Université de Paris

Hydrodynamic limits

From Boltzmann to Navier-Stokes

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ENS | PSL

Different levels of description

Macroscopic level

Time t, point x in space

- mass : $R_t(x) \geq 0$
- velocity : $U_t(x) \in \mathbb{R}^3$
- temperature : $T_t(x) \geq 0$
- viscosy, pressure, thermal conductivity...

Example (Incompressible Navier-Stokes)

$$\begin{cases} \partial_t U + U \cdot \nabla_x U = \nu \Delta_x U - \nabla_x P, \\ \mathrm{div}_x U = 0 \end{cases}$$

Different levels of description

Microscopic level

$N \approx 10^{26}$ particules, at position $x_i(t) \in \mathbb{R}^3$, velocities $v_i(t) \in \mathbb{R}^3$

$$\begin{cases} \dot{x}_i = v_i \\ \dot{v}_i = \text{interactions} \end{cases}$$

Different levels of description

"Mesoscopic" point of view

Kinetic theory of gases: micro. **statistical** behavior → macro. phenomena

$$\int_{\mathcal{V}_1 \times \mathcal{V}_2} F_t(x,v)\mathrm{d}x\mathrm{d}v = \begin{array}{c}\text{Nb of particules at position } x \in \mathcal{V}_1 \\ \text{and velocity } v \in \mathcal{V}_2\end{array}$$

- