From Boltzmann to Navier–Stokes with polynomial initial data

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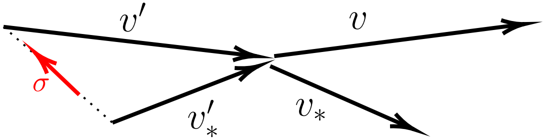
Introduction

The scaled Boltzmann equation

- $F^\varepsilon(t,x,v)$= particles density, $x \in \Omega = \mathbb{R}^d$ or $\mathbb{T}^d$, $d = 2$ or 3
- ε=mean free path (Knudsen number)

$$\varepsilon \partial_t F^\varepsilon + v \cdot \nabla_x F^\varepsilon = \frac{1}{\varepsilon} Q(F^\varepsilon, F^\varepsilon)$$

$$Q(F,G)(v) = \int_{\mathbb{R}^d_{v_*} \times \mathbb{S}^{d-1}_\sigma} |v - v_*| \Big(F(v')G(v'_*) - F(v)G(v_*) \Big) \mathrm{d}v_* \mathrm{d}\sigma$$



$$v' = \frac{v+v_*}{2} + \sigma \frac{|v-v_*|}{2}, \quad v'_* = \frac{v+v_*}{2} - \sigma \frac{|v-v_*|}{2}$$

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$$\varepsilon \partial_t F^\varepsilon + v \cdot \nabla_x F^\varepsilon = \frac{1}{\varepsilon} Q(F^\varepsilon, F^\varepsilon)$$

Conserved macroscopic observables:

- Mass : $R^\varepsilon = \displaystyle\int F^\varepsilon \,\mathrm{d}v$
- Momentum : $R^\varepsilon U^\varepsilon = \displaystyle\int F^\varepsilon \, v \,\mathrm{d}v$
- Energy : $\frac{1}{2} R^\varepsilon |U|^2 + \frac{d}{2} R^\varepsilon T^\varepsilon = \displaystyle\int F^\varepsilon \, \frac{|v|}{2}^2 \mathrm{d}v$

Introduction

Relation with the incompressible Navier–Stokes–Fourier system

Gas at thermodynamic equilibrium (constant heat, mass density, at rest) :

$$M = (2\pi)^{-d/2} \exp\left(-|v|^2/2\right)$$

Statistical fluctuation of order ε:

$$F^\varepsilon = M + \varepsilon f^\varepsilon, \; F^\varepsilon_{|t=0} = M + \varepsilon f_{\text{in}}$$

Macroscopic fluctuations of order ε:

$$\begin{aligned} R^\varepsilon(t,x) &\approx 1 + \varepsilon \rho^\varepsilon(t,x), \\ U^\varepsilon(t,x) &\approx 0 + \varepsilon u^\varepsilon(t,x), \\ T^\varepsilon(t,x) &\approx 1 + \varepsilon \theta^\varepsilon(t,x). \end{aligned}$$

Introduction

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"Linearized" equation:

$$\begin{cases} \partial_t f^\varepsilon = \frac{1}{\varepsilon^2} \left(\mathcal{L} + \varepsilon v \cdot \nabla_x\right) f^\varepsilon + \frac{1}{\varepsilon} Q(f^\varepsilon, f^\varepsilon), \\ f^\varepsilon_{|t=0} = f_{\text{in}}, \end{cases} \qquad (B^\varepsilon)$$

where

$$\mathcal{L} := Q(M, \cdot) + Q(\cdot, M)$$

Introduction

Relation with the incompressible Navier–Stokes–Fourier system

Definition-Theorem (microscopic, macroscopic)

- We say $f(x,v)$ is **macroscopic** if it satisfies the **equivalent conditions**
  - $\mathcal{L}f = 0$
  - $f(x,v) = \Big(\rho(x) + u(x)\cdot v + \frac{1}{2}(|v|^2 - d)\theta(x)\Big)M(v)$

  and **well-prepared** if $\nabla_x \cdot u(x) = 0,\ \rho(x) + \theta(x) = 0.$
- We say f is **microscopic** if

$$\int f(x,v)\varphi(v)\mathrm{d}v = 0,\ \varphi(v) = 1, v, |v|^2$$

Remark: There is a unique decomposition

$$\begin{aligned} f &= f_{\text{macro}} + f_{\text{micro}} \\ &= f_{\text{well-prepared}} + f_{\text{ill-prepared}} + f_{\text{micro}} \end{aligned}$$

Introduction

Relation with the incompressible Navier–Stokes–Fourier system

Theorem (1991-2004)

If $F^\varepsilon = M + \varepsilon f^\varepsilon$ is a **weak** (Diperna-Lions) solution to the Boltzmann equation, then f^ε converges in a **weak sense** to

$$f^0(t,x,v) = \left(\rho(t,x) + u(t,x)\cdot v + \frac{1}{2}\left(|v|^2 - d\right)\theta(t,x)\right)M(v),$$

where (ρ, u, θ) are **weak** (Leray) solutions to the Navier-Stokes-Fourier system

$$\begin{cases} \partial_t u + u\cdot\nabla_x u = \mu\Delta_x u - \nabla_x p, \\ \partial_t \theta + u\cdot\nabla_x \theta = \kappa\Delta_x\theta, \\ \nabla_x\cdot u = 0, \quad \rho+\theta = 0, \end{cases} \tag{INSF}$$

and $\mu, \kappa > 0$ depend only on Q and M.

Existence and convergence of strong solutions

Initial data with Gaussian decay (Bardos–Ukai/Gallagher–Tristani)

Functional space: $\mathbf{G} = L^\infty_v H^s_x \left(M^{-1/2} \langle v \rangle^\beta \mathrm{d}v \right)$, $s > \frac{d}{2}$, $\beta > \frac{d}{2} + 1$

$$\partial_t f^\varepsilon = \frac{1}{\varepsilon^2}\left(\mathcal{L} + \varepsilon v \cdot \nabla_x\right) f^\varepsilon + \frac{1}{\varepsilon} Q(f^\varepsilon, f^\varepsilon), \qquad (B^\varepsilon)$$

$$\big\downarrow \text{ Duhamel with } f^\varepsilon_{|t=0} = f_{\mathrm{in}}$$

$$f^\varepsilon(t) = U^\varepsilon(t) f_{\mathrm{in}} + \Psi^\varepsilon(t)(f^\varepsilon, f^\varepsilon), \qquad (B^\varepsilon)$$

Where we denote

$$U^\varepsilon(t) := \exp\left(\frac{1}{\varepsilon^2}\left(\mathcal{L} + \varepsilon v \cdot \nabla_x\right)\right),$$

$$\Psi^\varepsilon(t)(f^\varepsilon, f^\varepsilon) := \frac{1}{\varepsilon}\int_0^t U^\varepsilon(t - t') Q\left(f^\varepsilon(t'), f^\varepsilon(t')\right) \mathrm{d}t'$$

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- Existence of f^ε : Banach's fixed point theorem
  - on f^ε (Bardos-Ukai, 1991) if $\|f_{\text{in}}\|_{\mathbf{G}} \ll 1$
  - on $f^\varepsilon - f^0$ (Gallagher-Tristani, 2019) if $\varepsilon \ll 1$

...then $\varepsilon \to 0$

- Requires spectral study by Ellis, Pinsky, Ukai (1975-'86)
  - uniform bounds on U^ε and Ψ^ε
  - convergence of U^ε and Ψ^ε

Existence and convergence of strong solutions

Initial data with polynomial decay (Briant–Merino–Mouhot/G.)

Reminder

- Total mass: $\int F^\varepsilon \, \mathrm{d}v\mathrm{d}x = \|F^\varepsilon\|_{L^1_{x,v}}$
- Total energy: $\int F^\varepsilon \, \frac{|v|^2}{2} \mathrm{d}v\mathrm{d}x = \left\| |v|^2 F^\varepsilon \right\|_{L^1_{x,v}}$

Question: Can we only assume $f_{\text{in}} \in L^1\left(\left(1+|v|^2\right)\mathrm{d}v\mathrm{d}x\right)$?

Existence and convergence of strong solutions

Initial data with polynomial decay (Briant–Merino–Mouhot/G.)

Theorem (G. 2021)

For any $s > \frac{d}{2}$, $\delta > 0$, $f_{\text{in}} \in \mathbf{P} := L^1_v H^s_x \left(\langle v \rangle^{3+\delta} \mathrm{d}v \right)$, denote f^0 the **strong solution** of (INSF) generated by $f_{\text{in,well-prepared}}$ on $[0, T)$, then

- for $\varepsilon \ll 1$, the equation (B^ε) has a a **unique solution**

$$f^\varepsilon \in \mathcal{C}_b \left([0, T); L^1_v H^s_x \left(\langle v \rangle^{3+\delta} \right) \right) \cap L^1_t L^1_v H^s_x \left(\langle v \rangle^{4+\delta} \right)$$

- $f^\varepsilon = f^0 + u^\varepsilon_{\text{ac}} + u^\varepsilon_1 + u^\varepsilon_\infty$, where

$$u^\varepsilon_1(t) = \mathcal{O}(e^{-\gamma t / \varepsilon^2}), \; u^\varepsilon_\infty(t) = o(1), \; u^\varepsilon_{\text{ac}} \rightharpoonup 0,$$

- macroscopic part of f_{in} **well-prepared** $\Rightarrow u^\varepsilon_{\text{ac}} = 0$
- f_{in} **purely macroscopic** (micro. part = 0) $\Rightarrow u^\varepsilon_1 = 0$

Remark: $\Omega = \mathbb{R}^d \Rightarrow \|u^\varepsilon_{\text{ac}}\|_{L^p_t} \to 0$ for $\frac{2}{d-1} < p < \infty$.

Existence and convergence of strong solutions

Initial data with polynomial decay (Briant–Merino–Mouhot/G.)

- C. Mouhot (2005): Enlargement Theory
- M.P. Gualdani, S. Mischler, C. Mouhot (2017): strong solution for (B^{ε}) when $\varepsilon = 1$ and $\|f_{\text{in}}\| \ll 1$
- M. Briant, S. Merino, C. Mouhot (2019): weak hydrodynamic limit

Proof of the theorem

Gualdani–Mischler–Mouhot decomposition

Grad's decomposition: $\mathcal{L} = -\nu(v) + K$

$$\nu(v) \approx \langle v \rangle \textit{ and } \left\| \langle v \rangle^{k+1} K g \right\|_{\mathbf{G}} \lesssim \left\| \langle v \rangle^{k} g \right\|_{\mathbf{G}}$$

GMM decomposition: $\mathcal{L} = \mathcal{B} + \mathcal{A}$

$$\mathcal{B} = -\nu + \text{small non loc.op.} \textit{ and } \|\mathcal{A}h\|_{\mathbf{G}} \lesssim \|h\|_{\mathbf{P}}$$

$$\mathcal{A}f(v) := \int \Theta \left(M'_* f' + M' f'_* - M f_* \right) |v - v_*| \mathrm{d}v_* \mathrm{d}\sigma,$$

$$\Theta \in \mathcal{C}_c^\infty$$

Proof of the theorem

Splitting of the equation

- Use the GMM splitting $\mathcal{L} = \mathcal{B} + \mathcal{A}$ (where $\mathcal{A} : \mathbf{P} \xrightarrow{\text{bded.}} \mathbf{G}$)
- Write $f^\varepsilon(t) = h^\varepsilon(t) + g^\varepsilon(t) \in \mathbf{P} + \mathbf{G}$

$$\partial_t f^\varepsilon = \frac{1}{\varepsilon^2}\big(\mathcal{L} + \varepsilon v \cdot \nabla_x\big) f^\varepsilon + \frac{1}{\varepsilon} Q(f^\varepsilon, f^\varepsilon),\ f^\varepsilon_{|t=0} = f_{\text{in}}$$

$$\Uparrow$$

$$\mathbf{P} \ : \ \partial_t h^\varepsilon = \frac{1}{\varepsilon^2}\left(\mathcal{B} + \varepsilon v \cdot \nabla_x\right) h^\varepsilon + \frac{1}{\varepsilon} Q(h^\varepsilon, h^\varepsilon + 2 g^\varepsilon),\ h^\varepsilon_{|t=0} = f_{\text{in,mic}}$$

$$\mathbf{G} \ : \ \partial_t g^\varepsilon = \frac{1}{\varepsilon^2}\left(\mathcal{L} + \varepsilon v \cdot \nabla_x\right) g^\varepsilon \underbrace{+ \frac{1}{\varepsilon^2}\mathcal{A} h^\varepsilon}_{\in \mathbf{G}} + \frac{1}{\varepsilon} Q(g^\varepsilon, g^\varepsilon),\ g^\varepsilon_{|t=0} = f_{\text{in,mac}}$$

