Navier-Stokes limit of conservative bilinear kinetic equations

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Introduction

$F(t,x,v) = $ #particles with position $x \in \mathbb{R}^d$, velocity $v \in \mathbb{R}^d$ at time $t \in \mathbb{R}^+$

Local macroscopic (hydrodynamic) observables:

- Local density: $R(t,x) = \int_{\mathbb{R}^d} F(t,x,v)\mathrm{d}v$
- Local velocity: $U(t,x) = \frac{1}{R(t,x)} \int_{\mathbb{R}^d} vF(t,x,v)\mathrm{d}v$
- Local temperature: $T(t,x) = \frac{1}{R(t,x)} \int_{\mathbb{R}^d} \left|v - U(t,x)\right|^2 F(t,x,v)\mathrm{d}v$

Introduction

Dynamics of F:

$$(\partial_t + v \cdot \nabla_x) F(t,x,v) = \underbrace{\mathcal{C}\big[F(t,x,\cdot)\big](v)}_{\text{collisions}}$$

Collisions $\Rightarrow$ Dissipation of microscopic information (entropy):

$$F(t,x,v) \approx \mathcal{M}\Big[R(t,x); U(t,x); T(t,x)\Big](v), \qquad (t \to \infty)$$

$$\mathcal{C}[\mathcal{M}] = 0$$

$F(t,x,v)$ **asymptotically** characterized by its **macroscopic properties**.

Hydrodynamic limit: many collisions + scaling $\xrightarrow{\varepsilon \to 0}$ hydrodynamic equations:

$$(\varepsilon^{\boldsymbol{a}} \partial_t + \varepsilon^{\boldsymbol{b}} v \cdot \nabla_x) F(t,x,v) = \frac{\mathbf{1}}{\varepsilon} \mathcal{C}\big[F(t,x,\cdot)\big](v)$$

- Fokker-Planck $\to$ (fractional) diffusion
- Boltzmann, Landau $\to$ Euler, Navier-Stokes

Introduction

Our problem: fluctuaons $\propto \varepsilon$ + scaling $(\varepsilon^2 t, \varepsilon x)$ + conservtive binary collions $\propto \varepsilon^{-1}$:

$$\text{(prototypical equation)} \quad \underbrace{\partial_t f^\varepsilon + \frac{1}{\varepsilon} v \cdot \nabla_x f^\varepsilon}_{\text{transport}} = \underbrace{\frac{1}{\varepsilon^2} \mathcal{L} f^\varepsilon}_{\substack{\text{collisions}\\ \text{gas–medium}}} + \underbrace{\frac{1}{\varepsilon} \mathcal{Q}\big(f^\varepsilon, f^\varepsilon\big)}_{\text{collisions gas–gas}}$$

Goal: show that

$$f^\varepsilon(t,x,v) \xrightarrow{\varepsilon \to 0} \big(\varrho(t,x) + u(t,x) \cdot v + \theta(t,x)\big(|v|^2 - \text{cst.}\big)\big)\, \mu(v)$$

$$\begin{cases} \partial_t u + u \cdot \nabla_x u = \kappa_{\text{inc}} \Delta_x u, \quad \nabla_x \cdot u = 0, \\ \partial_t \theta + u \cdot \nabla_x \theta = \kappa_{\text{Bou}} \Delta_x \theta, \quad \varrho = -\theta. \end{cases}$$

Rough summary: if conservative binary collisions $\propto \varepsilon^{-1}$

$$F_{\text{in}}(x,v) = \mathcal{M} + \varepsilon f_{\text{in}}(\varepsilon x, v) \Rightarrow F \approx \mathcal{M} + \varepsilon\, \text{Navier-Stokes}(\varepsilon^2 t, \varepsilon x)$$

Introduction

Our structural assumptions

Consider $0 < \mu \in L^1\left(\langle v\rangle^k \mathrm{d}v\right)$ and $V^1 \subset V := L^2\left(\mu^{-1}\mathrm{d}v\right)$.

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1. **Isotropy:** $\forall O \in \mathbb{R}^{d\times d}$ orthogonal, denoting $(Of)(v) = f(Ov)$

$$O(\mathcal{L}f) = \mathcal{L}(Of), \qquad O\mathcal{Q}(f,f) = \mathcal{Q}(Of, Of),$$

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$$\ker \mathcal{L} = \left\{\varrho\mu + u\cdot v\mu + \theta\left(|v|^2 - \text{cst.}\right)\mu \,:\, (\varrho, u, \theta) \in \mathbb{R}^{d+2}\right\}$$

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4. **Control of the collisions by the energy and the dissipated entropy:**

$$\langle \mathcal{Q}(f,g), h\rangle_V \lesssim \|h\|_{V^1}\left(\|f\|_{V^1}\|g\|_V + \|f\|_V\|g\|_{V^1}\right)$$

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4. **Control of the collisions by the energy and the dissipated entropy:**

$$\|\mathcal{Q}(f,g)\|_{V^{-1}} \lesssim \|f\|_{V^1}\|g\|_V + \|f\|_V\|g\|_{V^1}, \quad V^{-1} := \left(V^1\right)' \text{w.r.t. V}$$

Strategy

Our problem: profile of small fluctuations f^ε around an equilibrium

$$\partial_t f^\varepsilon + \frac{1}{\varepsilon} v \cdot \nabla_x f^\varepsilon = \frac{1}{\varepsilon^2} \mathcal{L} f^\varepsilon + \frac{1}{\varepsilon} \mathcal{Q}\big(f^\varepsilon, f^\varepsilon\big), \quad f^\varepsilon(0,x,v) = f_{\text{in}}(x,v)$$

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and describe/quantify the convergence.

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$$f^\varepsilon(t) = U^\varepsilon(t) f_{\text{in}} + \Psi^\varepsilon\left(f^\varepsilon, f^\varepsilon\right)(t)$$

where

$$U^\varepsilon(t) := \exp\left(\frac{t}{\varepsilon^2}\left(\mathcal{L} - \varepsilon v \cdot \nabla_x\right)\right)$$

$$\Psi^\varepsilon(f,f)(t) := \frac{1}{\varepsilon}\int_0^t U^\varepsilon(t-\tau)\mathcal{Q}(f(\tau), f(\tau))\mathrm{d}\tau$$

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Strategy

Strategy for the existence of solutions

Identify some "hydrodynamic" and "kinetic" regimes:

$$\|f\|_{\mathcal{H}}^2 := \sup_{t\geq 0} \|f(t)\|_{H^\ell_x V^1_v}^2 + \int_0^\infty \|\nabla_x f(t)\|_{H^\ell_x V^1_v}^2 \mathrm{d}t, \quad \begin{pmatrix}\text{heat equation}\\ \text{Navier-Stokes}\end{pmatrix}$$

$$\|f\|_{\mathcal{K}^\varepsilon}^2 := \sup_{t\geq 0} e^{2\sigma t/\varepsilon^2} \|f(t)\|_{H^\ell_x V_v}^2 + \frac{1}{\varepsilon^2}\int_0^\infty e^{2\sigma t/\varepsilon^2}\|f(t)\|_{H^\ell_x V^1_v}^2 \mathrm{d}t, \quad \begin{pmatrix}\text{dissipative}\\ \text{equation}\end{pmatrix}$$

and a decomposition of the semigroup/nonlinearity:

$$U^\varepsilon(t) = \exp\left(\frac{t}{\varepsilon^2}\left(\mathcal{L} - \varepsilon v\cdot\nabla_x\right)\right) = U^\varepsilon_{\text{hydro}} \oplus U^\varepsilon_{\text{kine}}$$

$$\Psi^\varepsilon(f,f)(t) = \frac{1}{\varepsilon}\int_0^t U^\varepsilon(t-\tau)\mathcal{Q}(f(\tau),f(\tau))\mathrm{d}\tau = \Psi^\varepsilon_{\text{hydro}}(f,f) + \Psi^\varepsilon_{\text{kine}}(f,f)$$

compatible with the corresponding regimes:

$$\|U^\varepsilon_{\text{hydro}} f\|_{\mathcal{H}} \lesssim \|f\|_{H^\ell_x V_v}, \qquad \|U^\varepsilon_{\text{kine}} f\|_{\mathcal{K}^\varepsilon} \lesssim \|f\|_{H^\ell_x V_v}$$

$$\left\|\Psi^\varepsilon_{\text{hydro}}(f,f)\right\|_{\mathcal{H}} \lesssim \|f\|_{\mathcal{H}}^2, \qquad \|\Psi^\varepsilon_{\text{kine}}(f,f)\|_{\mathcal{K}^\varepsilon} \lesssim \|f\|_{\mathcal{K}^\varepsilon}^2$$

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Consider an **arbitrary** system of equations

$$\left.\begin{aligned} f^\varepsilon_{\text{hydro}}(t) &= U^\varepsilon_{\text{hydro}}(t) f_{\text{in}} + \Psi^\varepsilon_{\text{hydro}}(f^\varepsilon_{\text{hydro}}, f^\varepsilon_{\text{hydro}}) + \dots \\ f^\varepsilon_{\text{kine}}(t) &= U^\varepsilon_{\text{kine}}(t) f_{\text{in}} + \Psi^\varepsilon_{\text{kine}}(f^\varepsilon_{\text{kine}}, f^\varepsilon_{\text{kine}}) + \dots \\ &\dots \end{aligned}\right\} \xrightarrow{\text{Picard}} \begin{aligned} f^\varepsilon &:= f^\varepsilon_{\text{hydro}} + f^\varepsilon_{\text{kine}} + \dots \\ &\in \mathcal{H} + \mathcal{K}^\varepsilon + \dots \end{aligned}$$

is **a** solution

Strategy

Strategy for the convergence

To prove the convergence of

$$f^{\varepsilon}_{\text{hydro}}(t) = U^{\varepsilon}_{\text{hydro}}(t) f_{\text{in}} + \Psi^{\varepsilon}_{\text{hydro}}(f^{\varepsilon}_{\text{hydro}}, f^{\varepsilon}_{\text{hydro}}) + \dots$$

(1) Show the convergence of the “hydrodynamic” semigroup:

$$U^{\varepsilon}_{\text{hydro}} = U^0 + \mathcal{O}(\varepsilon) + \text{dispersive}, \quad \left(U^{\varepsilon}_{\text{hydro}}\right)_{|\ker(\mathcal{L})^{\perp}} = \varepsilon V^0 + \mathcal{O}\left(\varepsilon^2\right) + \text{dispersive}$$

and of the “hydrodynamic” nonlinearity (recall $\mathcal{Q} \perp \ker(\mathcal{L})$)

$$\Psi^{\varepsilon}_{\text{hydro}}(f, f) := \cancel{\frac{1}{\varepsilon}} \int_0^t \cancel{\varepsilon} V^0(t-\tau)\mathcal{Q}(f(\tau), f(\tau))\mathrm{d}\tau + \mathcal{O}(\varepsilon)$$

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and of the "hydrodynamic" nonlinearity (recall $\mathcal{Q} \perp \ker(\mathcal{L})$)

$$\Psi^\varepsilon_{\text{hydro}}(f, f) := \frac{1}{\varepsilon} \int_0^t \varepsilon V^0(t-\tau) \mathcal{Q}(f(\tau), f(\tau)) \mathrm{d}\tau + \mathcal{O}(\varepsilon)$$

(2) Check from **explicit** formulae of U^0 and V^0 that

$$f(t) = U^0(t) f_{\text{in}} + \int_0^t V^0(t-\tau) \mathcal{Q}\left(f(\tau), f(\tau)\right) \mathrm{d}\tau$$

$$\Leftrightarrow \begin{cases} f(t,x,v) = \big(\varrho(t,x) + u(t,x)\cdot v + \theta(t,x)\left(|v|^2 - \text{cst.}\right)\big)\mu(v), \\ \\ u(t) = e^{t\kappa_{\text{inc}}\Delta_x} u_{\text{in}} + \int_0^t e^{(t-\tau)\kappa_{\text{inc}}\Delta_x} \mathbb{P}\Big[\nabla_x \cdot \big(u(\tau)\otimes u(\tau)\big)\Big] \mathrm{d}\tau, \quad \nabla_x \cdot u = 0, \\ \theta(t) = e^{t\kappa_{\text{Bou}}\Delta_x}\theta_{\text{in}} + \int_0^t e^{(t-\tau)\kappa_{\text{Bou}}\Delta_x} \nabla_x \cdot \big(u(\tau)\theta(\tau)\big) \mathrm{d}\tau, \quad \varrho = -\theta. \end{cases}$$

Outline of the proof of the (non)linear bounds

Proposition (G, Lods)

The semigroup and Duhamelized nonlinearity split

$$U^\varepsilon(t) = \exp\left(\frac{t}{\varepsilon^2}\left(\mathcal{L} - \varepsilon v \cdot \nabla_x\right)\right) = U^\varepsilon_{\text{hydro}}(t) \oplus U^\varepsilon_{\text{kine}}(t)$$

$$\frac{1}{\varepsilon}\int_0^t U^\varepsilon(t-\tau)\mathcal{Q}(f(\tau), f(\tau))\mathrm{d}\tau = \Psi^\varepsilon_{\text{hydro}}(f,f)(t) + \Psi^\varepsilon_{\text{kine}}(f,f)(t)$$

and satisfy the continuity estimates

$$\|U^\varepsilon_{\text{hydro}} f\|_{\mathcal{H}} \lesssim \|f\|_{H^\ell_x V_v}, \qquad \|U^\varepsilon_{\text{kine}} f\|_{\mathcal{K}^\varepsilon} \lesssim \|f\|_{H^\ell_x V_v}$$

$$\left\|\Psi^\varepsilon_{\text{hydro}}(f,f)\right\|_{\mathcal{H}} \lesssim \|f\|^2_{\mathcal{H}}, \qquad \|\Psi^\varepsilon_{\text{kine}}(f,f)\|_{\mathcal{K}^\varepsilon} \lesssim \|f\|^2_{\mathcal{K}^\varepsilon}$$

Difficulties: Compensate $\frac{1}{\varepsilon}$ + deregularizing effect $\mathcal{Q} : V^1 \times V^1 \to V^{-1}$
Strategy: Finding stable subspaces $\to$ studying $\Sigma\big(\mathcal{L} - v \cdot \nabla_x\big) \to$ studying $\Sigma\big(\mathcal{L} - i(v \cdot \xi)\big)\ \forall \xi \in \mathbb{R}^d$

Outline of the proof of the (non)linear bounds

Finding stable subspaces of $\mathcal{L} - v \cdot \nabla_x$: localization of the spectrum of $\mathcal{L} - i(v \cdot \xi)$

$\mathcal{L} - i(v \cdot \xi)$ = perturbation of $\mathcal{L}$

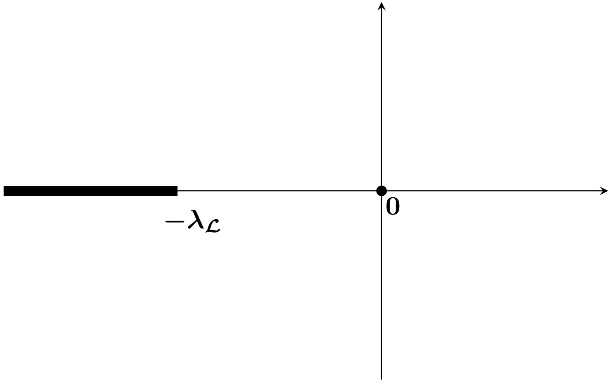

Figure: Localization of the spectrum of $\mathcal{L}$.

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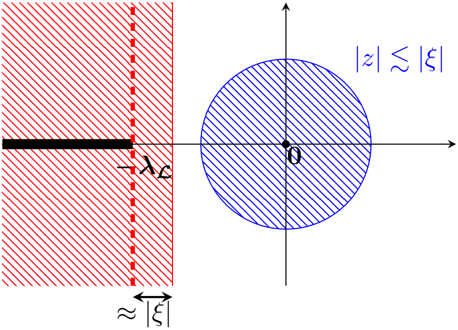

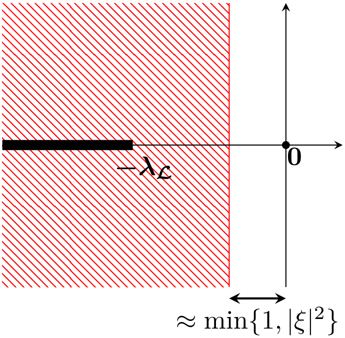

Figure: Localization of the spectrum of $\mathcal{L} - i(v \cdot \xi)$ for $|\xi| \ll 1$ and for $|\xi| \gtrsim 1$

