About stability of equilibria for the Vlasov-Fokker-Planck with general potentials

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Self-consistent Vlasov-Fokker-Planck

Consider a system of particles $\mathbb{R}^d \times \mathbb{R}^d$, described at time $t \geq 0$ by its phase-space distribution function $F(t, x, v)$, satisfying

$$\partial_t F + v \cdot \nabla_x F - \nabla_x (\Psi_F + V) \cdot \nabla_v F = \nabla_v \cdot (vF + \nabla_v F)$$

- Particles are moving in space
- Random fluctuations and damping of the velocity (Fokker-Planck)
- Particles localized in a region of space by an outside force $\nabla_x V$
- Particle at y affects particle at x with a force $\nabla_x k(x - y)$

$$\Psi_F(x) = \int_{\mathbb{R}^d} k(x-y)\rho_F(y)\mathrm{d}y\,, \qquad \rho_F(x) = \int_{\mathbb{R}^d} F(x,v)\mathrm{d}v\,.$$

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Why this equation is interesting/hard at first glance:

- Degeneracy: diffusion in v only and vanishes for $F = \rho(t,x)e^{-|v|^2/2}$
- Non-linearity is non-local

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A less obvious reason:

- Phase transition in the strongly non-linear (large mass) regime

$$\partial_t G + v \cdot \nabla_x G - \nabla_x (M\Psi_G + V) \cdot \nabla_v G = \nabla_v \cdot (vG + \nabla_v G)$$

where $M = \int_{\mathbb{R}^{2d}} F(t)\mathrm{d}x\mathrm{d}v$ is the (conserved) mass and $G = F/M$

Interaction potential

Even and odd parts of the interaction kernel:

$$k^{\mathrm{e}}(x) = \frac{k(x) + k(-x)}{2}, \quad k^{\mathrm{o}}(x) = \frac{k(x) - k(-x)}{2}, \quad k = k^e + k^o.$$

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"Well behaved" example (unique attractive steady state)

- In 3D plasma physics: k is symmetric with positive Fourier modes

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"Degenerate" examples (non-unique or unstable steady states)

- In particle accelerator physics: k is non-symmetric and $k \in W^{1,\infty}$:
- Kuramoto $k = -\cos(\omega x)$: k is symmetric but $\widehat{k}$ is negative

Positive symmetric potentials: example of a 3D plasma

3D Vlasov-Poisson-Fokker-Planck (Coulomb potential $k(x) \propto \lambda^{-2}|x|^{-1}$)

$$\begin{cases} \partial_t F + v \cdot \nabla_x F - \nabla_x (\Psi_F + V) \cdot \nabla_v F = \nabla_v \cdot (vF + \nabla_v F), \\ -\lambda^2 \Delta \Psi_F(t,x) = \rho_F, \\ F|_{t=0} = F_{\mathrm{in}}. \end{cases}$$

Theorem (Bouchut, Dolbeault '95 : <u>unconditional</u> cvg)

Assume that F_{in} satisfies physical bounds (mass, entropy, total energy) and $\nabla \Psi_F \in L^\infty_{t\,loc} L^\infty_x$, then

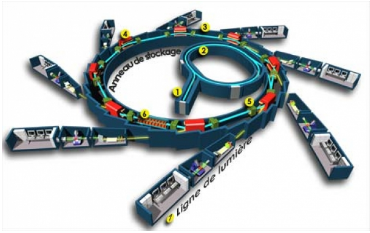
$$F(t) \xrightarrow{t \to \infty} F_\star \quad in \quad L^1(\mathbb{R}^3_x \times \mathbb{R}^3_v),$$

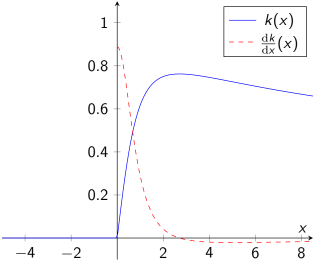
where $F_\star$ is the unique steady state.