Proof of the theorem

Splitting of the equation

- Use the GMM splitting $\mathcal{L} = \mathcal{B} + \mathcal{A}$ (where $\mathcal{A} : \mathbf{P} \xrightarrow{\text{bded.}} \mathbf{G}$)
- Write $f^\varepsilon(t) = h^\varepsilon(t) + g^\varepsilon(t) \in \mathbf{P} + \mathbf{G}$

$$\mathbf{P} \ : \ \partial_t \mathbf{h}^\varepsilon = \frac{1}{\varepsilon^2}\left(\mathcal{B} + \varepsilon v \cdot \nabla_x\right)\mathbf{h}^\varepsilon + \frac{1}{\varepsilon} Q(h^\varepsilon, h^\varepsilon + 2 g^\varepsilon), \ \mathbf{h}^\varepsilon_{|t=0} = f_{\text{in,mic}}$$

$$\mathbf{G} \ : \ \partial_t \mathbf{g}^\varepsilon = \frac{1}{\varepsilon^2}\left(\mathcal{L} + \varepsilon v \cdot \nabla_x\right)\mathbf{g}^\varepsilon \underbrace{+ \frac{1}{\varepsilon^2}\mathcal{A} h^\varepsilon}_{\in \mathbf{G}} + \frac{1}{\varepsilon} Q(g^\varepsilon, g^\varepsilon), \ \mathbf{g}^\varepsilon_{|t=0} = f_{\text{in,mac}}$$

$$T \ : X_{t,x,v} \ \to X_{t,x,v}, \ (h^\varepsilon, g^\varepsilon) \mapsto (\mathbf{h}^\varepsilon, \mathbf{g}^\varepsilon),$$
$$X = \{\text{functions } h^\varepsilon(t) \in \mathbf{P}\} \times \{\text{functions } g^\varepsilon(t) \in \mathbf{G}\}$$

- Fixed point on $T \to$ need contraction estimates = stability/a priori estimate for the equations

Proof of the theorem

Control of the polynomial part

$$\partial_t \mathbf{h}^\varepsilon = \frac{1}{\varepsilon^2}\left(\mathcal{B} + \varepsilon v \cdot \nabla_x\right)\mathbf{h}^\varepsilon + \frac{1}{\varepsilon} Q(h^\varepsilon, h^\varepsilon + 2g^\varepsilon),$$

- Energy inequality:

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\mathbf{h}^\varepsilon(t)\|_{\mathbf{P}} \leq -\frac{\Lambda}{\varepsilon^2}\|\langle v\rangle \mathbf{h}^\varepsilon(t)\|_{\mathbf{P}} + \frac{M}{\varepsilon}\|\langle v\rangle h^\varepsilon(t)\|_{\mathbf{P}}\|h^\varepsilon(t)\|_{\mathbf{P}} + (\dots)$$

- Grönwall for some $\lambda \in (0, \Lambda)$:

$$\sup_{0 \leq t < T}\left(e^{\lambda t/\varepsilon^2}\|\mathbf{h}^\varepsilon(t)\|_{\mathbf{P}} + \frac{\Lambda - \lambda}{\varepsilon^2}\int_0^t e^{\lambda t'/\varepsilon^2}\|\langle v\rangle \mathbf{h}^\varepsilon(t')\|_{\mathbf{P}}\,\mathrm{d}t'\right)$$
$$=: |\!|\!| \mathbf{h}^\varepsilon |\!|\!|_{\mathbf{P}^\varepsilon} \leq C\varepsilon |\!|\!| h^\varepsilon |\!|\!|_{\mathbf{P}^\varepsilon}\left(|\!|\!| h^\varepsilon |\!|\!|_{\mathbf{P}^\varepsilon} + \|g^\varepsilon\|_{L^\infty_t \mathbf{G}}\right) + \|f_{\mathrm{in,mic}}\|_{\mathbf{P}}$$

- Stability estimate (same initial condition)

$$|\!|\!| \mathbf{h}_1^\varepsilon - \mathbf{h}_2^\varepsilon |\!|\!|_{\mathbf{P}^\varepsilon} \lesssim \varepsilon\left(|\!|\!| h_1^\varepsilon - h_2^\varepsilon |\!|\!|_{\mathbf{P}^\varepsilon} + \|g_1^\varepsilon - g_2^\varepsilon\|_{L^\infty_t \mathbf{G}}\right) \times (\dots)$$