Localization for $|\xi| \ll 1$: Factorization methods from Tristani's work

Localization for $|\xi| \gtrsim 1$: Hypocoercivity

Disjoint parts of the spectrum $\Rightarrow$ stable subspaces (kinetic(ξ) and hydrodynamic(ξ))

Outline of the proof of the (non)linear bounds

The kinetic regime

Scaling $(t, x) \to (t/\varepsilon^2, x/\varepsilon)$ i.e. $\mathcal{L} - i(v\cdot\xi) \to \frac{1}{\varepsilon^2}\left(\mathcal{L} - i\varepsilon(v\cdot\xi)\right)$

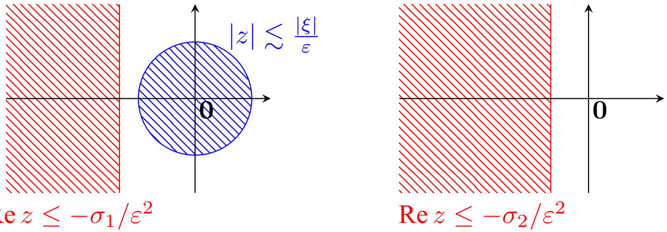

Figure: Localization of Spec$\left(\varepsilon^{-2}(\mathcal{L} - i\varepsilon(v\cdot\xi))\right)$ for $|\xi| \ll \varepsilon^{-1}$ and for $|\xi| \gtrsim \varepsilon^{-1}$

Localization + bounds + energy method (regularization $V, V^{-1} \to V, V^{1}$):

$$\|U^\varepsilon_{\text{kine}} f\|^2_{\mathcal{K}^\varepsilon} = \sup_{t\geq 0} e^{\frac{2\sigma t}{\varepsilon^2}} \|U^\varepsilon_{\text{kine}}(t) f\|^2_{H^\ell_x V_v} + \frac{1}{\varepsilon^2}\int_0^\infty e^{\frac{2\sigma t}{\varepsilon^2}} \|U^\varepsilon_{\text{kine}}(t) f\|^2_{H^\ell_x \boldsymbol{V}^1_v} \mathrm{d}t \lesssim \|f\|^2_{H^\ell_x V_v}$$

Convolution $\to$ better regularization:

$$\|\Psi^\varepsilon_{\text{kine}}(f, f)\|_{\mathcal{K}^\varepsilon} = \left\| \frac{1}{\varepsilon} U^\varepsilon_{\text{kine}} *_t \mathcal{Q}(f, f) \right\|_{\mathcal{K}^\varepsilon} \lesssim \varepsilon \|f\|^2_{\mathcal{K}^\varepsilon}$$

Outline of the proof of the (non)linear bounds

Properties of the hydrodynamic semigroup

Study the conjugated matrix $L(\xi) \underset{\Phi(\xi)}{\sim} \big(\mathcal{L} - i(v \cdot \xi)\big)_{|\text{hydro. space}(\xi)}$

Isotropy + perturb$^{\text{tion}}$ theory for matrices $\Rightarrow$ exp$^{\text{on}}$ of eigenval./eigenproj. of $L(\xi)$:

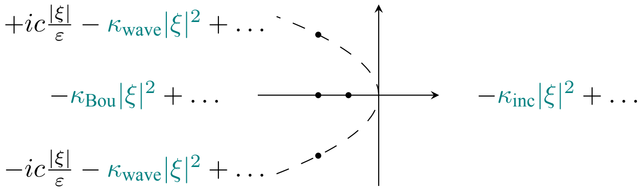

Figure: Spec $\left(\frac{1}{\varepsilon^2}\big(\mathcal{L} - i\varepsilon(v \cdot \xi)\big)_{|\text{hydro. space}(\varepsilon\xi)}\right)$ for $|\xi| \ll \varepsilon^{-1}$

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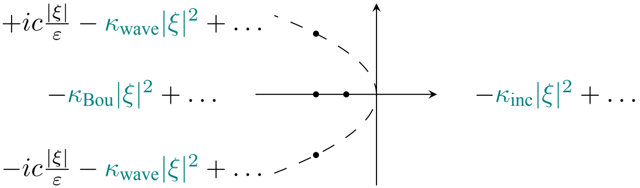

Figure: Spec $\left(\frac{1}{\varepsilon^2}\big(\mathcal{L} - i\varepsilon(v\cdot\xi)\big)_{|\text{hydro. space}(\varepsilon\xi)} \right)$ for $|\xi| \ll \varepsilon^{-1}$

+ expansion of $\Phi(\xi)$ in $\mathscr{B}\left(V^{-1} \to V^{1}\right)$ + Fourier representation of $U^{\varepsilon}_{\text{hydro}}(t)$

$$\Rightarrow e^{-t\Delta_x} U^{\varepsilon}_{\text{hydro}}(t) \in \mathscr{B}\left(H^{\ell}_x V^{-1}_v \to H^{\ell}_x V^{1}_v\right)$$

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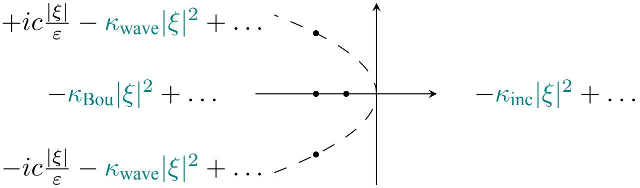

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+ expansion of $\Phi(\xi)$ in $\mathscr{B}\left(V^{-1} \to V^{1}\right)$ + Fourier representation of $U^{\varepsilon}_{\text{hydro}}(t)$

$$\Rightarrow \|U^{\varepsilon}_{\text{hydro}} f\|^2_{\mathcal{H}} = \sup_{t\geq 0} \left\|U^{\varepsilon}_{\text{hydro}}(t) f\right\|^2_{H^\ell_x \boldsymbol{V^1_v}} + \int_0^\infty \left\|\nabla_x U^{\varepsilon}_{\text{hydro}} f(t)\right\|^2_{H^\ell_x \boldsymbol{V^1_v}} \mathrm{d}t \lesssim \|f\|^2_{H^\ell_x V_v}$$

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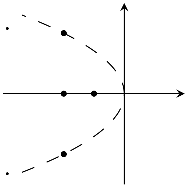

Figure: Spec $\left(\frac{1}{\varepsilon^2}\left(\mathcal{L} - i\varepsilon(v\cdot\xi)\right)_{|\text{hydro. space}(\varepsilon\xi)}\right)$ for $|\xi| \ll \varepsilon^{-1}$

+ expansion of $\Phi(\xi)$ in $\mathscr{B}\left(V^{-1} \to V^{1}\right)$ + Fourier representation of $U^{\varepsilon}_{\text{hydro}}(t)$

$$\Rightarrow \|U^{\varepsilon}_{\text{hydro}} f\|^2_{\mathcal{H}} = \sup_{t\geq 0} \left\|U^{\varepsilon}_{\text{hydro}}(t) f\right\|^2_{H^\ell_x \boldsymbol{V}^1_v} + \int_0^\infty \left\|\nabla_x U^{\varepsilon}_{\text{hydro}} f(t)\right\|^2_{H^\ell_x \boldsymbol{V}^1_v} \mathrm{d}t \lesssim \|f\|^2_{H^\ell_x V_v}$$

and $\Rightarrow \frac{1}{\varepsilon} e^{-t\Delta_x} U^{\varepsilon}_{\text{hydro}}(t)_{|(\ker \mathcal{L})^\perp} \in \mathscr{B}\left(H^\ell_x V^{-1}_v \to H^\ell_x V^1_v\right)$ (recall $\mathcal{Q} \perp \ker \mathcal{L}$)

Outline of the proof of the (non)linear bounds

Properties of the hydrodynamic semigroup

Study the conjugated matrix $L(\xi) \underset{\Phi(\xi)}{\sim} \left(\mathcal{L} - i(v\cdot\xi)\right)_{|\text{hydro. space}(\xi)}$

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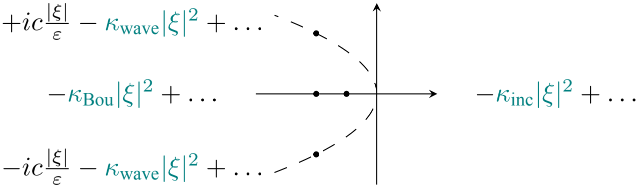

