onto $\mathrm{Ker}(L)$ the equation formally reads*

$$\partial_t u_h - \nu \Pi \Delta_h u_h = 0.$$

*That result is surprising as all non linear terms have disappeared in the limiting process. This can be understood as some sort of turbulent behaviour, where all scales are mixed due to the variation of $b$. Technically the result is due to the fact that the kernel of $L$ is very small as soon as $b$ is not a constant, which induces a lot of rigidity in the limit equation.*

The structure of the paper is as follows. In the next section, we present the operator $L$ and study its main properties (proof of Theorem 1, study of the kernel of $L$). The following section is devoted to the proof of Theorem 2. Although the result is not new, we present an alternative proof which holds regardless of the domain (with no boundary). This serves as a warm–up to the final section, in which the general variable case is presented, with the proof of Theorem 3.

**Remark 1.4.** *A more physical problem is the case when the direction of $B$ is not fixed, in other words when $B$ is a three component vector, depending on all three variables. Then geometrical problems appear, simply to determine the kernel of $L$ ; this will be dealt with in a forecoming paper.*

## 2. Study of the singular perturbation

### 2.1. **Energy estimate.** In this section we shall prove Theorem 1 stated in the introduction.

*Proof.* The structure of Equation (1.1) governing the rotating fluids is very similar to the one of the usual Navier-Stokes equation, since the singular perturbation is just a linear skew-symmetric operator. Therefore weak solutions “ à la Leray ” can be constructed by the approximation scheme of Friedrichs : approximate solutions are obtained by a standard truncation $J_n$ of high frequencies. In order to obtain uniform bounds on these approximate solutions, we have just to check that the energy inequality is still satisfied. Computing formally the $L^2$ scalar product of (1.1) by $u$ leads to

$$\frac{1}{2}\frac{d}{dt}\|u\|_{L^2}^2 = -\int\Big(\frac{1}{2}(u\cdot\nabla)|u|^2 - \nu u\cdot\Delta u + \frac{1}{\varepsilon}u\cdot u\wedge B + u\cdot\nabla p\Big)dx.$$

Integrating by parts (without boundary) and using the incompressibility constraint, we get

$$\frac{1}{2}\frac{d}{dt}\|u\|_{L^2}^2 = -\nu\|\nabla u\|_{L^2}^2,$$

which holds for any smooth solution of (1.1).

The energy inequality for weak solutions is obtained by taking limits in the approximation scheme. □

In particular, the energy estimate provides uniform bounds in $L^\infty(\mathbf{R}^+,\mathrm{H})\cap L^2(\mathbf{R}^+,\dot H^1)$ on any family $(u_\varepsilon)_{\varepsilon>0}$ of weak solutions of (1.1) provided that the initial data $u^0$ belongs to H.

**Corollary 2.1.** *Let $u^0$ be any vector field in* H. *For all $\varepsilon > 0$, denote by $u_\varepsilon$ a weak solution of (1.1). Then there exists $u \in L^\infty(\mathbf{R}^+,\mathrm{H})\cap L^2(\mathbf{R}^+,\dot H^1)$, such that, up to extraction of a subsequence,*

$$u_\varepsilon \rightharpoonup u \text{ weakly in } L^2_{loc}(\mathbf{R}^+;L^2(\Omega)) \text{ as } \varepsilon\to 0.$$

2.2. **Study of the kernel.** We are interested in describing the asymptotic behaviour of $(u_\varepsilon)$, i.e. in characterizing its limit points. Of course, the equations satisfied by such a limit point $u$ depend strongly on the structure of the singular perturbation

$$L : u\in\mathrm{H}\mapsto P(u\wedge B)\in\mathrm{H} \tag{2.1}$$

where $P$ denotes the Leray projection from $L^2(\Omega)$ onto its subspace H of divergence-free vector fields. In particular, we will prove that $u$ belongs to the kernel $\mathrm{Ker}(L)$ of $L$, which is characterized in the following proposition.

**Proposition 2.2.** *Define the linear operator $L$ by (2.1). If $u\in\mathrm{H}$ belongs to* $\mathrm{Ker}(L)$ *then $u$ is in $L^2(\Omega_h)$ and satisfies the following properties:*

$$\begin{cases} \mathrm{div}_h\, u_h = 0,\\ u_h\cdot\nabla_h b = 0,\\ \displaystyle\int_{\Omega_h} u_h\wedge B\, dx_h = 0.\end{cases}$$

**Remark 2.3.** *In the case when $\Omega_3 = \mathbf{R}$, Proposition 2.2 shows that the kernel of $L$ is reduced to zero. Indeed there are no vector fields other than 0 which are in $L^2(\Omega_h\times\mathbf{R})$ and which do not depend on the vertical variable.*

**Remark 2.4.** *The fact that $\mathrm{div}_h\, u_h = 0$ does not necessarily mean that $u_h$ can be written as $u_h = \nabla_h^\perp\varphi$ for some function $\varphi$ because the horizontal mean of $u_h$ is not preserved by the equation.*

*Proof.* If $u$ belongs to $\mathrm{Ker}(L)$ then we have

$$P(u\wedge B) = 0,$$

so in particular

$$\int_\Omega u_h\wedge B\, dx = 0.$$

Moreover in the sense of distributions,

$$\text{rot}\,(u \wedge B) = 0,$$

which can be rewritten

$$(\nabla\cdot B)u + (B\cdot\nabla)u - (u\cdot\nabla)B - (\nabla\cdot u)B = 0.$$

As $\nabla\cdot B = \nabla\cdot u = 0$ and $B = be_3$, we get

$$b\partial_3 u - (u\cdot\nabla)be_3 = 0. \tag{2.2}$$

In particular, $\partial_3 u_1 = \partial_3 u_2 = 0$ from which we deduce that

$$u_1, u_2 \in L^2(\Omega_h). \tag{2.3}$$

Note that in the case where $\Omega_3 = \mathbf{R}$, the invariance with respect to $x_3$ and the fact that $u$ belongs to $L^2(\Omega)$ imply that $u_1 = u_2 = 0$ (and therefore $u_3 = 0$ by the divergence free condition).

Differentiating the incompressibility constraint with respect to $x_3$ leads then to

$$\partial^2_{33}u_3 = -\partial^2_{13}u_1 - \partial^2_{23}u_2 = 0$$

in the sense of distributions. The function $\partial_3 u_3$ depends only on $x_1$ and $x_2$, and satisfies $\int \partial_3 u_3 dx_3 = 0$. So $\partial_3 u_3 = 0$ and

$$u_3 \in L^2(\Omega_h), \quad \partial_1 u_1 + \partial_2 u_2 = 0. \tag{2.4}$$

Finally we have proved that $\text{div}_h\, u_h = 0$, as well as the fact that by (2.2)

$$u_h \cdot \nabla_h b = 0$$

and

$$\int_{\Omega_h} u_h \wedge B\, dx_h = 0.$$

The proposition is proved. □

Before applying this result to the characterization of the weak limit $u$, let us just specify it in two important cases. If $\nabla b = 0$ almost everywhere, $u \in \mathrm{H}$ belongs to $\mathrm{Ker}(L)$ if and only if

$$u = \nabla_h^\perp \varphi + \alpha e_3,$$

for some $\nabla_h \varphi \in L^2(\Omega_h)$ and $\alpha \in L^2(\Omega_h)$. If $\nabla b \neq 0$ almost everywhere, then the condition arising on $u$ is much more restrictive : if $u \in \mathrm{H}$ belongs to $\mathrm{Ker}(L)$ then it can be written

$$u = \frac{u_h \cdot \nabla^\perp b}{|\nabla^\perp b|^2}\nabla^\perp b + \alpha e_3$$

for some $\alpha \in L^2(\Omega_h)$, with the additionnal condition that

$$\text{div}_h \left(\frac{u_h \cdot \nabla^\perp b}{|\nabla^\perp b|^2}\nabla^\perp b\right) = 0 \quad \text{and} \quad \int b\frac{u_h \cdot \nabla^\perp b}{|\nabla^\perp b|^2}\nabla^\perp b\, dx = 0.$$

From this characterization of $\mathrm{Ker}(L)$, we deduce some constraints on the weak limit $u$.

**Corollary 2.5.** *Let $u^0$ be any vector field in* H. *Denote by $(u_\varepsilon)_{\varepsilon>0}$ a family of weak solutions of (1.1), and by $u$ any of its limit points. Then*

$$u \in L^\infty(\mathbf{R}^+; L^2(\Omega_h)) \cap L^2(\mathbf{R}^+; \dot{H}^1(\Omega_h))$$

*and satisfies the following properties: for almost all $t \in \mathbf{R}^+$,*

$$\left\{ \begin{array}{r} \mathrm{div}_h\, u_h = 0, \\ u_h \cdot \nabla_h b = 0, \\ \displaystyle\int_{\Omega_h} u_h \wedge B\, dx_h = 0. \end{array} \right.$$

**Remark 2.6.** *Due to Remark 2.3, if $\Omega_3 = \mathbf{R}$ then necessarily all weak limit points $u$ are identically zero.*

*Proof.* Let $\chi \in \mathcal{D}(\mathbf{R}^+ \times \Omega)$ be any divergence-free test function. Multiplying (1.1) by $\varepsilon\chi$ and integrating with respect to all variables leads to

$$\iint u_\varepsilon(\varepsilon\partial_t\chi + \varepsilon u_\varepsilon \cdot \nabla\chi + \varepsilon\nu\Delta\chi + \chi \wedge B)dxdt = 0.$$

Because of the bounds coming from the energy estimate, we can take limits in the previous identity as $\varepsilon \to 0$ to get

$$\int u \wedge B \cdot \chi dxdt = 0.$$

This means that there exists some $p$ such that

$$u \wedge B = \nabla p.$$

As $u_\varepsilon$ satisfies the incompressibility relation for all $\varepsilon > 0$,

$$\nabla \cdot u = 0.$$

Then $u(t) \in \mathrm{Ker}(L)$ for almost all $t \in \mathbf{R}^+$, and we conclude by Proposition 2.2. □

## 2.3. Remarks concerning the regularity.

2.3.1. *Comparison with the gyrokinetic approximation.* As mentioned in the introduction, the study of the asymptotics for an inhomogeneous penalization is a natural question in the magnetohydrodynamic framework, when $B$ represents the magnetic field. Such a study has been performed for the gyrokinetic approximation [10], that is for a kinetic model perturbed by a singular magnetic constraint :

- in the case where $B = b(x_h)e_3$, the singular limit is exactly the same as in the constant case : the fast rotation has an averaging effect in the plane orthogonal to the magnetic lines;
- in the case where $B$ has constant modulus but variable direction, extra drift terms are obtained due to the curvature of the field.

A simplified version of this result can be written as follows.

**Theorem 4.** [10] *Let $f^0$ be a function of $L^\infty(\Omega\times\mathbf{R}^3)$, and $(f_\varepsilon)$ be a family of solutions of*

$$\partial_t f_\varepsilon + v\cdot\nabla_x f_\varepsilon + \frac{1}{\varepsilon} v\wedge B\cdot\nabla_v f_\varepsilon = 0, \quad t\in\mathbf{R}_+^*,\ (x,v)\in\Omega\times\mathbf{R}^3,$$

*with initial condition*

$$f_{\varepsilon|t=0} = f^0.$$

*Then the family $(f_\varepsilon)$ is relatively compact in $L^\infty(\mathbf{R}_+\times\Omega\times\mathbf{R}^3)$, as well as the family $(g_\varepsilon)$ defined by*

$$g_\varepsilon(t,x,w) = f_\varepsilon(t,x,R(x,-\frac{t}{\varepsilon})w)$$

*where $R(x,\theta)$ denotes the rotation of angle $\theta$ around the oriented axis of direction $B(x)$. Moreover,*

- *if $B = be_3$ with $b\in C^1(\Omega_h,\mathbf{R}_+^*)$, any limit point of $(g_\varepsilon)$ satisfies*

$$\partial_t g + v_3\partial_{x_3} g = 0;$$

- *if $B\in C^1(\Omega)$ with $\nabla_x\cdot B = 0$ and $|B|\equiv 1$, any of its limit points satisfies*

$$\partial_t g + (w\cdot B)B\cdot\nabla_x g = \frac{1}{2} w\wedge\big(3(w.B)(B\wedge\nabla_B B) - B\wedge\nabla_w B - \nabla_{B\wedge w}B\big)\cdot\nabla_w g$$

*with the notation $\nabla_V\Phi \overset{\text{def}}{=} V\cdot\nabla\Phi$.*

The result obtained in this paper is very different because of the incompressibility constraint, which imposes a lot of rigidity to the system. In particular, the kernel of the penalization is much smaller and the limiting system has less degrees of freedom.

2.3.2. *A remark in the inviscid case.* The weak compactness method used here allows to study the singular limit without regularity with respect to the time variable. However it uses crucially the strong compactness in $x$ given by the energy estimate (1.2). Implicitly we have actually considered the penalization

$$L_\varepsilon : u\in \mathrm{V}\mapsto P(u\wedge B) - \varepsilon\Delta u\in H^{-1}(\Omega).$$

That rules out the possibility to manage an analogous study for inviscid rotating fluids, the first obstacle being to prove the existence of solutions for

$$\partial_t u_\varepsilon + (u_\varepsilon\cdot\nabla)u_\varepsilon + \frac{1}{\varepsilon}u_\varepsilon\wedge B + \nabla p = 0, \quad \nabla\cdot u_\varepsilon = 0$$

on a uniform time interval $[0,T]$. Indeed it is not at all clear that the operator $\exp(tL/\varepsilon)$ is bounded on $H^s(\Omega)$ for $s>\frac{3}{2}$ (which is the regularity required to apply usual techniques for hyperbolic systems). Actually even a bound in $H^s(\Omega)$ for $s\geq\frac{1}{2}$ seems ruled out, although we will not pursue this issue here.

## 3. The case of a constant vector field $B$ : the 2D Navier-Stokes limit

In the previous section, we have obtained a constraint equation on the limiting velocity field, which expresses that $u$ belongs to the kernel of the singular perturbation $L$. This comes from the fact that the projection of $u_\varepsilon$ onto $(\operatorname{Ker} L)^\perp$ has fast oscillations with respect to time, and consequently converges weakly to 0. In the case where $\Omega_3 = \mathbf{R}$, this characterizes completely the weak limit $u = 0$.

Then it remains to get an evolution equation for $u$ in the case where $\Omega_3 = \mathbf{T}$. In the case of a constant vector field $B$, the action of the Coriolis operator on vector fields depending only on the horizontal variables is identically zero. It follows that all such, mean free vector fields are in the kernel of the Coriolis operator, hence oscillations are essentially due to vertical modes (and depend on $b$ of course). The first step of the proof of the convergence result consists in proving the compactness of the vertical average of $u_\varepsilon$. The second step then consists in proving a compensated-compactness type argument to show that there are no constructive interferences of $x_3$-dependent vector fields. This involves a precise description of the waves (see Lemma 3.3 below), which allows to derive formally the following limit (see Proposition 3.4 for a precise statement):

$$P \int_{\mathbf{T}} \operatorname{div}\,(u_\varepsilon \otimes u_\varepsilon)\, dx_3 \to \operatorname{div}_h(u \otimes u_h),$$

where $\partial_3 u = 0$. The proof of that result requires a preliminary smoothing in space, and is written in Sections 3.2 and 3.3.

The convergence result established here is not so precise as the ones given in [2]-[4], [8] or [12], since it does not describe the oscillating component and consequently does not provide any strong convergence. Nevertheless the proof is interesting in the sense that it does not require any knowledge on the spectral structure of $L$, which allows to consider more general cases in the sequel.

### 3.1. **Compactness of vertical averages.**

Let us start by proving the following proposition, which shows that the defect of compactness of the sequence of solutions $u_\varepsilon$ is due to functions depending on the vertical variable. In the following we normalize $\mathbf{T}$ so that $\displaystyle\int_{\mathbf{T}} dx_3 = 1$.

**Proposition 3.1.** *Let $u^0$ be any vector field in* H. *For all $\varepsilon > 0$, denote by $u_\varepsilon$ a weak solution of (1.1), and define*

$$\overline{u}_\varepsilon(x_h) \stackrel{\text{def}}{=} \int_{\mathbf{T}} u_\varepsilon(x)\, dx_3 \quad \text{and} \quad \overline{\overline{u}}_\varepsilon \stackrel{\text{def}}{=} \frac{1}{|\Omega_h|} \int_{\Omega_h \times \mathbf{T}} (\overline{u}_{\varepsilon,h}(x), 0)\, dx.$$

*Then the sequence $(\overline{u}_\varepsilon - \overline{\overline{u}}_\varepsilon)_{\varepsilon>0}$ is strongly compact in $L^2([0,T] \times \Omega_h)$, for all times $T$.*

**Remark 3.2.** *Note that in the case when $\Omega_h = \mathbf{R}^2$ then of course $\overline{\overline{u}}_\varepsilon = 0$.*

*Proof.* Let us take the vertical average of (1.1). In the case where $b$ is constant we have seen in the previous section that horizontal mean free, $x_3$-independent vector fields are in the kernel of $L$. We infer that

$$\int_{\mathbf{T}} P(u_\varepsilon \wedge B)\, dx_3 - \int_{\Omega_h \times \mathbf{T}} P(u_\varepsilon \wedge B)\, dx = P\left((\overline{u}_\varepsilon - \overline{\overline{u}}_\varepsilon) \wedge B\right) = 0.$$

It follows that

$$\partial_t(\overline{u}_\varepsilon - \overline{\overline{u}}_\varepsilon) - \nu\Delta_h\overline{u}_\varepsilon + P\int_{\mathbf{T}} u_\varepsilon(x)\cdot\nabla u_\varepsilon(x)\,dx_3 = 0. \tag{3.1}$$

First one notices that the energy estimate implies clearly that

$$\overline{u}_\varepsilon \text{ is uniformly bounded in } L^2([0,T],H^1)$$

for all times $T$, which provides regularity with respect to space variables.

The second step consists in getting regularity with respect to time. We claim that $u_\varepsilon\cdot\nabla u_\varepsilon$ is bounded in $L^2(\mathbf{R}^+;H^{-3/2}(\Omega))$. Indeed since $u_\varepsilon$ is divergence-free, we have

$$u_\varepsilon\cdot\nabla u_\varepsilon = \operatorname{div}(u_\varepsilon\otimes u_\varepsilon)$$

and by Sobolev embeddings we can write

$$\begin{aligned}
\|\nabla\cdot(u_\varepsilon\otimes u_\varepsilon)\|_{\dot H^{-3/2}(\Omega)} &\leq \|u_\varepsilon\otimes u_\varepsilon\|_{\dot H^{-1/2}(\Omega)}\\
&\leq C\|u_\varepsilon\otimes u_\varepsilon\|_{L^{3/2}(\Omega)}\\
&\leq C\|u_\varepsilon\|_{L^2(\Omega)}\|u_\varepsilon\|_{L^6(\Omega)}\\
&\leq C\|u_\varepsilon\|_{L^2(\Omega)}\|\nabla u_\varepsilon\|_{L^2(\Omega)}
\end{aligned}$$

which proves the claim.

Since of course $\Delta u_\varepsilon$ is bounded in $L^2(\mathbf{R}^+;\dot H^{-1}(\Omega))\subset L^2(\mathbf{R}^+;H^{-3/2}(\Omega))$, we infer finally that $\partial_t(\overline{u}_\varepsilon - \overline{\overline{u}}_\varepsilon)$ is uniformly bounded in $L^2(\mathbf{R}^+;H^{-3/2}(\Omega_h))$, which provides the expected regularity in $t$.

Aubin's lemma [1] then gives the following interpolation result

$$(\overline{u}_\varepsilon - \overline{\overline{u}}_\varepsilon)_{\varepsilon>0} \text{ is strongly compact in } L^2_{loc}(\mathbf{R}^+;L^2(\Omega_h)),$$

which proves the proposition. □

3.2. **Description of the oscillations.** In the previous section we proved that mean free, $x_3$-independent vector fields are compact. So the oscillations are due to $x_3$-dependent vector fields, and to prove Theorem 2 we need to show that such vector fields do not interfere constructively in the non linear term of the equation.

The proof of that result requires some preparation, which this section is devoted to: we will rewrite the equations in a convenient way for the algebraic computations of the next part, by introducing a regularization of the equations and getting a control of the source terms in some strong norm.

In the following for any vector field $f$ we will write

$$f(x) = \overline{f}(x_h) + \widetilde{f}(x), \quad \text{where} \quad \overline{f}(x_h) \stackrel{\text{def}}{=} \int_{\mathbf{T}} f(x)\,dx_3.$$

It will be useful to notice that $\widetilde{f}$ has a zero vertical average, hence can be written, for some $\widetilde{F}$,

$$\widetilde{f}(x) = \partial_3\widetilde{F}(x) \quad \text{with} \quad \int_{\mathbf{T}}\widetilde{F}(x)\,dx_3 = 0.$$

**Lemma 3.3.** *Let $u^0$ be any vector field in* H. *For all $\varepsilon > 0$, denote by $u_\varepsilon$ a weak solution of (1.1) in $L^\infty(\mathbf{R}^+, \mathrm{H}) \cap L^2(\mathbf{R}^+, \dot H^1)$. Then, for all $\varepsilon > 0$, there is a family $(u_\varepsilon^\delta)_{\delta>0}$ of smooth vector fields in $L^2(\mathbf{R}^+, \cap_s H^s(\Omega))$ such that*

$$\lim_{\delta\to 0} u_\varepsilon^\delta = u_\varepsilon \ \ \textit{in} \quad L^2_{loc}(\mathbf{R}^+, L^p(\Omega)) \textit{ for all } p \in [2,6[, \quad \textit{uniformly in } \varepsilon,$$

*and such that the functions*

$$\omega_\varepsilon^\delta \stackrel{\text{def}}{=} \partial_1 u_{\varepsilon,2}^\delta - \partial_2 u_{\varepsilon,1}^\delta \quad \text{and} \quad \partial_3 \widetilde\Omega_{\varepsilon,h}^\delta \stackrel{\text{def}}{=} \left(\mathrm{rot}\, \widetilde{u_\varepsilon}^\delta\right)_h, \ \textit{with} \quad \int_{\mathbf{T}} \widetilde\Omega_{\varepsilon,h}^\delta(x)\, dx_3 = 0$$

*satisfy the following equations (in the distribution sense):*

$$\begin{aligned} \varepsilon\partial_t \overline{\omega}_\varepsilon^\delta &= \varepsilon \overline{r}_\varepsilon^\delta, \\ \varepsilon\partial_t \widetilde{\omega}_\varepsilon^\delta - b\mathrm{div}_h \widetilde{u}_{\varepsilon,h}^\delta &= \varepsilon \widetilde{r}_\varepsilon^\delta, \\ \text{and} \quad \varepsilon\partial_t \widetilde\Omega_{\varepsilon,h}^\delta + b\widetilde{u}_{\varepsilon,h}^\delta &= \varepsilon \widetilde{R}_{\varepsilon,h}^\delta \end{aligned}$$

*where for all $\delta > 0$, the function $r_\varepsilon^\delta = \overline{r}_\varepsilon^\delta + \widetilde{r}_\varepsilon^\delta$ and the vector field $\widetilde{R}_{\varepsilon,h}^\delta$ are uniformly bounded in $\varepsilon$ in the space $L^2(\mathbf{R}^+, L^2(\Omega))$.*

*Proof.* The first step of the proof consists in taking the rotational of the equation and in computing the source terms, and the second step consists in the regularization of the equation obtained.

Taking the rotational of the equation is of course an easy matter. Let us define

$$\omega_\varepsilon \stackrel{\text{def}}{=} \partial_1 u_{\varepsilon,2} - \partial_2 u_{\varepsilon,1} \quad \text{and} \quad \partial_3 \widetilde\Omega_{\varepsilon,h} \stackrel{\text{def}}{=} (\mathrm{rot}\, \widetilde{u}_\varepsilon)_h = \nabla_h^\perp \widetilde{u}_{\varepsilon,3} - \partial_3 \widetilde{u}_{\varepsilon,h}^\perp,$$

with as usual $\displaystyle\int_{\mathbf{T}} \widetilde\Omega_{\varepsilon,h}(x)\, dx_3 = 0.$

Equation (3.1) derived in the previous section implies that

$$\varepsilon\partial_t \overline{\omega}_\varepsilon = \varepsilon(\partial_1 \overline{F}_{\varepsilon,2} - \partial_2 \overline{F}_{\varepsilon,1})$$

where $F_\varepsilon$ denotes the flux term

$$F_\varepsilon \stackrel{\text{def}}{=} \nu\Delta u_\varepsilon - P\nabla\cdot (u_\varepsilon \otimes u_\varepsilon). \tag{3.2}$$

As in the previous section $(\partial_1 F_{\varepsilon,2} - \partial_2 F_{\varepsilon,1})$ is bounded in $L^2(\mathbf{R}^+, H^{-5/2}(\Omega))$. So we can write

$$\varepsilon\partial_t \overline{\omega}_\varepsilon = \varepsilon \overline{r}_\varepsilon, \quad \text{where } \overline{r}_\varepsilon \text{ is uniformly bounded in } L^2(\mathbf{R}^+, H^{-5/2}(\Omega)). \tag{3.3}$$

Similarly an easy computation joint with the above bounds yields the following equation for $\widetilde{\omega}_\varepsilon$:

$$\varepsilon\partial_t \widetilde{\omega}_\varepsilon - \mathrm{div}_h \widetilde{u}_{\varepsilon,h} b = \varepsilon \widetilde{r}_\varepsilon, \quad \text{where } \widetilde{r}_\varepsilon \text{ is uniformly bounded in } L^2(\mathbf{R}^+, H^{-5/2}(\Omega)). \tag{3.4}$$

For the other components of the vorticity vector, the computations are similar: since

$$\nabla \wedge (u \wedge b) = b\partial_3 u,$$

we find after integration in the vertical variable

$$\varepsilon\partial_t \widetilde\Omega_{\varepsilon,h} + b\widetilde{u}_{\varepsilon,h} = \varepsilon \widetilde{R}_\varepsilon, \quad \text{where } \widetilde{R}_\varepsilon \text{ is uniformly bounded in } L^2(\mathbf{R}^+, H^{-5/2}(\Omega)). \tag{3.5}$$

Now let us proceed with the regularization: let $\kappa \in C_c^\infty(\mathbf{R}^3;\mathbf{R}^+)$ such that $\kappa(x)=0$ if $|x|\geq 1$ and $\displaystyle\int \kappa dx = 1$. We define

$$\kappa_\delta : x \mapsto \frac{1}{\delta^3}\kappa\left(\frac{\cdot}{\delta}\right)$$

as well as

$$\omega_\varepsilon^\delta \stackrel{\rm def}{=} \omega_\varepsilon * \kappa_\delta = \overline{\omega}_\varepsilon^\delta + \widetilde{\omega}_\varepsilon^\delta, \quad \text{and} \quad \widetilde{\Omega}_\varepsilon^\delta \stackrel{\rm def}{=} \widetilde{\Omega}_\varepsilon * \kappa_\delta.$$

We clearly have $\omega_\varepsilon^\delta = \partial_1 u_{\varepsilon,2}^\delta - \partial_2 u_{\varepsilon,1}^\delta$ and $\partial_3 \widetilde{\Omega}_{\varepsilon,h}^\delta = \left(\operatorname{rot} \widetilde{u_\varepsilon}^\delta\right)_h$, where

$$u_\varepsilon^\delta = u_\varepsilon * \kappa_\delta.$$

Let us prove the strong convergence of $u_\varepsilon^\delta$ towards $u_\varepsilon$ in $L^2_{loc}(\mathbf{R}^+, L^p(\Omega))$ for any $p\in[2,6[$ as $\delta$ goes to zero, uniformly in $\varepsilon$: by the energy estimate, for all $T>0$, the sequence $u_\varepsilon$ is uniformly bounded in $L^2([0,T],H^1(\Omega))$. It follows that

$$\|u_\varepsilon^\delta(t) - u_\varepsilon(t)\|_{L^2(\Omega)} \leq \sup_{|h|\leq\delta} \|\tau_h u_\varepsilon(t) - u_\varepsilon(t)\|_{L^2(\Omega)},$$

where $\tau_h$ denotes the space-translation operator $\tau_h u_\varepsilon(t,x) = u_\varepsilon(t,x+h)$. In particular we can write that

$$\forall \varepsilon>0, \quad \|u_\varepsilon^\delta - u_\varepsilon\|_{L^2([0,T]\times\Omega)} \leq \delta \|\nabla u_\varepsilon\|_{L^2([0,T]\times\Omega)}$$

and the result follows for $p=2$. The result for $p\in[2,6[$ simply follows from the fact that $u_\varepsilon^\delta$ and $u_\varepsilon$ are both uniformly bounded in $\varepsilon$ and in $\delta$, in the space $L^2([0,T],H^1(\Omega))$ which is continuously embedded in $L^2([0,T],L^6(\Omega))$. So for all $p\in[2,6[$, we can write

$$\begin{aligned}\forall\varepsilon>0, \quad \|u_\varepsilon^\delta - u_\varepsilon\|_{L^2([0,T],L^p(\Omega))} &\leq C(p)\|u_\varepsilon^\delta - u_\varepsilon\|_{L^2([0,T]\times\Omega)}^{\frac{3}{p}-\frac12}\|u_\varepsilon^\delta - u_\varepsilon\|_{L^2([0,T],H^1(\Omega))}^{\frac32-\frac3p} \\ &\leq C(p)\delta^{\frac3p-\frac12}.\end{aligned}$$

That proves the convergence of $u_\varepsilon^\delta$ towards $u_\varepsilon$ in $L^2_{loc}(\mathbf{R}^+, L^p(\Omega))$ for any $p\in[2,6[$ as $\delta$ goes to zero, uniformly in $\varepsilon$.

Regularizing (3.3), (3.4) and (3.5) leads to

$$\begin{aligned}\varepsilon\partial_t \overline{\omega}_\varepsilon^\delta &= \varepsilon \overline{r}_\varepsilon * \kappa_\delta, \\ \varepsilon\partial_t \widetilde{\omega}_\varepsilon^\delta - b\operatorname{div}_h \widetilde{u}_{\varepsilon,h}^\delta &= \varepsilon\widetilde{r}_\varepsilon * \kappa_\delta, \\ \varepsilon\partial_t \widetilde{\Omega}_{\varepsilon,h}^\delta + b\widetilde{u}_{\varepsilon,h}^\delta &= \varepsilon \widetilde{R}_\varepsilon * \kappa_\delta,\end{aligned}$$

because $b$ is homogeneous. Then we notice that for all $T>0$ and for $\delta$ small enough,

$$\begin{aligned}\|r_\varepsilon^\delta\|_{L^2([0,T],L^2(\Omega))} &= \|r_\varepsilon * \kappa_\delta\|_{L^2([0,T],L^2(\Omega))} \\ &\leq C\|\kappa_\delta\|_{W^{5/2,1}(\mathbf{R}^3)}\|r_\varepsilon\|_{L^2([0,T],H^{-5/2}(\Omega))} \\ &\leq \frac{C}{\delta^{5/2}}\|r_\varepsilon\|_{L^2([0,T],H^{-5/2}(\Omega))}.\end{aligned}$$

And, in the same way,

$$\|R_\varepsilon^\delta\|_{L^2([0,T],L^2(\Omega))} \leq \frac{C}{\delta^{5/2}}\|R_\varepsilon\|_{L^2([0,T],H^{-5/2}(\Omega))}.$$

For any fixed $\delta$, the uniform bounds derived above on $r_\varepsilon$ and $R_\varepsilon$ provide the expected convergences. Lemma 3.3 is proved. □

3.3. **Computation of the coupling term.** Equipped with this preliminary result, we are now ready to study the coupling between the oscillating terms and to prove the following proposition.

**Proposition 3.4.** *Let $u^0$ be any vector field in* H*. For all $\varepsilon > 0$, denote by $u_\varepsilon$ a weak solution of (1.1), and by $(u_\varepsilon^\delta)_{\delta>0}$ the approximate family of Lemma 3.3. Then for any $\varepsilon > 0$ and any $\delta > 0$, the vertical average of the nonlinear term in (1.1) can be decomposed as follows:*

$$\int_{\mathbf{T}} \left(u_\varepsilon^\delta \cdot \nabla u_\varepsilon^\delta\right)_h dx_3 = \overline{\omega}_{\varepsilon,h}^\delta (\overline{u}_{\varepsilon,h}^\delta)^\perp + \nabla_h \int_{\mathbf{T}} \frac{|u_\varepsilon^\delta|^2}{2} dx_3 + \nabla_h \frac{|\overline{u}_{\varepsilon,3}^\delta|^2}{2} + \frac{1}{b}\varepsilon\partial_t \int_{\mathbf{T}} \widetilde{\omega}_\varepsilon^\delta (\widetilde{\Omega}_{\varepsilon,h}^\delta)^\perp dx_3 + \varepsilon\rho_{\varepsilon,h}^\delta,$$

*and*

$$\int_{\mathbf{T}} \left(u_\varepsilon^\delta \cdot \nabla u_\varepsilon^\delta\right)_3 dx_3 = \mathrm{div}_h(\overline{u}_{\varepsilon,3}^\delta \overline{u}_{\varepsilon,h}^\delta) - \frac{1}{2b}\varepsilon\partial_t \int_{\mathbf{T}} (\widetilde{\Omega}_{\varepsilon,h}^\delta \cdot \partial_3 (\widetilde{\Omega}_{\varepsilon,h}^\delta)^\perp)\, dx_3 + \varepsilon\rho_{\varepsilon,3}^\delta,$$

*where the vector field $\rho_\varepsilon^\delta$ satisfies*

$$\forall \delta > 0, \quad \forall T > 0, \quad \sup_{\varepsilon > 0} \|\rho_\varepsilon^\delta\|_{L^1([0,T],L^{6/5}(\Omega))} < +\infty.$$

*Proof.* Since $u_\varepsilon^\delta$ is divergence free, we have

$$u_\varepsilon^\delta \cdot \nabla u_\varepsilon^\delta = \nabla \cdot (u_\varepsilon^\delta \otimes u_\varepsilon^\delta) = \nabla \frac{|u_\varepsilon^\delta|^2}{2} - u_\varepsilon^\delta \wedge (\nabla \wedge u_\varepsilon^\delta),$$

so we shall now restrict our attention to the term $u_\varepsilon^\delta \wedge (\nabla \wedge u_\varepsilon^\delta)$.

That term can in turn be separated into three different types of terms :

$$u_\varepsilon^\delta \wedge (\nabla \wedge u_\varepsilon^\delta) = \overline{u}_\varepsilon^\delta \wedge (\nabla \wedge \overline{u}_\varepsilon^\delta) + \Big(\overline{u}_\varepsilon^\delta \wedge (\nabla \wedge \widetilde{u}_\varepsilon^\delta) + \widetilde{u}_\varepsilon^\delta \wedge (\nabla \wedge \overline{u}_\varepsilon^\delta)\Big) + \widetilde{u}_\varepsilon^\delta \wedge (\nabla \wedge \widetilde{u}_\varepsilon^\delta).$$

Obviously the second term in the decomposition is of vanishing vertical mean, and therefore

$$\int_{\mathbf{T}} u_\varepsilon^\delta \wedge (\nabla \wedge u_\varepsilon^\delta)\, dx_3 = \overline{u}_\varepsilon^\delta \wedge (\nabla \wedge \overline{u}_\varepsilon^\delta) + \int_{\mathbf{T}} \widetilde{u}_\varepsilon^\delta \wedge (\nabla \wedge \widetilde{u}_\varepsilon^\delta)\, dx_3. \tag{3.6}$$

Let us concentrate first on the first term in (3.6). A direct computation gives

$$\overline{u}_\varepsilon^\delta \wedge (\nabla \wedge \overline{u}_\varepsilon^\delta) = \frac{1}{2}\nabla|\overline{u}_{\varepsilon,3}^\delta|^2 + \overline{\omega}_{\varepsilon,h}^\delta (\overline{u}_{\varepsilon,h}^\delta)^\perp - \mathrm{div}_h(\overline{u}_{\varepsilon,3}^\delta \overline{u}_{\varepsilon,h}^\delta) e_3.$$

To compute the second term in (3.6), we will use the equations derived in Lemma 3.3. Indeed we have

$$\widetilde{u}_\varepsilon^\delta \wedge (\nabla \wedge \widetilde{u}_\varepsilon^\delta) = \begin{pmatrix} (\widetilde{u}_{\varepsilon,h}^\delta)^\perp \widetilde{\omega}_\varepsilon^\delta - \partial_3(\widetilde{u}_{\varepsilon,3}^\delta (\widetilde{\Omega}_{\varepsilon,h}^\delta)^\perp) + \mathrm{div}_h \widetilde{u}_{\varepsilon,h}^\delta (\widetilde{\Omega}_{\varepsilon,h}^\delta)^\perp \\ -(\widetilde{u}_{\varepsilon,h}^\delta)^\perp \cdot \partial_3 \widetilde{\Omega}_{\varepsilon,h}^\delta \end{pmatrix}. \tag{3.7}$$

Let us study first the horizontal components in (3.7): by Lemma 3.3 we have

$$(\widetilde{u}_{\varepsilon,h}^\delta)^\perp \widetilde{\omega}_\varepsilon^\delta - \mathrm{div}_h \widetilde{u}_{\varepsilon,h}^\delta (\widetilde{\Omega}_{\varepsilon,h}^\delta)^\perp = -\frac{1}{b}\varepsilon\partial_t (\widetilde{\Omega}_{\varepsilon,h}^\delta)^\perp \widetilde{\omega}_\varepsilon^\delta + \varepsilon (R_{\varepsilon,h}^\delta)^\perp \widetilde{\omega}_\varepsilon^\delta - \mathrm{div}_h \widetilde{u}_{\varepsilon,h}^\delta (\widetilde{\Omega}_{\varepsilon,h}^\delta)^\perp.$$

Now on the one hand we can estimate the remainder term in the following way :

$$\|(R_{\varepsilon,h}^\delta)^\perp \widetilde{\omega}_\varepsilon^\delta\|_{L^1([0,T],L^{6/5}(\Omega))} \leq \|(R_{\varepsilon,h}^\delta)^\perp\|_{L^2([0,T],L^2(\Omega))} \|\widetilde{\omega}_\varepsilon^\delta\|_{L^2([0,T],L^3(\Omega))}.$$

By Sobolev embeddings we have $\|\widetilde{\omega}_\varepsilon^\delta\|_{L^2([0,T],L^3(\Omega))} \leq C\|\widetilde{\omega}_\varepsilon^\delta\|_{L^2([0,T],H^{1/2}(\Omega))}$, hence by the regularization kernel and the energy estimate we infer that

$$\|(R_{\varepsilon,h}^\delta)^\perp \widetilde{\omega}_\varepsilon^\delta\|_{L^1([0,T],L^{6/5}(\Omega))} \leq C\delta^{-1/2} \|(R_{\varepsilon,h}^\delta)^\perp\|_{L^2([0,T],L^2(\Omega))} \leq C(\delta)$$

where the constant depends on $\delta$ but is uniform in $\varepsilon$. The term $(R^\delta_{\varepsilon,h})^\perp \widetilde{\omega}^\delta_\varepsilon$ can therefore generically be written $\rho^\delta_{\varepsilon,h}$ as in the statement of the proposition.

On the other hand, still by Lemma 3.3 we have

$$-\mathrm{div}_h \widetilde{u}^\delta_{\varepsilon,h} = \frac{\varepsilon}{b} r^\delta_{\varepsilon,h} - \frac{1}{b}\varepsilon\partial_t \widetilde{\omega}^\delta_\varepsilon.$$

Noticing that exactly as above, the term $\frac{1}{b} r^\delta_{\varepsilon,h}(\widetilde{\Omega}^\delta_{\varepsilon,h})^\perp$ can generically be written $\rho^\delta_{\varepsilon,h}$, we have therefore

$$\int_{\mathbf{T}} \left( \widetilde{u}^{\delta,\perp}_{\varepsilon,h} \widetilde{\omega}^\delta_\varepsilon - \mathrm{div}_h \widetilde{u}^\delta_{\varepsilon,h} \widetilde{\Omega}^{\delta,\perp}_{\varepsilon,h} \right) dx_3 = -\frac{1}{b}\varepsilon\partial_t \int_{\mathbf{T}} \widetilde{\Omega}^{\delta,\perp}_{\varepsilon,h}\, dx_3 \widetilde{\omega}^\delta_\varepsilon + \varepsilon\rho^\delta_{\varepsilon,h} - \frac{1}{b}\varepsilon\partial_t \int_{\mathbf{T}} \widetilde{\omega}^\delta_\varepsilon\, dx_3.$$

Finally we have proved that

$$\int_{\mathbf{T}} \left( \widetilde{u}^{\delta,\perp}_{\varepsilon,h} \widetilde{\omega}^\delta_\varepsilon - \mathrm{div}_h \widetilde{u}^\delta_{\varepsilon,h} (\widetilde{\Omega}^\delta_{\varepsilon,h})^\perp \right) dx_3 = -\frac{1}{b}\varepsilon\partial_t \int_{\mathbf{T}} (\widetilde{\Omega}^\delta_{\varepsilon,h})^\perp \widetilde{\omega}^\delta_\varepsilon\, dx_3 + \varepsilon\rho^\delta_{\varepsilon,h}.$$

Now we are left with the last term in (3.7), which is the third component: we can write, by Lemma 3.3,

$$\widetilde{u}^\delta_{\varepsilon,h} = \varepsilon R^\delta_{\varepsilon,h} - \frac{1}{b}\varepsilon\partial_t \widetilde{\Omega}^\delta_{\varepsilon,h},$$

so

$$(\widetilde{u}^\delta_{\varepsilon,h})^\perp \cdot \partial_3 \widetilde{\Omega}^\delta_{\varepsilon,h} = -\frac{1}{b}\varepsilon\partial_t (\widetilde{\Omega}^\delta_{\varepsilon,h})^\perp \cdot \partial_3 \widetilde{\Omega}^\delta_{\varepsilon,h} + \varepsilon (R^\delta_{\varepsilon,h})^\perp \cdot \partial_3 \widetilde{\Omega}^\delta_{\varepsilon,h}.$$

As before, the term $(R^\delta_{\varepsilon,h})^\perp \cdot \partial_3 \widetilde{\Omega}^\delta_{\varepsilon,h}$ is a $\rho^\delta_{\varepsilon,h}$-type remainder term. Moreover we notice that

$$\varepsilon\partial_t (\widetilde{\Omega}^\delta_{\varepsilon,h})^\perp \cdot \partial_3 \widetilde{\Omega}^\delta_{\varepsilon,h} = -\frac{1}{2}\varepsilon\partial_t \left( \widetilde{\Omega}^\delta_{\varepsilon,h} \cdot (\partial_3 \widetilde{\Omega}^\delta_{\varepsilon,h})^\perp \right) + \frac{1}{2}\partial_3 \left( \widetilde{\Omega}^\delta_{\varepsilon,h} \cdot (\varepsilon\partial_t \widetilde{\Omega}^\delta_{\varepsilon,h})^\perp \right).$$

Putting those computations together yields finally

$$\int_{\mathbf{T}} (\widetilde{u}^\delta_{\varepsilon,h})^\perp \cdot \partial_3 \widetilde{\Omega}^\delta_{\varepsilon,h}\, dx_3 = -\frac{1}{b} \int_{\mathbf{T}} \varepsilon\partial_t (\widetilde{\Omega}^\delta_{\varepsilon,1} \cdot \partial_3 \widetilde{\Omega}^\delta_{\varepsilon,2})\, dx_3 + \varepsilon\rho^\delta_{\varepsilon,h},$$

and the proposition is proved. □

3.4. **Passage to the limit.** Now we can prove Theorem 2. In order to do so we need to take the limit of Equation (3.1). The nonlinear term will be dealt with using the following proposition.

**Proposition 3.5.** *Let $u^0$ be any vector field in* H. *For all $\varepsilon > 0$, denote by $u_\varepsilon$ a weak solution of (1.1). Then for any vector field $\phi \in \mathrm{V} \cap Ker(L)$, we have the following limit in $W^{-1,1}([0,T])$:*

$$\lim_{\varepsilon\to 0} \left( \int_\Omega \nabla \cdot (u_\varepsilon \otimes u_\varepsilon) \cdot \phi(x_h)\, dx - \int_{\Omega_h} \nabla_h \cdot (\overline{u}_\varepsilon \otimes \overline{u}_\varepsilon) \cdot \phi(x_h)\, dx_h \right) = 0.$$

*Proof.* Let us use the same regularization procedure as previously and split the integral in the following way:

$$
\begin{aligned}
&\int_\Omega \nabla\cdot(u_\varepsilon\otimes u_\varepsilon)\cdot\phi(x_h)\,dx-\int_{\Omega_h}\nabla_h\cdot(\overline{u}_\varepsilon\otimes\overline{u}_\varepsilon)\cdot\phi(x_h)\,dx_h\\
&\quad=\int_\Omega \nabla\cdot(u_\varepsilon^\delta\otimes u_\varepsilon^\delta)\cdot\phi(x_h)\,dx-\int_{\Omega_h}\nabla_h\cdot(\overline{u}_\varepsilon^\delta\otimes\overline{u}_\varepsilon^\delta)\cdot\phi(x_h)\,dx_h\\
&\quad+\int_\Omega \nabla\cdot((u_\varepsilon-u_\varepsilon^\delta)\otimes u_\varepsilon)\cdot\phi(x_h)dx-\int_\Omega\nabla_h\cdot\Big((\overline{u}_\varepsilon-\overline{u}_\varepsilon^\delta)\otimes\overline{u}_\varepsilon\Big)\cdot\phi(x_h)\,dx_h\\
&\quad+\int_\Omega \nabla\cdot(u_\varepsilon^\delta\otimes(u_\varepsilon^\delta-u_\varepsilon))\cdot\phi(x_h)\,dx-\int_\Omega\nabla_h\cdot(\overline{u}_\varepsilon^\delta\otimes(\overline{u}_\varepsilon-\overline{u}_\varepsilon^\delta))\cdot\phi(x_h)\,dx_h.
\end{aligned}
\tag{3.8}
$$

Let us start by noticing that the four last terms converge to 0 as $\delta\to 0$ uniformly in $\varepsilon$ : indeed, a Hölder estimate yields

$$
\left\|\int_\Omega \nabla\cdot((u_\varepsilon-u_\varepsilon^\delta)\otimes u_\varepsilon)\cdot\phi dx\right\|_{L^1([0,T])}\le\|\nabla\phi\|_{L^2(\Omega)}\|u_\varepsilon\|_{L^2([0,T],L^6(\Omega))}\|u_\varepsilon^\delta-u_\varepsilon\|_{L^2([0,T],L^3(\Omega))}
$$

and Lemma 3.3, along with the energy bound on $u_\varepsilon$, implies the result.

Now let us compute the difference between the first two terms. We can use Proposition 3.4 to find that

$$
\begin{aligned}
&\int_\Omega \nabla\cdot(u_\varepsilon^\delta\otimes u_\varepsilon^\delta)\cdot\phi(x_h)dx-\int_{\Omega_h}\nabla_h\cdot(\overline{u}_\varepsilon^\delta\otimes\overline{u}_\varepsilon^\delta)\cdot\phi(x_h)\,dx_h\\
&=\int_\Omega\Big(\varepsilon\rho_{\varepsilon,h}^\delta+\varepsilon\partial_t(\widetilde{\omega}_\varepsilon^\delta(\widetilde{\Omega}_\varepsilon^\delta)^\perp)\Big)\cdot\phi_h(x_h)dx+\int_\Omega\left(\varepsilon\rho_{\varepsilon,3}^\delta-\frac{1}{2b}\varepsilon\partial_t(\widetilde{\Omega}_{\varepsilon,h}^\delta\cdot(\partial_3\widetilde{\Omega}_{\varepsilon,h}^\delta)^\perp)\right)\cdot\phi_3(x_h)dx
\end{aligned}
\tag{3.9}
$$

where we have used the fact that $\phi$ is divergence free and does not depend on the third variable.

The terms involving the remainder $\rho_\varepsilon^\delta$ are easily proved to go to zero, simply as

$$
\|\int_\Omega\rho_\varepsilon^\delta\cdot\phi(x_h)dx\|_{L^1([0,T])}\le\varepsilon\|\phi\|_{L^6}\|\rho_\varepsilon^\delta\|_{L^1([0,T],L^{6/5})}\le C(\delta)\varepsilon.
$$

We now just have to consider the two remaining terms, which we can easily prove go to zero in the space $W^{-1,1}([0,T])$. A Hölder estimate in space gives for instance

$$
\varepsilon\left|\int_\Omega\frac{1}{2b}\widetilde{\Omega}_{\varepsilon,h}^\delta\cdot(\partial_3\widetilde{\Omega}_{\varepsilon,h}^\delta)^\perp\cdot\phi(x_h)dx\right|\le C\varepsilon\|\phi\|_{L^6}\|\widetilde{\Omega}_\varepsilon^\delta\|_{L^3}\|\partial_3\widetilde{\Omega}_\varepsilon^\delta\|_{L^2}\le C(\delta)\varepsilon.
$$

The result follows. □

The proof of Theorem 2 follows in an obvious manner from the previous results: we have seen in Section 2.2 that $u_\varepsilon$ converges weakly in $L^2([0,T]\times\Omega)$ towards a vector field $u$ depending only on the horizontal variable. Then in Section 3.1 we proved that $\overline{u}_\varepsilon-\overline{\overline{u}}_\varepsilon$ converges strongly towards $u$ in the space $L^2([0,T]\times\Omega)$. Finding the equation satisfied by the limit is therefore a matter of computing the limit of Equation (3.1). The linear terms converge in the sense of distributions of course, so

$$
\partial_t(\overline{u}_\varepsilon-\overline{\overline{u}}_\varepsilon)\rightharpoonup\partial_t u\quad\text{and}\quad\Delta_h\overline{u}_\varepsilon\rightharpoonup\Delta_h u.
$$

Finally to find the limit of the nonlinear term we use Proposition 3.5 as well as the following weak-strong limit argument: we have

$$\nabla\cdot(\overline{u}_\varepsilon\otimes\overline{u}_\varepsilon)=\nabla\cdot(\overline{\overline{u}}_\varepsilon\otimes(\overline{u}_\varepsilon-\overline{\overline{u}}_\varepsilon))+\nabla\cdot((\overline{u}_\varepsilon-\overline{\overline{u}}_\varepsilon)\otimes\overline{\overline{u}}_\varepsilon)))+\nabla\cdot((\overline{u}_\varepsilon-\overline{\overline{u}}_\varepsilon)\otimes(\overline{u}_\varepsilon-\overline{\overline{u}}_\varepsilon)).$$

The two first terms converge towards zero in $\mathcal{D}'(\Omega)$ since $\overline{u}_\varepsilon-\overline{\overline{u}}_\varepsilon$ is compact and $\overline{\overline{u}}_\varepsilon$ converges weakly to zero, whereas the last term satisfies

$$\nabla\cdot((\overline{u}_\varepsilon-\overline{\overline{u}}_\varepsilon)\otimes(\overline{u}_\varepsilon-\overline{\overline{u}}_\varepsilon))\to\nabla\cdot(u\otimes u)\quad\text{in}\quad\mathcal{D}'(\Omega).$$

That gives the expected result: the limit $u$ satisfies the two dimensional Navier-Stokes equation

$$\partial_t u-\nu\Delta_h u+P\nabla_h\cdot(u\otimes u)=0.$$

Theorem 2 is proved.