Proof of the theorem

Study of the Gaussian part

$$\partial_t g^\varepsilon = \frac{1}{\varepsilon^2}(\mathcal{L} + \varepsilon v\cdot\nabla_x) g^\varepsilon + \frac{1}{\varepsilon} Q(g^\varepsilon, g^\varepsilon) + \frac{1}{\varepsilon^2}\mathcal{A} h^\varepsilon,$$

$$\Big\downarrow \text{ Duhamel with } g^\varepsilon_{|t=0} = f_{\text{in,mac}}$$

$$g^\varepsilon = U^\varepsilon f_{\text{in,mac}} + \Psi^\varepsilon(g^\varepsilon, g^\varepsilon) + \frac{1}{\varepsilon^2}\mathcal{A}U * h^\varepsilon,$$

- Already well understood by BU/GT
- ...but convolution term not small in $L^\infty_t \mathbf{G}$

$$U^\varepsilon(t) := \exp\left(\frac{1}{\varepsilon^2}(\mathcal{L} + \varepsilon v \cdot \nabla_x)\right),$$

$$\Psi^\varepsilon(t)(f^\varepsilon, f^\varepsilon) := \frac{1}{\varepsilon}\int_0^t U^\varepsilon(t-t')Q\left(f^\varepsilon(t'), f^\varepsilon(t')\right) \mathrm{d}t'$$

Proof of the theorem

Study of the Gaussian part

Lemma - Convolution splitting (G. 2021)

If $\mathbf{h}^\varepsilon$ is the solution to

$$\begin{cases} \partial_t \mathbf{h}^\varepsilon = \frac{1}{\varepsilon^2}\left(\mathcal{B} + \varepsilon v \cdot \nabla_x\right)\mathbf{h}^\varepsilon + \frac{1}{\varepsilon} Q(h^\varepsilon, h^\varepsilon + 2g^\varepsilon), \\ h^\varepsilon_{|t=0} = f_{\text{in,mic}}, \end{cases}$$

then, uniformly in t and ε,

$$\frac{1}{\varepsilon^2} U^\varepsilon * \mathcal{A}\mathbf{h}^\varepsilon(t) = o(1) + \mathcal{O}\left(e^{-\lambda t/\varepsilon^2}\right)$$

Ingredients of the proof:

- Generalize to $\mathbf{P}$ some properties of U^ε and Ψ^ε known on $\mathbf{G}$
- Duhamel applied to $\mathcal{L} = \mathcal{A} + \mathcal{B}$ and $\mathbf{h}^\varepsilon$ + some algebra:

$$\frac{1}{\varepsilon^2} U^\varepsilon * \mathcal{A}\mathbf{h}^\varepsilon(t) = U^\varepsilon(t) f_{\text{in,mic}} + \dots$$

Proof of the theorem

Study of the Gaussian part

$$g^\varepsilon = U^\varepsilon f_{\text{in,mac}} + \Psi^\varepsilon(g^\varepsilon, g^\varepsilon) + \frac{1}{\varepsilon^2} U^\varepsilon * \mathcal{A} h^\varepsilon,$$

$$\Big\downarrow \; \overline{g}^\varepsilon := g^\varepsilon - f^0 - \mathcal{O}\left(e^{-\lambda t/\varepsilon^2}\right)$$

$$\overline{g}^\varepsilon = o(1) + \underbrace{\{\text{Linear}\}}_{\text{contraction}}(\overline{g}^\varepsilon) + \underbrace{\{\text{Bilinear}\}}_{\text{bounded}}(\overline{g}^\varepsilon, \overline{g}^\varepsilon),$$

- {Linear} and {Bilinear} **depend on $\boldsymbol{f^0}$**
  $\to$ use norm equivalent to $\| \cdot \|_{L^\infty_t \mathbf{G}}$ $\to$ {Linear} is a contraction
- ... **and on $\boldsymbol{h^\varepsilon}$** $\to$ generalize to $\mathbf{P}$ some estimates/convergence on U^ε and Ψ^ε known on $\mathbf{G}$.

Thank you for your attention!