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$$\text{and } \Rightarrow \|\Psi^{\varepsilon}_{\text{hydro}}(f,f)\|^2_{\mathcal{H}} = \left\|\frac{1}{\varepsilon} U^{\varepsilon}_{\text{hydro}} *_t \boldsymbol{\mathcal{Q}}(f,f)\right\|^2_{\mathcal{H}} \lesssim \int_0^\infty \|\boldsymbol{\mathcal{Q}}(f,f)(t)\|^2_{H^{\ell}_x \boldsymbol{V^{-1}_v}} \mathrm{d}t$$

Outline of the proof of the (non)linear bounds

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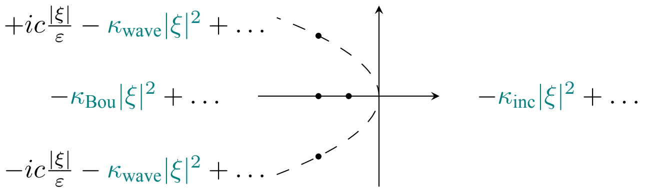

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Conclusion

Theorem (G, Lods) — ArXiv:2304.11698

Consider any **(non-small)** $f_{\text{in}} \in H^\ell_x L^2_v\left(\mu^{-1}\mathrm{d}v\right)$, for any $\varepsilon \ll 1$, $\exists!$ solution to

$$\partial_t f^\varepsilon + \varepsilon^{-1} v \cdot \nabla_x f^\varepsilon = \varepsilon^{-2}\mathcal{L} + \varepsilon^{-1}\mathcal{Q}\left(f^\varepsilon, f^\varepsilon\right), \quad f^\varepsilon \in L^\infty_t H^\ell_x L^2\left(\mu^{-1}\mathrm{d}v\mathrm{d}x\right),$$

with the **same lifespan** as the Navier-Stokes limit f^0, and for smooth initial data

$$f^\varepsilon = f^0 + \mathcal{O}\left(e^{-t\sigma/\varepsilon^2}\right) + \mathcal{O}\left(\sqrt{\varepsilon}\right) + \mathcal{O}\left(\left(\frac{\varepsilon}{t}\right)^{\frac{d-1}{2}}\right)$$

and for incompressible initial data

$$f^\varepsilon = f^0 + \mathcal{O}\left(e^{-t\sigma/\varepsilon^2}\right) + \mathcal{O}\left(\varepsilon\right) \quad \text{(optimal)}$$

- $d = 2 \Rightarrow f^0$ global $\Rightarrow f^\varepsilon$ global
- Applies to Boltzmann/Landau ($\gamma + 2s \geq 0$) and $\int |f_{\text{in}}(x,v)|^2 \langle v \rangle^k \mathrm{d}v < \infty$
- Modern strategy for spectral study. No compactness $\Rightarrow$ constructive constants
- Use of isotropy $\to$ adaptable to relativistic Boltzmann/Landau
- Spectral study works for quantum Boltzmann/Landau but $\mathcal{Q}$ is trilinear

Thank you for your attention

Extra

Kato's reduction process: eigen. pbm. in Banach $\to$ eigen. pbm. in finite dimension

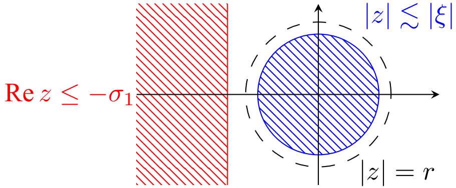

Figure: Localization of $\mathrm{Spec}\big(\mathcal{L} - i(v \cdot \xi)\big)$ for $|\xi| \ll 1$

Goal: expansion of eigenvalues and eigenfunctions of $\big(\mathcal{L} - i(v \cdot \xi)\big)_{|\text{hydro. space}(\xi)}$

Difficulty: hydro. space(ξ) depends on ξ

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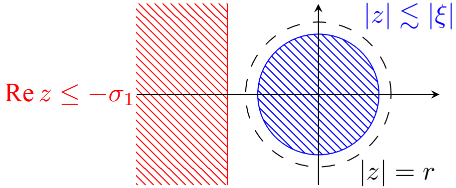