## 4. The case of a variable vector field $B$ : a turbulent behaviour

In this section we shall prove Theorem 3 stated in the introduction, concerning the case when the rotation vector $B=b(x_h)e_3$ is inhomogeneous. If $\Omega_3=\mathbf{R}$, then $u=0$ simply because it is in $L^2(\Omega)$ but only depends on the horizontal variables. So from now on we can suppose that $\Omega_3=\mathbf{T}$.

The strategy of proof is quite similar to the constant case : we have first to give a precise description of the different oscillating modes, and then to prove that these oscillations do not occur in the limiting equation.

As in the constant case, vertical modes generate fast oscillations in the system, meaning that the whole part of the velocity field corresponding to Fourier modes with $k_3\neq 0$ converges weakly to zero. The corresponding vertical oscillations depend directly on the order of magnitude of $b$. The main difference comes then from the fact that, in the case of a heterogeneous rotation, the kernel of the penalization is much smaller : restricting our attention to the horizontal modes ($k_3=0$), we see that the Coriolis term penalizes all the fields which are parallel to $\nabla b$, which implies in particular that the vertical average of the horizontal velocity is no longer strongly compact. The corresponding two-dimensional oscillations are then governed by $\nabla b$, and possibly become singular if $\nabla b$ cancels.

In the following we will therefore only be able to prove that the vertical average of the vertical velocity is strongly compact, and the use of that information alone, coupled with some compensated compactness argument, will enable us to establish the equation satisfied by the weak limit of the velocity field, in a similar way to the constant case studied previously. The additional difficulty due to the possible cancellation of $\nabla b$ is dealt with by using a truncation operator around those cancellation points, as well as the non degeneracy assumption (1.3).

### 4.1. Strong compactness of the averaged vertical velocity.

**Lemma 4.1.** *Let $u^0$ be a vector field in $\mathrm{H}(\Omega)$. For all $\varepsilon>0$, denote by $u_\varepsilon$ a weak solution of (1.1) and by* $\overline{u}_\varepsilon\stackrel{\text{def}}{=}\int_{\mathbf{T}}u_\varepsilon dx_3$.

*Then, for all $T>0$, $(\overline{u}_{\varepsilon,3})$ is strongly compact in $L^2([0,T]\times\Omega)$.*

*Proof.* By the energy estimate, $u_\varepsilon$ and consequently $\overline{u}_\varepsilon$ are uniformly bounded in $L^2([0,T],\mathrm{V})$.

The computation is similar to the constant case studied in Section 3.1, only for the fact that one must restrict one's attention to the vertical component only. Integrating with respect to $x_3$ the vertical component of the penalized Navier-Stokes equation leads to

$$\partial_t \overline{u}_{\varepsilon,3} + \int \nabla \cdot (u_\varepsilon u_{\varepsilon,3}) dx_3 - \nu \Delta_h \overline{u}_{\varepsilon,3} = 0,$$

from which we deduce that $\partial_t \overline{u}_{\varepsilon,3}$ is uniformly bounded in $L^2([0,T], H^{-3/2}(\Omega))$.

Aubin's lemma [1] then gives the following interpolation result

$$(\overline{u}_{\varepsilon,3})_{\varepsilon>0} \text{ is strongly compact in } L^2([0,T]\times\Omega),$$

and Lemma 4.1 is proved. □

## 4.2. **Description of the oscillations and regularization.**

**Lemma 4.2.** *Let $u^0$ be a vector field in $\mathrm{H}(\Omega)$. For all $\varepsilon > 0$, denote by $u_\varepsilon$ a weak solution of (1.1), by $\overline{u}_\varepsilon = \int u_\varepsilon dx_3$ and by $\widetilde{u}_\varepsilon = u_\varepsilon - \overline{u}_\varepsilon$.*

*Define as previously*

$$\omega_\varepsilon = \partial_1 u_{\varepsilon,2} - \partial_2 u_{\varepsilon,1}, \quad \overline{\omega}_\varepsilon = \int_{\mathbf{T}} \omega_\varepsilon dx_3, \quad \widetilde{\omega}_\varepsilon = \omega_\varepsilon - \overline{\omega}_\varepsilon,$$
$$\partial_3 \widetilde{\Omega}_{\varepsilon,h} = \nabla_h^\perp \widetilde{u}_{\varepsilon,3} - \partial_3 \widetilde{u}_{\varepsilon,h}^\perp, \quad \int_{\mathbf{T}} \widetilde{\Omega}_{\varepsilon,h} dx_3 = 0.$$

*Then, regularizing by a kernel $\kappa_\delta$, we get the following description of the oscillations*

$$\begin{aligned}
\varepsilon\partial_t \overline{\omega}_\varepsilon^\delta - \overline{u}_{\varepsilon,h}^\delta \cdot \nabla b &= -\varepsilon r_{\varepsilon,\delta} - s_{\varepsilon,\delta}\\
\nabla_h \cdot \overline{u}_{\varepsilon,h}^\delta &= 0\\
\varepsilon\partial_t \widetilde{\Omega}_{\varepsilon,h}^\delta + b\widetilde{u}_{\varepsilon,h}^\delta &= -\varepsilon r_{\varepsilon,\delta} - s_{\varepsilon,\delta}\\
\varepsilon\partial_t \widetilde{\omega}_\varepsilon^\delta - \nabla\cdot(b\widetilde{u}_{\varepsilon,h}^\delta) &= -\varepsilon r_{\varepsilon,\delta} - s_{\varepsilon,\delta}
\end{aligned} \tag{4.1}$$

*denoting generically by $r_{\varepsilon,\delta}$ and $s_{\varepsilon,\delta}$ some quantities satisfying*

$$\forall \delta > 0, \quad \forall T > 0, \quad \sup_\varepsilon \|r_{\varepsilon,\delta}\|_{L^2([0,T]\times\Omega)} < +\infty,$$

$$\forall T > 0, \quad \sup_{\varepsilon,\delta} \delta^{-1} \|s_{\varepsilon,\delta}\|_{L^2([0,T]\times\Omega)} < +\infty.$$

*Proof.* Denote, as in (3.2), by $F_\varepsilon$ the flux term

$$F_\varepsilon = -\nabla\cdot(u_\varepsilon \otimes u_\varepsilon) + \nu\Delta u_\varepsilon.$$

As in Section 3, the energy inequality and standard bilinear estimates yield

$$\|F_\varepsilon\|_{L^2([0,T],H^{-3/2}(\Omega))} \le C_0.$$

Using this notation, (1.1) can be simply rewritten

$$\begin{aligned}\varepsilon\partial_t\overline{u}_\varepsilon + \overline{u}_\varepsilon\wedge b + \nabla_h\overline{p}_\varepsilon &= \varepsilon\overline{F}_\varepsilon,\\ \nabla_h\cdot\overline{u}_{\varepsilon,h} &= 0,\\ \varepsilon\partial_t\widetilde{u}_\varepsilon + \widetilde{u}_\varepsilon\wedge b + \nabla\widetilde{p}_\varepsilon &= \varepsilon\widetilde{F}_\varepsilon,\\ \nabla\cdot\widetilde{u}_\varepsilon &= 0,\end{aligned}$$

splitting the purely 2D modes ($k_3 = 0$) and the vertical Fourier modes ($k_3 \neq 0$).

Using the vorticity formulation for the horizontal component of $\overline{u}_\varepsilon$, we get

$$\begin{aligned}\varepsilon\partial_t\overline{\omega}_\varepsilon + \overline{u}_{\varepsilon,h}\cdot\nabla b &= -\varepsilon\nabla_h^\perp\cdot\overline{F}_{\varepsilon,h},\\ \nabla_h\cdot\overline{u}_{\varepsilon,h} &= 0.\end{aligned}\tag{4.2}$$

Then taking the rotational of the other part of the equation

$$\varepsilon\partial_t\nabla\wedge\widetilde{u}_\varepsilon + \nabla\wedge(\widetilde{u}_\varepsilon\wedge b) = \varepsilon\nabla\wedge\widetilde{F}_\varepsilon$$

and integrating the horizontal component with respect to $x_3$ leads to

$$\begin{aligned}\varepsilon\partial_t\widetilde{\Omega}_{\varepsilon,h} + b\widetilde{u}_{\varepsilon,h} &= \varepsilon(\nabla\wedge\widetilde{G}_\varepsilon)_h,\\ \varepsilon\partial_t\widetilde{\omega}_\varepsilon + \nabla\cdot(\widetilde{u}_{\varepsilon,h}b) &= -\varepsilon\nabla_h^\perp\cdot\widetilde{F}_{\varepsilon,h}\end{aligned}\tag{4.3}$$

where $\widetilde{G}_\varepsilon$ is just defined by $\partial_3\widetilde{G}_\varepsilon = \widetilde{F}_\varepsilon$ and $\displaystyle\int_{\mathbf{T}}\widetilde{G}_\varepsilon dx_3 = 0$, and thus satisfies the same uniform estimates as $\widetilde{F}_\varepsilon$.

The second step of the proof consists then in regularizing the previous wave equations (4.2) and (4.3). We therefore introduce - as in the previous section - a smoothing family $\kappa_\delta$ defined by

$$\kappa_\delta(x) = \delta^{-3}\kappa(\delta^{-1}x)$$

where $\kappa$ is a function of $C_c^\infty(\mathbf{R}^3,\mathbf{R}^+)$ such that $\kappa(x) = 0$ if $|x|\geq 1$ and $\int\kappa dx = 1$.

By convolution, we then obtain

$$\begin{aligned}\varepsilon\partial_t\overline{\omega}_\varepsilon^\delta + \overline{u}_{\varepsilon,h}^\delta\cdot\nabla b &= -\varepsilon\nabla_h^\perp\cdot\overline{F}_{\varepsilon,h}^\delta + \overline{u}_{\varepsilon,h}^\delta\cdot\nabla b - (\overline{u}_{\varepsilon,h}\cdot\nabla b)^\delta,\\ \nabla_h\cdot\overline{u}_{\varepsilon,h}^\delta &= 0.\end{aligned}\tag{4.4}$$

and

$$\begin{aligned}\varepsilon\partial_t\widetilde{\Omega}_{\varepsilon,h}^\delta + b\widetilde{u}_{\varepsilon,h}^\delta &= \varepsilon(\nabla\wedge\widetilde{G}_\varepsilon^\delta)_h + b\widetilde{u}_{\varepsilon,h}^\delta - (b\widetilde{u}_{\varepsilon,h})^\delta,\\ \varepsilon\partial_t\widetilde{\omega}_\varepsilon^\delta - \nabla_h\cdot(\widetilde{u}_{\varepsilon,h}^\delta b) &= -\varepsilon\nabla_h^\perp\cdot\tilde{F}_{\varepsilon,h}^\delta + \nabla_h\cdot(\widetilde{u}_{\varepsilon,h}^\delta b) - \nabla_h\cdot(\widetilde{u}_{\varepsilon,h}b)^\delta\end{aligned}\tag{4.5}$$

It remains only to check that the source terms satisfy the convenient a priori estimates.

From the uniform bound on $F_\varepsilon$ and the definition of $\widetilde{G}_\varepsilon$, we deduce that

$$\nabla_h^\perp\cdot\overline{F}_{\varepsilon,h},\quad (\nabla\wedge\tilde{G}_\varepsilon)_h \text{ and } \nabla_h^\perp\cdot\tilde{F}_{\varepsilon,h} \text{ are uniformly bounded in } L^2([0,T],H^{-5/2}(\Omega)).$$

By a scaling argument,

$$\|\kappa_\delta\|_{W^{5/2,1}(\mathbf{R}^3)} \leq \delta^{-5/2}\|\kappa\|_{W^{5/2,1}(\mathbf{R}^3)},$$

and consequently the terms generically called $r_{\varepsilon,\delta}$ satisfy a uniform bound for any fixed $\delta$ :

$\nabla_h^\perp\cdot\overline{F}_{\varepsilon,h}^\delta$, $\quad(\nabla\wedge\widetilde{G}_\varepsilon^\delta)_h$ and $\nabla_h^\perp\cdot\widetilde{F}_{\varepsilon,h}^\delta$ are uniformly bounded in $L^2([0,T]\times\Omega)$ (of order $\delta^{-5/2}$).

