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From Boltzmann to Navier–Stokes with slowly decaying initial data

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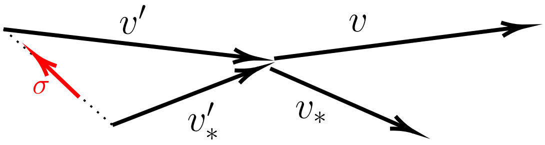
Introduction

The scaled Boltzmann equation

- $F(t,x,v)$ = particles number density, $x \in \Omega = \mathbb{R}^d, \mathbb{T}^d$, $d = 2,3$
- Kn = mean free path, Ma = Mach number

$$\mathrm{Ma}\,\partial_t F + v\cdot\nabla_x F = \frac{1}{\mathrm{Kn}} Q(F,F),$$

$$Q(F,G)(v) = \int_{\mathbb{R}^d_{v_*}\times\mathbb{S}^{d-1}_\sigma} |v-v_*|\Big(F(v')G(v'_*) - F(v)G(v_*)\Big)\mathrm{d}v_*\mathrm{d}\sigma$$



$$v' = \frac{v+v_*}{2} + \sigma\frac{|v-v_*|}{2}, \quad v'_* = \frac{v+v_*}{2} - \sigma\frac{|v-v_*|}{2}$$

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The scaled Boltzmann equation

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- Kn = mean free path= Ma = Mach number= $\varepsilon \ll 1$

$$\varepsilon \partial_t F^\varepsilon + v \cdot \nabla_x F^\varepsilon = \frac{1}{\varepsilon} Q(F^\varepsilon, F^\varepsilon),$$

Conserved macroscopic observables:

- Mass : $R^\varepsilon := \int F^\varepsilon \, \mathrm{d}v$
- Momentum : $R^\varepsilon U^\varepsilon := \int F^\varepsilon \, v \, \mathrm{d}v$
- Energy : $\underbrace{\frac{1}{2} R^\varepsilon |U|^2}_{\text{kinetic}} + \underbrace{\frac{d}{2} R^\varepsilon T^\varepsilon}_{\text{thermal}} := \int F^\varepsilon \, \frac{|v|^2}{2} \, \mathrm{d}v$

Introduction

Relation with the incompressible Navier–Stokes–Fourier system

Gas at thermodynamic equilibrium (constant heat, mass density, at rest) :

$$M = (2\pi)^{-d/2} \exp\big(-|v|^2/2\big),\ Q(M, M) = 0,$$

Statistical fluctuation of order ε:

$$F^\varepsilon_{|t=0} = M + \varepsilon f_{\mathrm{in}},\ F^\varepsilon = M + \varepsilon f^\varepsilon,$$

Macroscopic fluctuations of order ε:

- Mass: $R^\varepsilon(t,x) = \int (M + \varepsilon f^\varepsilon)\,\mathrm{d}v = 1 + \varepsilon \rho^\varepsilon(t,x)$
- Velocity: $U^\varepsilon(t,x) = (\dots) = 0 + \varepsilon u^\varepsilon(t,x)$
- Temperature: $T^\varepsilon(t,x) = (\dots) = 1 + \varepsilon \theta^\varepsilon(t,x)$

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$$F^\varepsilon_{|t=0} = M + \varepsilon f_{\text{in}},\ F^\varepsilon = M + \varepsilon f^\varepsilon,$$

Equation on f^ε:

$$\varepsilon \partial_t F^\varepsilon + v \cdot \nabla_x F^\varepsilon = \frac{1}{\varepsilon} Q(F^\varepsilon, F^\varepsilon),$$

$$\Downarrow$$

$$\varepsilon \partial_t \left(M + \varepsilon f^\varepsilon\right) + v \cdot \nabla_x \big(M + \varepsilon f^\varepsilon\big) = \frac{1}{\varepsilon} Q(M + \varepsilon f^\varepsilon, M + \varepsilon f^\varepsilon),$$

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Equation on f^ε:

$$\varepsilon \partial_t F^\varepsilon + v \cdot \nabla_x F^\varepsilon = \frac{1}{\varepsilon} Q(F^\varepsilon, F^\varepsilon),$$

$$\Downarrow \quad Q(M,M) = 0$$

$$\partial_t f^\varepsilon = \frac{1}{\varepsilon^2}(\mathcal{L} + \varepsilon v \cdot \nabla_x) f^\varepsilon + \frac{1}{\varepsilon} Q(f^\varepsilon, f^\varepsilon)$$

where

$$\mathcal{L} := Q(M, \cdot) + Q(\cdot, M)$$

Introduction

Relation with the incompressible Navier–Stokes–Fourier system

Definition (microscopic, macroscopic)

- f is **macroscopic**
  $\overset{\text{def}}{\equiv} f(x,v) = \Big(1\rho(x) + u(x)\cdot v + \frac{1}{2}(|v|^2 - d)\theta(x)\Big)M(v)$
- f is **well-prepared** $\overset{\text{def}}{\equiv} \begin{cases} f \text{ is macroscopic,} \\ \nabla_x \cdot u(x) = 0, \\ \rho(x) + \theta(x) = 0 \end{cases}$
- f is **microscopic** $\overset{\text{def}}{\equiv} \displaystyle\int f(x,v)\varphi(v)\mathrm{d}v = 0,\ \varphi(v) = 1, v, |v|^2$

Remark: Unique decomposition $f = f_{\text{macro}} + f_{\text{micro}}$

$$= f_{\text{well-prepared}} + f_{\text{ill-prepared}} + f_{\text{micro}}$$

Introduction

Weak theory: state of the art

Theorem: weak theory of hydrodynamic limits (1989-2004)

- [DiPerna-Lions] : $\exists$ (at least one) **weak** solution f^ε to Boltz.
- [Golse, Masmoudi, Saint-Raymond...] : $f^\varepsilon \rightharpoonup f^0$
- [Bardos, Golse, Levermore] : $f^0_{|t=0} = f_{\text{in,well-p.}}$ and

$$f^0 = \left(\rho(t,x) + u(t,x)\cdot v + \frac{1}{2}\left(|v|^2 - d\right)\theta(t,x)\right)M(v),$$

$$\begin{cases} \partial_t u + u\cdot\nabla_x u = \mu\Delta_x u - \nabla_x p, \\ \nabla_x \cdot u = 0, \\ \partial_t \theta + u\cdot\nabla_x\theta = \kappa\Delta_x\theta, \\ \rho + \theta = 0, \end{cases}$$

$$\mu = \mu(Q,M) > 0,\ \kappa = \kappa(Q,M) > 0$$

Introduction

Weak theory: state of the art

Question: Strong solutions ? Strong convergence ? What functional space ?

1. Construct strong solutions for $f_{\text{in}} \in \mathbf{X}_{x,v}$
2. Prove strong convergence in $\mathbf{X}_{x,v}$

Existence and convergence of strong solutions

Initial data with Gaussian decay (Bardos–Ukai/Gallagher–Tristani)

Functional space: $\mathbf{G} = L^\infty_v H^s_x \left(M^{-1/2} \langle v \rangle^\beta \mathrm{d}v \right)$,

$$\partial_t f^\varepsilon = \frac{1}{\varepsilon^2}\big(\mathcal{L} + \varepsilon v \cdot \nabla_x\big) f^\varepsilon + \frac{1}{\varepsilon} Q(f^\varepsilon, f^\varepsilon),$$

$$\Big\downarrow \text{ Duhamel with } f^\varepsilon_{|t=0} = f_{\mathrm{in}}$$

$$f^\varepsilon(t) = U^\varepsilon(t) f_{\mathrm{in}} + \Psi^\varepsilon(t)(f^\varepsilon, f^\varepsilon),$$