Quantitative exponential convergence rate:

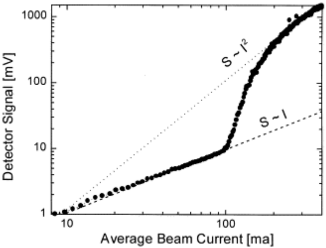
- [Hérau, Thomann '16] (weakly nonlinear $\lambda \gg 1$)
- [Toshpulatov, '23], [Gervais, Herda, '24] (strongly nonlinear $\lambda \ll 1$)

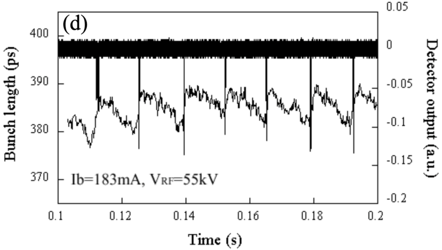
Asymmetric potentials: example of a particle accelerator





High currents (large mass) ⇒ cyclical/instable behavior (microbunching)





[Roussel, PhD, '14], [Evain et al., Nature Physics '19]

Assumptions on the confining potential

Assumption on the confinement: (eg. $V(x) \approx |x|^a$ with $a > 1$)

We make the following regularity assumption for any $\varepsilon \in (0, 1)$:

$$\left(1 + |\nabla V|^2\right) e^{-V} \in L^1 \cap L^\infty\,, \qquad |\nabla^2 V(\cdot)| \leq \varepsilon |\nabla V(\cdot)| + C_\varepsilon\,,$$

and assume the measure $\mathrm{d}\mu = e^{-V} \mathrm{d}x$ admits a Poincaré inequality:

$$\int_{\mathbb{R}^d} |u|^2 \,\mathrm{d}\mu - \left(\int_{\mathbb{R}^d} u \mathrm{d}\mu\right)^2 \lesssim \int_{\mathbb{R}^d} |\nabla_x u|^2 \mathrm{d}\mu\,.$$

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Assumption on the interactions:

The interaction operator is regularizing: for $p \in [2,\infty]$ and $q \in (d,\infty]$

$$\|k^\alpha * \rho\|_{L^p} + \|\nabla k^\alpha * \rho\|_{L^q} \le \overline{\kappa}^\alpha \|\rho\|_{L^1 \cap L^2}, \qquad \alpha = e, o\,.$$

The interaction kernel has bounded negative Fourier modes

$$\langle k * \rho, \rho \rangle \ge -\underline{\kappa}^{\mathrm{e}} \|\rho\|^2_{L^1 \cap L^2}, \quad \forall \rho \in L^1 \cap L^2 \text{ s.t. } \int \rho = 0.$$

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Quantitative local asymptotic stability

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Existence and uniqueness: *The equation has at least one equilibrium, which is unique if the interactions are almost positive* ($\underline{\kappa}^e \ll 1$).

Stability: *If furthermore the interactions are almost symmetric* ($\overline{\kappa}^o \ll 1$), *it is stable: for any* $s \in [0,1]$ *and intial datum such that*

$$s > s_c := \frac{3}{2}\left(\frac{d}{q} - \frac{1}{3}\right), \qquad \|F_{in} - F_\star\|_{H^s_x L^2_v(F_\star^{-1})} \ll 1,$$

VFP has a unique solution $F \in \mathcal{C}(\mathbb{R}^+; H^s_x L^2_v\left(F_\star^{-1}\right))$, *and*

$$\|F(t) - F_\star\|_{H^s_x L^2_v(F_\star^{-1})} \lesssim \|F_{in} - F_\star\|_{H^s_x L^2_v(F_\star^{-1})} e^{-\lambda t},$$

and is instantly H^1 *in space:*

$$\|F(t) - F_\star\|_{H^1_x L^2_v(F_\star^{-1})} \lesssim \|F_{in} - F_\star\|_{H^s_x L^2_v(F_\star^{-1})} t^{-\frac{3}{2}(1-s)} e^{-\lambda t}.$$

Every constant is constructive and symmetric part can be large ($\overline{\kappa}^e \gg 1$).

Corollary: Vlasov-Poisson-FP $k * (\cdot) = (-\lambda^2 \Delta)^{-1}$

Hypotheses on the potential

- Regularity on $k, \nabla k$: Hardy-Littlewood-Sobolev or elliptic regularity.
- Valid for $\lambda \ll 1$ and $\lambda \gg 1$:

$$k * (\cdot) = k^e * (\cdot) = \left(-\lambda^2 \Delta\right)^{-1} \geq 0 \quad \Rightarrow \quad \overline{\kappa}^o = \underline{\kappa}^e = 0$$

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Consequences of our result

- Constructive estimates but constants degenerate as $\lambda \to 0$
- Regularity on initial data $H^{\frac{1}{2}+}_{x,v}$ [Hérau, Thomann '16], [Toshpulatov '23] lowered to $H^{\frac{1}{4}+}_x L^2_v$ (in particular, no regularity in v)

Stability analysis: A natural Hilbert norm

The free energy functional

$$\mathcal{F}[F] = \int F(x,v)\left(\underbrace{\frac{|v|^2}{2}}_{\substack{\text{kinetic}\\ \text{energy}}} + \underbrace{V(x)}_{\substack{\text{confinement}\\ \text{energy}}} + \underbrace{\Psi_F(x)}_{\substack{\text{interaction}\\ \text{energy}}} + \underbrace{\log F(x,v)}_{\text{entropy}} \right) \mathrm{d}x\mathrm{d}v$$

is a Lyapunov functional for symmetric interactions $(\overline{\kappa}^o = 0)$

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{F}[F] + \mathcal{D}[F] = \mathcal{O}\left(\overline{\kappa}^o\right), \qquad \mathcal{D}[F] \geq 0\,.$$

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Functional framework for stability: fluctuation f and Hilbert norm

$$F = F_\star(1+f)\,, \qquad \mathcal{F}[F] \approx \mathrm{d}^2\mathcal{F}[F_\star].(F_\star f, F_\star f) =: |||f|||^2$$

$$|||f||| = \left(\int F_\star(x,v) f(x,v)^2\,\mathrm{d}x\mathrm{d}v + \int k^e * \rho_f(x)\rho_f(x)\,\mathrm{d}x\right)^{1/2}.$$

where $|||\cdot|||$ is well defined because k^e is almost positive $(\underline{\kappa}^e \ll 1)$.

Idea to use $|||\cdot|||$ originally from [Addala, Dolbeault, Li, Tayeb, '19].

Stability analysis: hypocoercivity

The fluctuation f satisfies

$$\partial_t f + Tf = Lf + \mathcal{O}(\overline{\kappa}^o)\,, \quad \text{where} \quad L \leq 0\,, \quad T^* = -T$$

- Problem: $\ker(L) \neq 0 \Rightarrow$ incomplete energy estimate
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2D toy-model for hypocoercivity

$$\frac{\mathrm{d}y}{\mathrm{d}t} = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix} y$$

$$\text{Eigenvalues} = -\frac{1}{2} \pm i\frac{\sqrt{3}}{2}$$

- Incomplete energy estimate $\frac{\mathrm{d}}{\mathrm{d}t}|y(t)|^2 = -2y_2^2(t) \not\Rightarrow$ decay $\mathcal{O}\left(e^{-t/2}\right)$
- Introduce the equivalent (squared) norm ($|\eta| < 1$)

$$H(y) = y_1^2 + y_2^2 + 2\eta y_1 y_2 \quad \Rightarrow \quad \frac{\mathrm{d}}{\mathrm{d}t}H(y(t)) + H(y(t)) \leq 0\,.$$

Stability analysis: linear study

Exponential decay: DMS strategy [Dolbeault Mouhot, Schmeiser, '15]:

$$A = A(T, \Pi_{\ker(L)}), \qquad \mathcal{E}(f) := |||f|||^2 + \eta \langle\langle Af, f \rangle\rangle \approx \|f\|^2_{L^2_{x,v}(F_\star)}$$

We recover exponential decay for $\overline{\kappa}^o \ll 1$:

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}(f) + \mu\,\mathcal{E}(f) \lesssim \overline{\kappa}^o \mathcal{E}(f) \Rightarrow \|f(t)\|_{L^2_{x,v}(F_\star)} \lesssim e^{-\lambda t}\|f_{\text{in}}\|_{L^2_{x,v}(F_\star)}$$

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Regularization estimate: Hypoellipticity strategy of Hérau,Villani... :

$$\mathcal{H}(f) := \mathcal{E}(f) + \alpha_1(t)\|\nabla_v f\|^2 + \alpha_2(t)\,\langle \nabla_x f, \nabla_v f\rangle + \alpha_3(t)\|\nabla_x f\|^2$$

For the right $\alpha_i(t)$ with $\alpha_i(0) = 0$, uniform regularization estimate:

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{H}(f) \le 0 \quad \Rightarrow \quad \|\nabla_x f\|_{L^2_{x,v}(F_\star)} \lesssim t^{-3/2} e^{-\lambda t}\|f_{\text{in}}\|_{L^2_{x,v}(F_\star)}$$

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Interpolation: Combine all estimates by interpolation for $s \in [0,1]$:

$$\|f(t)\|_{H^s_x L^2_v(F_\star)} + t^{\frac{3}{2}(1-s)}\|f(t)\|_{H^1_x L^2_v(F_\star)} \lesssim e^{-\lambda t}\|f_{\mathrm{in}}\|_{H^s_x L^2_v(F_\star)}$$

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Stability analysis: source term and nonlinear estimates

VFP with a source: $\partial_t f + Tf = Lf + \mathcal{O}(\overline{\kappa}^o) - (\nabla_v - v)\varphi$ measured by

$$\|\varphi\|_{\mathbb{H}^s}^2 := \int_0^\infty e^{2\lambda t} \left(t^{3(1-s)} \|\varphi\|_{H_x^1 L_v^2(F_\star)}^2 + \dots \right) \mathrm{d}t$$

where $\varphi = f \nabla_x \psi_f$ in the original perturbation equation.

Proposition

For any given $s \in [0,1]$ there holds

$$\|f\|_{\mathcal{X}^s} \lesssim \|f_{in}\|_{H_x^s L_v^2(F_\star)} + \|\varphi\|_{\mathbb{H}^s}.$$

If additionally $s > s_c := \frac{3}{2}\left(\frac{d}{q} - \frac{1}{3}\right)$, then

$$\|f \nabla_x \psi_g\|_{\mathbb{H}^s} \lesssim \|f\|_{\mathcal{X}^s} \|g\|_{\mathcal{X}^s}.$$

Taking $\varphi = f \nabla_x \psi_f$ and f_{in} small $\Rightarrow \exists!$ solution $f \in \mathcal{X}^s$ by fixed point.

- Phase transition in the strongly non-linear regime

  Example: Kuramoto $k(x) = -\underline{\kappa}^e \cos(\omega x) \Rightarrow$ negative modes at $\pm\omega$

  Three steady states for $\underline{\kappa}^e \gg 1$

  Q1: Stability/instability ?

  Q2: Infinite modes ?

  Q3: Non-symmetric k ?

  Q4: Numerics ?

  See [Carrillo et al. '20] for the torus $x \in \mathbb{R}/\mathbb{Z}$ with $V = 0$.
- Diffusive approximation: long-time and strong randomness/damping
- Numerical schemes for McKean-Vlasov

Perspectives

- Phase transition in the strongly non-linear regime
- Diffusive approximation: long-time and strong randomness/damping

$$\varepsilon\partial_t F^\varepsilon + v \cdot \nabla_x F^\varepsilon - \nabla_x(\Psi_{F^\varepsilon} + V) \cdot \nabla_v F = \frac{1}{\varepsilon}\nabla_v \cdot (vF^\varepsilon + \nabla_v F^\varepsilon)$$

Then $F^\varepsilon(t,x,v) \xrightarrow{\varepsilon \to 0} \rho(t,x)e^{-|v|^2/2}$ where

$$\partial_t \rho - \nabla_x \cdot (\nabla_x \rho + \rho \nabla(\psi_\rho + V)) = 0 \qquad \text{(McKean-Vlasov)}$$

NB: Same steady state equation for MV and VFP

- Numerical schemes for McKean-Vlasov

Thank you for your attention!

[G, Herda., *Well-posedness and long-time behavior for self-consistent Vlasov-Fokker-Planck equations with general potentials*. arXiv:2408.16468]