We then have to estimate quantities of the form $u_\varepsilon^\delta\psi-(u_\varepsilon\psi)^\delta$ for smooth functions $\psi$

$$\begin{aligned}|u_\varepsilon^\delta\psi(x)-(u_\varepsilon\psi)^\delta(x)| &= \left|\int \kappa_\delta(y)u_\varepsilon(x-y)(\psi(x)-\psi(x-y))dy\right|\\ &\leq \delta\|\nabla\psi\|_{L^\infty(\Omega)}(\kappa^\delta * |u_\varepsilon|)(x).\end{aligned}$$

In particular,

$$\begin{aligned}\|\overline{u}_{\varepsilon,h}^\delta\cdot\nabla b-(\overline{u}_{\varepsilon,h}\cdot\nabla b)^\delta\|_{L^2([0,T]\times\Omega)} &\leq \delta\|D^2b\|_{L^\infty(\Omega)}\|u_\varepsilon\|_{L^2([0,T]\times\Omega)}\\ \|b\widetilde{u}_{\varepsilon,h}^\delta-(b\widetilde{u}_{\varepsilon,h})^\delta\|_{L^2([0,T]\times\Omega)} &\leq \delta\|Db\|_{L^\infty(\Omega)}\|u_\varepsilon\|_{L^2([0,T]\times\Omega)}\\ \|\nabla_h\cdot(\widetilde{u}_{\varepsilon,h}^\delta b)-\nabla_h\cdot(\widetilde{u}_{\varepsilon,h}b)^\delta\|_{L^2([0,T]\times\Omega)} &\leq \delta(\|Db\|_{L^\infty(\Omega)}+\|D^2b\|_{L^\infty(\Omega)})\|u_\varepsilon\|_{L^2([0,T],H^1(\Omega))}\end{aligned}$$

meaning that the terms generically called $s_{\varepsilon,\delta}$ converge to 0 as $\delta\to 0$ uniformly in $\varepsilon$, according to the bound

$$\forall T>0,\quad \sup_{\varepsilon,\delta}\delta^{-1}\|s_{\varepsilon,\delta}\|_{L^2([0,T]\times\Omega)}<+\infty.$$

Lemma 4.2 is proved. □

### 4.3. **Computation of the coupling term and truncation.**

**Proposition 4.3.** *Let $u^0$ be any vector field in* H. *For all $\varepsilon>0$, denote by $u_\varepsilon$ a weak solution of (1.1).*

*Define the truncation $\chi_\delta$ by*

$$\chi_\delta(x)=\chi(\delta^{-1/4}\nabla b(x))$$

*where $\chi$ is a function of $C_c^\infty(\mathbf{R}^3,\mathbf{R}^+)$ such that $\chi(x)=1$ if $|x|\leq 1$.*

*Then, with the same notations as in Lemma 4.2, the averaged nonlinear term in (1.1) can be rewritten*

$$\begin{aligned}&\int\left(\nabla\cdot(u_\varepsilon^\delta\otimes u_\varepsilon^\delta)-\nabla\frac{|u_\varepsilon^\delta|^2}{2}\right)dx_3\\ =&\nabla_h\cdot(\overline{u}_{\varepsilon,h}^\delta\overline{u}_{\varepsilon,3}^\delta)e_3-\nabla\frac{|\overline{u}_{\varepsilon,3}^\delta|^2}{2}+\varepsilon\rho_{\varepsilon,\delta}+\sigma_{\varepsilon,\delta}\\ &-\frac{\varepsilon}{2}\partial_t|\overline{\omega}_\varepsilon^\delta|^2(1-\chi_\delta)\frac{\nabla^\perp b}{|\nabla b|^2}+(1-\chi_\delta)(\overline{u}_{\varepsilon,h}^\delta\cdot\nabla^\perp b)\frac{\nabla b}{|\nabla b|^2}\\ &+\frac{\varepsilon}{b}\partial_t\int\widetilde{\omega}_\varepsilon^\delta(\widetilde{\Omega}_{\varepsilon,h}^\delta)^\perp dx_3+\frac{\varepsilon}{2b^2}\partial_t\int(\widetilde{\Omega}_{\varepsilon,h}^\delta\cdot\nabla b)^2dx_3\frac{\nabla^\perp b}{|\nabla b|^2}\\ &-\frac{1}{b^2}\int(\widetilde{\Omega}_{\varepsilon,h}^\delta\cdot\nabla^\perp b)(\varepsilon\partial_t\widetilde{\Omega}_{\varepsilon,h}^\delta\cdot\nabla b)dx_3\frac{\nabla b}{|\nabla b|^2}-\frac{\varepsilon}{2b}\partial_t\int(\widetilde{\Omega}_{\varepsilon,h}^\delta\cdot(\partial_3\widetilde{\Omega}_{\varepsilon,h}^\delta)^\perp)dx_3e_3\end{aligned}\tag{4.6}$$

*where $\rho_{\varepsilon,\delta}$ and $\sigma_{\varepsilon,\delta}$ are quantities satisfying the following estimates*

$$\forall\delta>0,\quad\forall T>0,\quad\sup_{\varepsilon\to 0}\|\rho_{\varepsilon,\delta}\|_{L^1([0,T],L^{6/5}\Omega))}<+\infty,$$

$$\text{and}\quad\forall T>0,\quad\lim_{\delta\to 0}\sup_{\varepsilon}\|\sigma_{\varepsilon,\delta}\|_{L^1([0,T],L^{6/5}\Omega))}=0.$$

*Proof.* Proposition 4.3 lies on a proper decomposition of the nonlinear term $\nabla\cdot(u_\varepsilon^\delta\otimes u_\varepsilon^\delta)$ and on the identities derived in Lemma 4.2, in a similar way to Section 3.2 (though the analysis is more complicated due to the variations of $b$).

Let us first remark that

$$\int \nabla\cdot(u_\varepsilon^\delta\otimes u_\varepsilon^\delta)dx_3=\nabla\cdot(\overline{u}_\varepsilon^\delta\otimes\overline{u}_\varepsilon^\delta)+\int\nabla\cdot(\widetilde{u}_\varepsilon^\delta\otimes\widetilde{u}_\varepsilon^\delta)dx_3$$

which allows us to consider separately purely 2D modes and vertical modes.

Because of the identity

$$\nabla\cdot(u\otimes u)=\nabla\frac{|u|^2}{2}-u\wedge(\nabla\wedge u)$$

which holds for any divergence-free vector field $u$, we can further restrict our attention to both quantities $-\overline{u}_\varepsilon^\delta\wedge(\nabla\wedge\overline{u}_\varepsilon^\delta)$ and $\int\overline{u}_{\varepsilon,3}^\delta\wedge(\nabla\wedge\overline{u}_{\varepsilon,3}^\delta)dx_3$.

(i) We start with the study of the purely 2D modes. A simple computation leads to

$$\begin{aligned}-\overline{u}_\varepsilon^\delta\wedge(\nabla\wedge\overline{u}_\varepsilon^\delta)&=-\overline{u}_\varepsilon^\delta\wedge(\nabla_h^\perp\overline{u}_{\varepsilon,3}^\delta+\overline{\omega}_\varepsilon^\delta e_3)\\&=-\overline{\omega}_\varepsilon^\delta(\overline{u}_{\varepsilon,h}^\delta)^\perp-\nabla_h\frac{|\overline{u}_{\varepsilon,3}^\delta|^2}{2}+\nabla_h\cdot(\overline{u}_{\varepsilon,h}^\delta\overline{u}_{\varepsilon,3}^\delta)e_3\end{aligned}\tag{4.7}$$

We can decompose $\overline{u}_{\varepsilon,h}^\delta$ as follows

$$\overline{u}_{\varepsilon,h}^\delta=(\overline{u}_{\varepsilon,h}^\delta\cdot\nabla b)\frac{\nabla b}{|\nabla b|^2}+(\overline{u}_{\varepsilon,h}^\delta\cdot\nabla^\perp b)\frac{\nabla^\perp b}{|\nabla b|^2}$$

as soon as $\nabla b\neq 0$, and we will actually do so, using the truncation $\chi$, only if $|\nabla b|\geq\delta^{1/4}$.

Using the first identity in (4.1), we obtain

$$\overline{u}_{\varepsilon,h}^\delta=(\varepsilon\partial_t\overline{\omega}_\varepsilon^\delta+\varepsilon r_{\varepsilon,\delta}+s_{\varepsilon,\delta})\frac{\nabla b}{|\nabla b|^2}+(\overline{u}_{\varepsilon,h}^\delta\cdot\nabla^\perp b)\frac{\nabla^\perp b}{|\nabla b|^2}$$

and replacing in (4.7) provides finally

$$\begin{aligned}-\overline{u}_\varepsilon^\delta\wedge(\nabla\wedge\overline{u}_\varepsilon^\delta)=&-(1-\chi_\delta)\left(\varepsilon\partial_t\frac{|\overline{\omega}_\varepsilon^\delta|^2}{2}+\overline{\omega}_\varepsilon^\delta(\varepsilon r_{\varepsilon,\delta}+s_{\varepsilon,\delta})\right)\frac{\nabla^\perp b}{|\nabla b|^2}\\&+(1-\chi_\delta)(\overline{u}_{\varepsilon,h}^\delta\cdot\nabla^\perp b)\frac{\nabla b}{|\nabla b|^2}\\&-\chi_\delta\overline{\omega}_\varepsilon^\delta(\overline{u}_{\varepsilon,h}^\delta)^\perp-\nabla_h\frac{|\overline{u}_{\varepsilon,3}^\delta|^2}{2}+\nabla_h\cdot(\overline{u}_{\varepsilon,h}^\delta\overline{u}_{\varepsilon,3}^\delta)e_3.\end{aligned}\tag{4.8}$$

That concludes the first step of the proof since

$$\begin{aligned}\left\|\varepsilon\overline{\omega}_\varepsilon^\delta r_{\varepsilon,\delta}\frac{(1-\chi_\varepsilon)}{|\nabla b|}\right\|_{L^1([0,T],L^{6/5}(\Omega))}&\leq\|\overline{\omega}_\varepsilon^\delta\|_{L^2([0,T],L^3(\Omega))}\|\varepsilon r_{\varepsilon,\delta}\|_{L^2([0,T]\times\Omega)}\left\|\frac{(1-\chi_\delta)}{|\nabla b|}\right\|_{L^\infty(\Omega)}\\&\leq C\varepsilon\delta^{-13/4},\\\left\|\overline{\omega}_\varepsilon^\delta s_{\varepsilon,\delta}\frac{(1-\chi_\delta)}{|\nabla b|}\right\|_{L^1([0,T],L^{6/5}(\Omega))}&\leq\|\overline{\omega}_\varepsilon^\delta\|_{L^2([0,T],L^3(\Omega))}\|s_{\varepsilon,\delta}\|_{L^2([0,T]\times\Omega)}\left\|\frac{(1-\chi_\delta)}{|\nabla b|}\right\|_{L^\infty(\Omega)}\\&\leq C\delta^{1/4},\end{aligned}$$

and

$$\left\|\chi_\delta\overline{\omega}^\delta_\varepsilon\overline{u}^\delta_{\varepsilon,h}\right\|_{L^1([0,T],L^{6/5}(\Omega))} \leq \|\chi_\delta\|_{L^6(\Omega)}\|\overline{\omega}^\delta_\varepsilon\|_{L^2([0,T]\times\Omega)}\|\overline{u}^\delta_{\varepsilon,h}\|_{L^2([0,T],L^6(\Omega))}$$
$$\leq C\left(\mu\{x\in\Omega_h\ /\ |\nabla b(x)|\leq\delta^{\frac14}\}\right)^{\frac16},$$

which goes to zero with $\delta$ according to Assumption (1.3).

(ii) We have now to deal with the vertical modes. A simple computation leads to

$$\begin{aligned}-\widetilde{u}^\delta_\varepsilon\wedge(\nabla\wedge\widetilde{u}^\delta_\varepsilon) &= -\widetilde{u}^\delta_\varepsilon\wedge(\partial_3\widetilde{\Omega}^\delta_{\varepsilon,h}+\widetilde{\omega}^\delta_\varepsilon e_3)\\ &= -\widetilde{\omega}^\delta_\varepsilon(\widetilde{u}^\delta_{\varepsilon,h})^\perp+\widetilde{u}^\delta_{\varepsilon,3}\partial_3(\widetilde{\Omega}^\delta_{\varepsilon,h})^\perp+(\widetilde{u}^\delta_{\varepsilon,h}\cdot(\partial_3\widetilde{\Omega}^\delta_{\varepsilon,h})^\perp)e_3\end{aligned}$$

so that

(4.9)
$$-\int\widetilde{u}^\delta_\varepsilon\wedge(\nabla\wedge\widetilde{u}^\delta_\varepsilon)dx_3=\int\left(-\widetilde{\omega}^\delta_\varepsilon(\widetilde{u}^\delta_{\varepsilon,h})^\perp+(\widetilde{\Omega}^\delta_{\varepsilon,h})^\perp(\nabla_h\cdot\widetilde{u}^\delta_{\varepsilon,h})\right)dx_3+\int(\widetilde{u}^\delta_{\varepsilon,h}\cdot(\partial_3\widetilde{\Omega}^\delta_{\varepsilon,h})^\perp)dx_3e_3.$$

In order to determine the horizontal component, we then use the last two identities in (4.1)

$$\begin{aligned}-\int(\widetilde{u}^\delta_\varepsilon\wedge(\nabla\wedge\widetilde{u}^\delta_\varepsilon))_hdx_3 =&\int\widetilde{\omega}^\delta_\varepsilon\frac1b(\varepsilon\partial_t\widetilde{\Omega}^\delta_{\varepsilon,h}+\varepsilon r_{\varepsilon,\delta}+s_{\varepsilon,\delta})^\perp dx_3\\ &+\int(\widetilde{\Omega}^\delta_{\varepsilon,h})^\perp\frac1b(\varepsilon\partial_t\widetilde{\omega}^\delta_\varepsilon-\widetilde{u}^\delta_{\varepsilon,h}\cdot\nabla b+\varepsilon r_{\varepsilon,\delta}+s_{\varepsilon,\delta})dx_3\\ =&\frac{\varepsilon}{b}\partial_t\int\widetilde{\omega}^\delta_\varepsilon(\widetilde{\Omega}^\delta_{\varepsilon,h})^\perp dx_3\\ &+\int(\widetilde{\Omega}^\delta_{\varepsilon,h})^\perp\frac{1}{b^2}(\varepsilon\partial_t\widetilde{\Omega}^\delta_{\varepsilon,h}+\varepsilon r_{\varepsilon,\delta}+s_{\varepsilon,\delta})\cdot\nabla b dx_3\\ &+\int\widetilde{\omega}^\delta_\varepsilon\frac1b(\varepsilon r_{\varepsilon,\delta}+s_{\varepsilon,\delta})^\perp dx_3+\int(\widetilde{\Omega}^\delta_{\varepsilon,h})^\perp\frac1b(\varepsilon r_{\varepsilon,\delta}+s_{\varepsilon,\delta})dx_3\end{aligned}$$

We can decompose $\widetilde{\Omega}^\delta_{\varepsilon,h}$ as follows

$$\widetilde{\Omega}^\delta_{\varepsilon,h}=(\widetilde{\Omega}^\delta_{\varepsilon,h}\cdot\nabla b)\frac{\nabla b}{|\nabla b|^2}+(\widetilde{\Omega}^\delta_{\varepsilon,h}\cdot\nabla^\perp b)\frac{\nabla^\perp b}{|\nabla b|^2}$$

as soon as $\nabla b\neq 0$, that is almost everywhere by assumption. Finally we get

$$\begin{aligned}-\int(\widetilde{u}^\delta_\varepsilon\wedge(\nabla\wedge\widetilde{u}^\delta_\varepsilon))_hdx_3 =&\frac{\varepsilon}{b}\partial_t\int\widetilde{\omega}^\delta_\varepsilon(\widetilde{\Omega}^\delta_{\varepsilon,h})^\perp dx_3+\frac{\varepsilon}{2b^2}\partial_t\int(\widetilde{\Omega}^\delta_{\varepsilon,h}\cdot\nabla b)^2dx_3\frac{\nabla^\perp b}{|\nabla b|^2}\\ &-\frac{1}{b^2}\int(\widetilde{\Omega}^\delta_{\varepsilon,h}\cdot\nabla^\perp b)(\varepsilon\partial_t\widetilde{\Omega}^\delta_{\varepsilon,h}\cdot\nabla b)dx_3\frac{\nabla b}{|\nabla b|^2}\\ &+\int(\widetilde{\Omega}^\delta_{\varepsilon,h})^\perp\frac{1}{b^2}(\varepsilon r_{\varepsilon,\delta}+s_{\varepsilon,\delta})\cdot\nabla b dx_3\\ &+\int\widetilde{\omega}^\delta_\varepsilon\frac1b(\varepsilon r_{\varepsilon,\delta}+s_{\varepsilon,\delta})^\perp dx_3+\int(\widetilde{\Omega}^\delta_{\varepsilon,h})^\perp\frac1b(\varepsilon r_{\varepsilon,\delta}+s_{\varepsilon,\delta})dx_3\end{aligned}\quad(4.10)$$

which is the expected formula since

$$\begin{aligned}
&\left\|\varepsilon r_{\varepsilon,\delta}\widetilde\Omega^\delta_{\varepsilon,h}\frac{|\nabla b|}{b^2}\right\|_{L^1([0,T],L^{6/5}(\Omega))}+\left\|\varepsilon r_{\varepsilon,\delta}\widetilde\omega^\delta_\varepsilon\frac{1}{b}\right\|_{L^1([0,T],L^{6/5}(\Omega))}+\left\|\varepsilon r_{\varepsilon,\delta}\widetilde\Omega^\delta_{\varepsilon,h}\frac{1}{b}\right\|_{L^1([0,T],L^{6/5}(\Omega))}\\
&\le\|\varepsilon r_{\varepsilon,\delta}\|_{L^2([0,T]\times\Omega)}\left(\left\|\widetilde\Omega^\delta_{\varepsilon,h}\right\|_{L^2([0,T],L^3(\Omega))}\left(\left\|\frac{\nabla b}{b^2}\right\|_{L^\infty(\Omega)}+\left\|\frac{1}{b}\right\|_{L^\infty(\Omega)}\right)+\left\|\widetilde\omega^\delta_\varepsilon\right\|_{L^2([0,T],L^3(\Omega))}\left\|\frac1b\right\|_{L^\infty(\Omega)}\right)\\
&\le C\varepsilon\delta^{-3},
\end{aligned}$$

and

$$\begin{aligned}
&\left\|s_{\varepsilon,\delta}\widetilde\Omega^\delta_{\varepsilon,h}\frac{|\nabla b|}{b^2}\right\|_{L^1([0,T],L^{6/5}(\Omega))}+\left\|s_{\varepsilon,\delta}\widetilde\omega^\delta_\varepsilon\frac{1}{b}\right\|_{L^1([0,T],L^{6/5}(\Omega))}+\left\|s_{\varepsilon,\delta}\widetilde\Omega^\delta_{\varepsilon,h}\frac{1}{b}\right\|_{L^1([0,T],L^{6/5}(\Omega))}\\
&\le\|s_{\varepsilon,\delta}\|_{L^2([0,T]\times\Omega)}\left(\left\|\widetilde\Omega^\delta_{\varepsilon,h}\right\|_{L^2([0,T],L^3(\Omega))}\left(\left\|\frac{\nabla b}{b^2}\right\|_{L^\infty(\Omega)}+\left\|\frac{1}{b}\right\|_{L^\infty(\Omega)}\right)+\left\|\widetilde\omega^\delta_\varepsilon\right\|_{L^2([0,T],L^3(\Omega))}\left\|\frac1b\right\|_{L^\infty(\Omega)}\right)\\
&\le C\delta^{1/2}.
\end{aligned}$$

In order to determine the vertical component, we use the third identity in (4.1) and an integration by parts with respect to $x_3$

$$\begin{aligned}
\int\widetilde u^\delta_{\varepsilon,h}\cdot(\partial_3\widetilde\Omega^\delta_{\varepsilon,h})^\perp dx_3&=-\int\frac1b(\varepsilon\partial_t\widetilde\Omega^\delta_{\varepsilon,h}+\varepsilon r_{\varepsilon,\delta}+s_{\varepsilon,\delta})\cdot(\partial_3\widetilde\Omega^\delta_{\varepsilon,h})^\perp dx_3\\
&=-\int\frac{1}{2b}((\varepsilon\partial_t\widetilde\Omega^\delta_{\varepsilon,h})\cdot(\partial_3\widetilde\Omega^\delta_{\varepsilon,h})^\perp-(\varepsilon\partial_t\partial_3\widetilde\Omega^\delta_{\varepsilon,h})\cdot(\widetilde\Omega^\delta_{\varepsilon,h})^\perp)dx_3\\
&\quad-\int\frac1b(\varepsilon r_{\varepsilon,\delta}+s_{\varepsilon,\delta})\cdot(\partial_3\widetilde\Omega^\delta_{\varepsilon,h})^\perp dx_3,
\end{aligned}$$

from which we deduce

$$\int\widetilde u^\delta_{\varepsilon,h}\cdot(\partial_3\widetilde\Omega^\delta_{\varepsilon,h})^\perp dx_3=-\frac{\varepsilon}{2b}\partial_t\int(\widetilde\Omega^\delta_{\varepsilon,h}\cdot(\partial_3\widetilde\Omega^\delta_{\varepsilon,h})^\perp)dx_3-\int\frac1b(\varepsilon r_{\varepsilon,\delta}+s_{\varepsilon,\delta})\cdot(\partial_3\widetilde\Omega^\delta_{\varepsilon,h})^\perp dx_3,\tag{4.11}$$

which is the expected formula since

$$\left\|\varepsilon r_{\varepsilon,\delta}\frac1b\partial_3\widetilde\Omega^\delta_{\varepsilon,h}\right\|_{L^1([0,T],L^{6/5}(\Omega))}\le\|\varepsilon r_{\varepsilon,\delta}\|_{L^2([0,T]\times\Omega)}\left\|\partial_3\widetilde\Omega^\delta_{\varepsilon,h}\right\|_{L^2([0,T],L^3(\Omega))}\left\|\frac1b\right\|_{L^\infty(\Omega)}\le C\varepsilon\delta^{-3},$$

and

$$\left\|s_{\varepsilon,\delta}\frac1b\partial_3\widetilde\Omega^\delta_{\varepsilon,h}\right\|_{L^1([0,T],L^{6/5}(\Omega))}\le\|s_{\varepsilon,\delta}\|_{L^2([0,T]\times\Omega)}\left\|\partial_3\widetilde\Omega^\delta_{\varepsilon,h}\right\|_{L^2([0,T],L^3(\Omega))}\left\|\frac1b\right\|_{L^\infty(\Omega)}\le C\delta^{1/2}.$$

Combining (4.8), (4.10) and (4.11) gives finally the proper decomposition of the averaged nonlinear term. Proposition 4.3 is proved. □

4.4. **Passage to the limit.**

**Proposition 4.4.** *Let $u^0$ be any vector field in $\mathrm{H}(\Omega)$. For all $\varepsilon > 0$, denote by $u_\varepsilon$ a weak solution of (1.1) and by $\overline{u}_\varepsilon = \int u_\varepsilon dx_3$.*

*Then, for all $\phi \in H^1(\Omega) \cap \mathrm{Ker}(L)$, we have the following limit in $W^{-1,1}([0,T])$:*

$$\int_\Omega \nabla \cdot (u_\varepsilon \otimes u_\varepsilon) \cdot \phi dx - \int_\Omega \nabla_h \cdot (\overline{u}_{\varepsilon,h}\overline{u}_{\varepsilon,3})\phi_3 dx \to 0 \text{ as } \varepsilon \to 0. \tag{4.12}$$

*Proof.* We first introduce the same regularization as in the previous paragraphs, and split the integral as follows

$$\begin{aligned}
&\int_\Omega \nabla \cdot (u_\varepsilon \otimes u_\varepsilon) \cdot \phi dx - \int_\Omega \nabla_h \cdot (\overline{u}_{\varepsilon,h}\overline{u}_{\varepsilon,3})\phi_3 dx \\
&\quad = \int_\Omega \nabla \cdot (u_\varepsilon^\delta \otimes u_\varepsilon^\delta) \cdot \phi dx - \int_\Omega \nabla_h \cdot (\overline{u}^\delta_{\varepsilon,h}\overline{u}^\delta_{\varepsilon,3})\phi_3 dx \\
&\quad + \int_\Omega \nabla \cdot ((u_\varepsilon - u_\varepsilon^\delta) \otimes u_\varepsilon) \cdot \phi dx - \int_\Omega \nabla_h \cdot ((\overline{u}_{\varepsilon,h} - \overline{u}^\delta_{\varepsilon,h})\overline{u}_{\varepsilon,3})\phi_3 dx \\
&\quad + \int_\Omega \nabla \cdot (u_\varepsilon^\delta \otimes (u_\varepsilon^\delta - u_\varepsilon)) \cdot \phi dx - \int_\Omega \nabla_h \cdot (\overline{u}^\delta_{\varepsilon,h}(\overline{u}_{\varepsilon,3} - \overline{u}^\delta_{\varepsilon,3}))\phi_3 dx.
\end{aligned} \tag{4.13}$$

From the energy estimate, we deduce that the four last terms converge to 0 as $\delta \to 0$ uniformly in $\varepsilon$ : indeed,

$$\begin{aligned}
\left\| \int_\Omega \nabla \cdot ((u_\varepsilon - u_\varepsilon^\delta) \otimes u_\varepsilon) \cdot \phi dx \right\|_{L^1([0,T])} &\leq \|\nabla \phi\|_{L^2(\Omega)} \|u_\varepsilon\|_{L^2([0,T],L^6(\Omega))} \|u_\varepsilon^\delta - u_\varepsilon\|_{L^2([0,T],L^3(\Omega))} \\
&\leq \omega(\delta) \|\nabla \phi\|_{L^2(\Omega)} \|u_\varepsilon\|^2_{L^2([0,T],H^1(\Omega))},
\end{aligned}$$

where according to Lemma 3.3, the function $\omega(\delta)$ goes to zero as $\delta$ goes to zero.