Where we denote

$$U^\varepsilon(t) := \exp\left(\frac{1}{\varepsilon^2} \left(\mathcal{L} + \varepsilon v \cdot \nabla_x \right) \right),$$

$$\Psi^\varepsilon(t)(f^\varepsilon, f^\varepsilon) := \frac{1}{\varepsilon} \int_0^t U^\varepsilon(t - t') Q\left(f^\varepsilon(t'), f^\varepsilon(t') \right) \mathrm{d}t'$$

Existence and convergence of strong solutions

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$$f^\varepsilon(t) = U^\varepsilon(t) f_{\text{in}} + \Psi^\varepsilon(t)(f^\varepsilon, f^\varepsilon),$$

- Spectral study by Ellis, Pinsky, Ukai (1975-'86) + Fourier
  - uniform bounds on U^ε and Ψ^ε
  - prove $U^\varepsilon \to U^0$, $\Psi^\varepsilon \to \Psi^0$
- Existence of f^ε: Banach's fixed point theorem
  - on f^ε (Bardos-Ukai, 1991) if $\|f_{\text{in}}\|_{\mathbf{G}} \ll 1$
  - on $f^\varepsilon - f^0$ (Gallagher-Tristani, 2019) if $\varepsilon \ll 1$
- Let $\varepsilon \to 0$ in the equation

Existence and convergence of strong solutions

Initial data with polynomial decay (Briant–Merino–Mouhot/G.)

If F^ε =physical gas:

- Mass: $\displaystyle\int F^\varepsilon \, \mathrm{d}v = \|F^\varepsilon\|_{L^1_v} < \infty$
- Energy: $\displaystyle\int F^\varepsilon \, |v|^2 \mathrm{d}v = \big\| |v|^2 F^\varepsilon \big\|_{L^1_v} < \infty$

i.e. $f_{\text{in}} \in L^1\left(\left(1 + |v|^2\right) \mathrm{d}v\right)$

Question: Is it enough for existence/uniqueness ? Convergence ?

Existence and convergence of strong solutions

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Question: Is it enough for existence/uniqueness ? Convergence ?

Partial answer: $\mathbf{P} := L^1_v H^s_x \left(\langle v \rangle^{3+\delta} \mathrm{d}v \right)$

Theorem (G. 2021)

Let $f_{\text{in}} \in \mathbf{P}$, Boltzmann has a **unique strong solution** for $\varepsilon \ll 1$

$$f^\varepsilon \in \mathcal{C}_b \left([0, T); L^1_v H^s_x \left(\langle v \rangle^{3+\delta} \right) \right) \cap L^1_t L^1_v H^s_x \left(\langle v \rangle^{4+\delta} \right),$$

$$T = \text{lifespan of Navier-Stokes } f^0 \text{ with } f^0_{|t=0} = f_{\text{in,well-p.}}$$

Existence and convergence of strong solutions

Initial data with polynomial decay (Briant–Merino–Mouhot/G.)

Theorem (G. 2021)

$$f^\varepsilon = f^0 + u^\varepsilon_\infty + u^\varepsilon_1 + u^\varepsilon_{ac}$$

- Navier-Stokes: f^0 with $f^0_{|t=0} = f_{\text{in,well-p.}}$
- Error term: $\sup_t \|u^\varepsilon_\infty(t)\| \to 0, \ (\varepsilon \to 0)$
- Initial layer: $\|u^\varepsilon_1(t)\| \lesssim e^{-\gamma t/\varepsilon^2} \|f_{\text{in},\perp}\|$
- Acoustic waves: $u^\varepsilon_{ac} \rightharpoonup 0$ and $f_{\text{in,ill-p.}} = 0 \Leftrightarrow u^\varepsilon_{ac} = 0$

Remark: $\Omega = \mathbb{R}^d \Rightarrow \int \|u^\varepsilon_{ac}(t)\|^q \mathrm{d}t \to 0$, with $0 < q < \frac{2}{d-1}$

Existence and convergence of strong solutions

Initial data with polynomial decay (Briant–Merino–Mouhot/G.)