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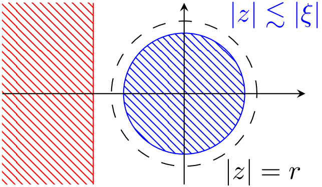
Solution: Rectify $\mathcal{L} - i(v \cdot \xi)$ to a matrix $L(\xi)$ by conjugating with

$$\Phi(\xi) : \text{hydro. space}(\xi) \xrightarrow{\text{iso.}} \ker \mathcal{L} \approx \mathbb{C}^{d+2}$$

Extra

Kato's reduction process



Projection on the hydrodynamic spectrum(ξ) = R$(\mathcal{P}(\xi))$:

$$\mathcal{P}(\xi) = \frac{1}{2i\pi} \oint_{|z|=r} \left(z - \mathcal{L} + i(v \cdot \xi)\right)^{-1} \mathrm{d}z \in \mathscr{B}\left(V^{-1} \to V^{1}\right)$$

Kato's isomorphism:

$$\frac{\mathcal{P}(0)\mathcal{P}(\xi) + \mathcal{P}(0)^{\perp}\mathcal{P}(\xi)^{\perp}}{\sqrt{\mathrm{Id} - (\mathcal{P}(\xi) - \mathcal{P}(0))^2}} =: \Phi(\xi) \ : \ \text{hydro. space}(\xi) \xrightarrow{\text{iso}} \ker(\mathcal{L})$$

$$\Phi(\xi) = \mathrm{Id} + |\xi|\Phi^{(1)} + |\xi|^2\Phi^{(2)}, \qquad \Phi^{(j)} \in \mathscr{B}\left(V^{-1} \to V^{1}\right)$$

Rectified operator:

$$L(\xi) := \Phi(\xi)\big(\mathcal{L} - i(v \cdot \xi)\big)\Phi(\xi)^{-1} \in \mathscr{B}\left(\ker \mathcal{L}\right) \sim \mathbb{C}^{(d+2)\times(d+2)}$$

Extra

Diagonalization of $L(\xi) = \Phi(\xi)^{-1}\left(\mathcal{L} - i(v\cdot\xi)\right)\Phi(\xi)$

Problem: $|\xi|^{-1}L(\xi) \xrightarrow{\xi\to 0}$ matrix with **non-simple** eigenvalues

Solution: Isotropy of $\mathcal{L} \Rightarrow$ block representation of $L(\xi)$:

$$L(\xi) = \begin{pmatrix} \lambda_{\mathrm{inc}}(\xi)\mathrm{Id} & 0 \\ 0 & |\xi| M(\xi) \end{pmatrix}, \qquad \lambda_{\mathrm{inc}}(\xi) = -\kappa_{\mathrm{inc}}|\xi|^2 + \mathcal{O}\left(|\xi|^3\right),$$

in the decomposition

$$\{u\cdot v\mu \,|\, u \perp \xi\} \oplus \left\{\varrho\mu + \alpha\xi\cdot v\mu + e|v|^2\mu \;:\; (\varrho,\alpha,e)\in\mathbb{R}^3\right\}$$

where

$$M(\xi) = \begin{pmatrix} +ic & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -ic \end{pmatrix} + |\xi| \begin{pmatrix} -\kappa_{\mathrm{wave}} & * & * \\ * & -\kappa_{\mathrm{Bou}} & * \\ * & * & -\kappa_{\mathrm{wave}} \end{pmatrix} + \mathcal{O}\left(|\xi|^2\right)$$

Conclusion: $L(\xi)$ diagonalizable + expansion of eigenprojector and eigenvalues:

$$\lambda_{\pm\mathrm{wave}}(\xi) = \pm ic|\xi| - \kappa_{\mathrm{wave}}|\xi|^2 + \mathcal{O}\left(|\xi|^3\right), \qquad \lambda_{\mathrm{Bou}}(\xi) = -\kappa_{\mathrm{Bou}}|\xi|^2 + \mathcal{O}\left(|\xi|^3\right)$$

$$L(\xi) = \sum_{\star} \lambda_\star(\xi) P_\star(\xi), \qquad P_\star(\xi) = P_\star^{(0)} + |\xi| P_\star^{(1)} + |\xi|^2 P_\star^{(2)}$$

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