We are then interested in the difference between the first two terms. By Proposition 4.3, it can be rewritten

$$\begin{aligned}
&\int_\Omega \nabla \cdot (u_\varepsilon^\delta \otimes u_\varepsilon^\delta) \cdot \phi dx - \int_\Omega \nabla_h \cdot (\overline{u}^\delta_{\varepsilon,h}\overline{u}^\delta_{\varepsilon,3})\phi_3 dx \\
&= \int_\Omega \phi \cdot (\varepsilon \rho_{\varepsilon,\delta} + \sigma_{\varepsilon,\delta}) dx - \frac{\varepsilon}{2}\partial_t \int_\Omega |\overline{\omega}_\varepsilon^\delta|^2 (1 - \chi_\delta) \frac{\nabla^\perp b}{|\nabla b|^2} \cdot \phi dx \\
&\quad + \varepsilon \partial_t \int_\Omega \frac{1}{b} \widetilde{\omega}_\varepsilon^\delta (\widetilde{\Omega}^\delta_{\varepsilon,h})^\perp \cdot \phi dx + \varepsilon \partial_t \int_\Omega \frac{1}{2b^2} (\widetilde{\Omega}^\delta_{\varepsilon,h} \cdot \nabla b)^2 \frac{\nabla^\perp b}{|\nabla b|^2} \cdot \phi dx \\
&\quad - \varepsilon \partial_t \int_\Omega \frac{1}{2b} (\widetilde{\Omega}^\delta_{\varepsilon,h} \cdot (\partial_3 \widetilde{\Omega}^\delta_{\varepsilon,h})^\perp) \phi_3 dx
\end{aligned} \tag{4.14}$$

because

$$\partial_3 \phi = 0, \quad \nabla \cdot \phi = 0 \text{ and } \phi \cdot \nabla b = 0,$$

using as previously the notations $\rho_{\varepsilon,\delta}$ and $\sigma_{\varepsilon,\delta}$ for quantities satisfying the following estimates

$$\forall \delta > 0, \quad \forall T > 0, \quad \sup_{\varepsilon \to 0} \|\rho_{\varepsilon,\delta}\|_{L^1([0,T],L^{6/5}\Omega))} < +\infty,$$

$$\forall T > 0, \quad \lim_{\delta \to 0} \sup_\varepsilon \|\sigma_{\varepsilon,\delta}\|_{L^1([0,T],L^{6/5}\Omega))} = 0.$$

In particular, the second term in the right-hand side of (4.14) converges to 0 as $\delta \to 0$ uniformly in $\varepsilon$ :

$$\left\| \int_\Omega \sigma_{\varepsilon,\delta} \cdot \phi dx \right\|_{L^1([0,T])} \leq \|\phi\|_{L^6(\Omega)} \|\sigma_{\varepsilon,\delta}\|_{L^1([0,T],L^{6/5}(\Omega))} \leq C\delta^{1/24}.$$

It remains only to check that, for any fixed $\delta > 0$, the other terms in the right-hand side of (4.14) converge to 0 as $\varepsilon \to 0$ in the sense of distributions. We have

$$\left\| \int_\Omega \phi \cdot (\varepsilon \rho_{\varepsilon,\delta}) dx \right\|_{L^1([0,T])} \leq \varepsilon \|\phi\|_{L^6(\Omega)} \|\rho_{\varepsilon,\delta}\|_{L^1([0,T],L^{6/5}(\Omega))} \leq C\varepsilon\delta^{-13/4},$$

$$\begin{aligned}\left\| \varepsilon \partial_t \int_\Omega |\overline{\omega}_\varepsilon^\delta|^2 (1-\chi_\delta) \frac{\nabla^\perp b}{|\nabla b|^2} \cdot \phi dx \right\|_{W^{-1,1}([0,T])} &\leq \varepsilon \|\phi\|_{L^6(\Omega)} \|\overline{\omega}_\varepsilon^\delta\|^2_{L^2([0,T],L^{12/5}(\Omega))} \left\| \frac{(1-\chi_\delta)}{|\nabla b|} \right\|_{L^\infty(\Omega)} \\ &\leq C\varepsilon\delta^{-3/4},\end{aligned}$$

and the three other terms are bounded in $W^{-1,1}([0,T])$ by some constant $C_b$ (depending on the $L^\infty$-norm of $b^{-1}$ and $\nabla b$) times

$$\varepsilon \|\phi\|_{L^6(\Omega)} \|\nabla \wedge u_\varepsilon\|^2_{L^2([0,T],L^{12/5}(\Omega))} \leq C\varepsilon\delta^{-1/4}.$$

Taking limits as $\varepsilon \to 0$ and then as $\delta \to 0$ in (4.13)-(4.14) shows that, for all $\phi \in H^1(\Omega) \cap \mathrm{Ker}(L)$,

$$\int_\Omega \nabla \cdot (u_\varepsilon \otimes u_\varepsilon) \cdot \phi dx - \int_\Omega \nabla_h \cdot (\overline{u}_{\varepsilon,h} \overline{u}_{\varepsilon,3}) \phi_3 dx \to 0 \text{ as } \varepsilon \to 0$$

in $W^{-1,1}([0,T])$.

□

4.5. **End of the proof of Theorem 3.** Theorem 3 is an easy consequence of the various results established up to now.

From Corollaries 2.1 and 2.5 in Section 2, we deduce that, up to extraction of a subsequence,

$$u_\varepsilon \rightharpoonup u \text{ weakly in } L^2([0,T] \times \Omega)$$

where $u$ belongs to $L^2([0,T], H^1(\Omega) \cap \mathrm{Ker}(L))$. And, by Lemma 4.1,

$$\overline{u}_{\varepsilon,3} \to \overline{u}_3 = u_3 \text{ strongly in } L^2([0,T] \times \Omega).$$

Let $\phi$ be any vector-field in $H^1(\Omega) \cap \mathrm{Ker}(L)$. Integrating the penalized Navier-Stokes equation against $\phi$ leads to

$$\partial_t \int_\Omega u_\varepsilon \cdot \phi dx + \int_\Omega \nabla \cdot (u_\varepsilon \otimes u_\varepsilon) \cdot \phi dx - \nu \int_\Omega \Delta u_\varepsilon \cdot \phi dx = 0,$$

which can be rewritten

$$\begin{aligned} &\partial_t \int_\Omega u_\varepsilon \cdot \phi dx + \int_\Omega \nabla \cdot (\overline{u}_{\varepsilon,h} \overline{u}_{\varepsilon,3}) \phi_3 dx - \nu \int_\Omega \Delta u_\varepsilon \cdot \phi dx \\ &= \int_\Omega \nabla \cdot (\overline{u}_{\varepsilon,h}\ \overline{u}_{\varepsilon,3}) \phi_3 dx - \int_\Omega \nabla \cdot (u_\varepsilon \otimes u_\varepsilon) \cdot \phi dx. \end{aligned} \tag{4.15}$$

By Proposition 4.4, the right-hand side in (4.15) converges to 0 in the sense of distributions as $\varepsilon \to 0$. The weak convergence of $(u_\varepsilon)$ and the strong convergence of $(\overline{u}_{\varepsilon,3})$ allow then to take limits in the left-hand side :

$$\forall \phi \in H^1(\Omega) \cap \mathrm{Ker}(L), \quad \partial_t \int_\Omega u \cdot \phi dx + \int_\Omega \nabla \cdot (u_h u_3) \phi_3 dx - \nu \int_\Omega \Delta u \cdot \phi dx = 0. \tag{4.16}$$

Therefore any weak limit point of $(u_\varepsilon)$ is the unique $u \in L^2([0,T], H^1(\Omega) \cap \mathrm{Ker}(L))$ solution of (4.16). The whole sequence is converging, which concludes the proof of Theorem 3.

## References


[1] J.-P. Aubin, Un théorème de compacité, *Notes aux Comptes–Rendus de l'Académie des Sciences de Paris*, **309**, (1963), pages 5042-5044.

[2] A. Babin, A. Mahalov, and B. Nicolaenko, Global splitting, integrability and regularity of 3D Euler and Navier–Stokes equations for uniformly rotating fluids, *European Journal of Mechanics*, **15**, (1996), pages 291-300.

[3] A. Babin, A. Mahalov, and B. Nicolaenko, Resonances and regularity for Boussinesq equations, *Russian Journal of Mathematical Physics*, **4**, (1996), pages 417-428.

[4] A. Babin, A. Mahalov, and B. Nicolaenko, Global regularity of 3D rotating Navier–Stokes equations for resonant domains, *Indiana University Mathematics Journal*, **48**, (1999), pages 1133–1176.

[5] J.-Y. Chemin, B. Desjardins, I. Gallagher and E. Grenier, Anisotropy and dispersion in rotating fluids, *Nonlinear Partial Differential Equations and their applications, Collège de France Seminar*, Studies in Mathematics and its Applications, **31**, pages 171–191.

[6] J.-Y. Chemin, B. Desjardins, I. Gallagher and E. Grenier, Ekman boundary layers in rotating fluids, *ESAIM Contrôle Optimal et Calcul des Variations, Special Tribute issue to Jacques–Louis Lions*, **8**, (2002), pages 441-466.

[7] P. Embid and A. Majda, Averaging over fast gravity waves for geophysical flows with arbitrary potential vorticity, *Communications in Partial Differential Equations*, **21** (1996), pages 619–658.

[8] I. Gallagher, Applications of Schochet's methods to parabolic equations, *Journal de Mathématiques Pures et Appliquées*, **77**, (1998), pages 989-1054.

[9] A. E. Gill, Atmosphere-Ocean Dynamics, *International Geophysics Series*, **Vol. 30**, 1982.

[10] F. Golse and L. Saint-Raymond, The Vlasov-Poisson system with strong magnetic field, *Journal de Mathématiques Pures et Appliquées*, **78**, (1999), pages 791–817.

[11] H.P. Greenspan, The theory of rotating fluids, *Cambridge monographs on mechanics and applied mathematics*, 1969.

[12] E. Grenier, Oscillatory perturbations of the Navier–Stokes equations. *Journal de Mathématiques Pures et Appliquées*, **76**, (1997), pages 477-498.

[13] E. Grenier and N. Masmoudi, Ekman layers of rotating fluids, the case of well prepared initial data, *Comm. Partial Differential Equations* **22** (1997), no. 5-6, pages 953–975.

[14] J. Leray, Essai sur le mouvement d'un liquide visqueux emplissant l'espace, *Acta Matematica*, **63**, (1933), pages 193-248.

[15] P.-L. Lions and N. Masmoudi, Une approche locale de la limite incompressible, *C. R. Acad. Sci. Paris* Sér. I Math. **329**, (1999), pages 387–392.

[16] P.-L. Lions and N. Masmoudi, Incompressible limit for a viscous compressible fluid, *Journal de Mathématiques Pures et Appliquées*, **77**, (1998), pages 585–627.

[17] N. Masmoudi, Ekman layers of rotating fluids: the case of general initial data, *Comm. Pure Appl. Math.*, **53**, (2000), pages 432–483.

[18] J. Pedlosky, Geophysical fluid dynamics, *Springer*, 1979.

[19] S. Schochet, Fast singular limits of hyperbolic PDEs. *J. Diff. Equ.* **114**, 1994, pages $476-512$.

[20] R. Temam and M. Ziane, Some mathematical problems in geophysical fluid dynamics,*Handbook of mathematical fluid dynamics. Vol. III,* pages 535–657, North-Holland, Amsterdam, 2004.

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