- [Mouhot, '05]: Enlargement Theory
- [Gualdani, Mischler, Mouhot, '17]: strong solution for Boltzman when $\varepsilon = 1$ and $\|f_{\text{in}}\| \ll 1$
- [Briant, Merino, Mouhot, '19]: weak hydrodynamic limit

Proof of the theorem
Strategy

$$f_{\text{in}} = f_{\text{in,well-p.}} + f_{\text{in,ill-p.}} + f_{\text{in},\perp} \rightsquigarrow f^\varepsilon = f^0 + u^\varepsilon_{\text{ac}} + u^\varepsilon_1 + u^\varepsilon_\infty$$

- [Fujita-Kato] $f_{\text{in,well-p.}} \rightsquigarrow f^0$ on $[0, T]$
- [Ellis-Pinsky] Spectral analysis of $\mathcal{L} + v \cdot \nabla_x$: $f_{\text{in,ill-p.}} \rightsquigarrow u^\varepsilon_{\text{ac}}$

[Gualdani, Mischler, Mouhot] Decomposition $\mathcal{L} = \mathcal{B} + \mathcal{A}$:

$$\text{Boltzmann} \longrightarrow \text{Coupled system on } u^\varepsilon_1 \text{ and } u^\varepsilon_\infty$$

- Energy method : $f_{\text{in},\perp} \rightsquigarrow u^\varepsilon_1$
- Fixed point $\rightsquigarrow u^\varepsilon_\infty$: source term $u^\varepsilon_1 \Rightarrow$ need to extends results of Gallagher-Tristani from **G** to **P**

Proof of the theorem

Splitting of the equation

- GMM splitting $\mathcal{L} = \mathcal{B} + \mathcal{A}$

$$\mathcal{B} = -c\langle v\rangle + \text{non-loc. pertu.}\ ,\ \mathcal{A} : \mathbf{P} \to \mathbf{G} \text{ bounded}$$

- $f^\varepsilon(t) = h^\varepsilon(t) + g^\varepsilon(t) \in \mathbf{P} + \mathbf{G}$,
  $h^\varepsilon_{|t=0} = f_{\text{in,mic}} \in \mathbf{P}$, $g^\varepsilon_{|t=0} = f_{\text{in,mac}} \in \mathbf{G}$

$$\partial_t f^\varepsilon = \frac{1}{\varepsilon^2}\big(\mathcal{L} + \varepsilon v\cdot\nabla_x\big) f^\varepsilon + \frac{1}{\varepsilon} Q(f^\varepsilon, f^\varepsilon),\ f^\varepsilon_{|t=0} = f_{\text{in}}$$

$$\Uparrow$$

$$\partial_t h^\varepsilon = \frac{1}{\varepsilon^2}\left(\mathcal{B} + \varepsilon v\cdot\nabla_x\right) h^\varepsilon + \frac{1}{\varepsilon} Q(h^\varepsilon, h^\varepsilon + 2g^\varepsilon),$$

$$\partial_t g^\varepsilon = \frac{1}{\varepsilon^2}\left(\mathcal{L} + \varepsilon v\cdot\nabla_x\right) g^\varepsilon \underbrace{+ \frac{1}{\varepsilon^2}\mathcal{A} h^\varepsilon}_{\in \mathbf{G}} + \frac{1}{\varepsilon} Q(g^\varepsilon, g^\varepsilon),$$

Proof of the theorem

Splitting of the equation

- GMM splitting $\mathcal{L} = \mathcal{B} + \mathcal{A}$
- $f^\varepsilon(t) = h^\varepsilon(t) + g^\varepsilon(t) \in \mathbf{P} + \mathbf{G}$,
  $h^\varepsilon_{|t=0} = f_{\text{in,mic}} \in \mathbf{P}$, $g^\varepsilon_{|t=0} = f_{\text{in,mac}} \in \mathbf{G}$

$$\partial_t f^\varepsilon = \frac{1}{\varepsilon^2}\big(\mathcal{L} + \varepsilon v \cdot \nabla_x\big) f^\varepsilon + \frac{1}{\varepsilon} Q(f^\varepsilon, f^\varepsilon),\ f^\varepsilon_{|t=0} = f_{\text{in}}$$

$$\Uparrow$$

$$\text{Fixed point of } T : (h^\varepsilon, g^\varepsilon) \mapsto \left(\overline{h}^\varepsilon, \overline{g}^\varepsilon\right),$$

$$\partial_t \overline{h}^\varepsilon = \frac{1}{\varepsilon^2}\left(\mathcal{B} + \varepsilon v \cdot \nabla_x\right) \overline{h}^\varepsilon + \frac{1}{\varepsilon} Q(h^\varepsilon, h^\varepsilon + 2g^\varepsilon),$$

$$\partial_t \overline{g}^\varepsilon = \frac{1}{\varepsilon^2}\left(\mathcal{L} + \varepsilon v \cdot \nabla_x\right) \overline{g}^\varepsilon \underbrace{+ \frac{1}{\varepsilon^2} \mathcal{A} h^\varepsilon}_{\in \mathbf{G}} + \frac{1}{\varepsilon} Q(g^\varepsilon, g^\varepsilon)$$

- Need contraction estimates = stability+a priori estimates

Proof of the theorem

Control of the polynomial part

$$\partial_t \overline{h}^\varepsilon = \frac{1}{\varepsilon^2}\left(\mathcal{B} + \varepsilon v \cdot \nabla_x\right) \overline{h}^\varepsilon + \frac{1}{\varepsilon} Q(h^\varepsilon, h^\varepsilon + 2g^\varepsilon),$$

- Energy inequality:

$$\frac{\mathrm{d}}{\mathrm{d}t} \|\overline{h}^\varepsilon(t)\|_{\mathbf{P}} + \frac{1}{\varepsilon^2} \|\langle v \rangle \overline{h}^\varepsilon(t)\|_{\mathbf{P}} \lesssim \frac{1}{\varepsilon} \|\langle v \rangle h^\varepsilon(t)\|_{\mathbf{P}} \|h^\varepsilon(t)\|_{\mathbf{P}} + (\dots)$$

- Grönwall:

$$\sup_{0 \le t < T} \left(e^{t/2\varepsilon^2} \|\overline{h}^\varepsilon(t)\|_{\mathbf{P}} + \frac{1}{2\varepsilon^2} \int_0^t e^{t'/2\varepsilon^2} \|\langle v \rangle \overline{h}^\varepsilon(t')\|_{\mathbf{P}} \, \mathrm{d}t' \right)$$

$$=: \left|\!\left|\!\left| \overline{h}^\varepsilon \right|\!\right|\!\right|_{\mathbf{P}^\varepsilon} \lesssim \varepsilon |\!|\!| h^\varepsilon |\!|\!|_{\mathbf{P}^\varepsilon} \left(|\!|\!| h^\varepsilon |\!|\!|_{\mathbf{P}^\varepsilon} + \|g^\varepsilon\|_{L^\infty_t \mathbf{G}} \right) + \|f_{\mathrm{in,mic}}\|_{\mathbf{P}}$$

- Stability estimate (same initial condition)

$$\left|\!\left|\!\left| \overline{h}_1^\varepsilon - \overline{h}_2^\varepsilon \right|\!\right|\!\right|_{\mathbf{P}^\varepsilon} \lesssim \varepsilon \left(|\!|\!| h_1^\varepsilon - h_2^\varepsilon |\!|\!|_{\mathbf{P}^\varepsilon} + \|g_1^\varepsilon - g_2^\varepsilon\|_{L^\infty_t \mathbf{G}} \right) \times (\dots)$$

Proof of the theorem

Study of the Gaussian part

$$\partial_t g^\varepsilon = \frac{1}{\varepsilon^2}(\mathcal{L} + \varepsilon v \cdot \nabla_x) g^\varepsilon + \frac{1}{\varepsilon} Q(g^\varepsilon, g^\varepsilon) + \frac{1}{\varepsilon^2} \mathcal{A} h^\varepsilon,$$

$$\Big\downarrow \text{ Duhamel with } g^\varepsilon_{|t=0} = f_{\text{in,mac}}$$

$$g^\varepsilon = U^\varepsilon f_{\text{in,mac}} + \Psi^\varepsilon(g^\varepsilon, g^\varepsilon) + \frac{1}{\varepsilon^2} \mathcal{A} U * h^\varepsilon,$$

- Already well understood by BU/GT
- ...but convolution term not small in $L^\infty_t \mathbf{G}$

$$U^\varepsilon(t) := \exp\left(\frac{1}{\varepsilon^2}(\mathcal{L} + \varepsilon v \cdot \nabla_x)\right),$$

$$\Psi^\varepsilon(t)(f^\varepsilon, f^\varepsilon) := \frac{1}{\varepsilon} \int_0^t U^\varepsilon(t - t') Q\left(f^\varepsilon(t'), f^\varepsilon(t')\right) \mathrm{d}t'$$

Proof of the theorem

Study of the Gaussian part

Lemma - Convolution splitting (G. 2021)

$$\begin{cases} \partial_t \overline{h}^\varepsilon = \frac{1}{\varepsilon^2}\left(\mathcal{B} + \varepsilon v \cdot \nabla_x\right) \overline{h}^\varepsilon + \frac{1}{\varepsilon} Q(h^\varepsilon, h^\varepsilon + 2g^\varepsilon), \\ \overline{h}^\varepsilon_{|t=0} = f_{\text{in,mic}}, \end{cases}$$

$$\Downarrow$$

$$\frac{1}{\varepsilon^2} U^\varepsilon * \mathcal{A}\overline{h}^\varepsilon(t) = o(1) + \mathcal{O}\left(e^{-t/\varepsilon^2}\right) \in \mathbf{G}$$

Idea of proof: Duhamel on $\mathcal{B} = \mathcal{L} - \mathcal{A}$ and equation for $\overline{h}^\varepsilon$

$$\left.\begin{aligned} \frac{1}{\varepsilon^2} U^\varepsilon * \mathcal{A} V^\varepsilon &= U^\varepsilon - V^\varepsilon \\ \overline{h}^\varepsilon &= V^\varepsilon f_{\text{in,mic}} + \dots \end{aligned}\right\} \Rightarrow \frac{1}{\varepsilon^2} U^\varepsilon * \mathcal{A}\overline{h}^\varepsilon(t) = U^\varepsilon(t) f_{\text{in,mic}} + \dots$$

$$V^\varepsilon = \exp\left(t\varepsilon^{-2}\left(\mathcal{B} + \varepsilon v \cdot \nabla_x\right)\right)$$

Requirements: generalize behavior of U^ε on micro. fluctuations $\in \mathbf{P}$

Proof of the theorem

Study of the Gaussian part

$$g^\varepsilon = U^\varepsilon f_{\text{in,mac}} + \Psi^\varepsilon(g^\varepsilon, g^\varepsilon) + \frac{1}{\varepsilon^2} U^\varepsilon * \mathcal{A} h^\varepsilon,$$

$$\Big\downarrow \; u_\infty^\varepsilon := g^\varepsilon - f^0 - u_{\text{ac}}^\varepsilon - \mathcal{O}\left(e^{-t/\varepsilon^2}\right)$$

$$u_\infty^\varepsilon = o(1) + \underbrace{\{\text{Linear}\}}_{\text{contraction}}(u_\infty^\varepsilon) + \underbrace{\Psi^\varepsilon}_{\text{bounded}}(u_\infty^\varepsilon, u_\infty^\varepsilon),$$

Problem: {Linear} depends on f^0 **which may be large**

Solution: equivalent norm $\rightarrow$ {Linear} is a contraction

Possible extensions

- Other kinetic models: (Landau, Boltzmann with hard potentials...)
- Other limiting hydrodynamic models

Thank you for your attention!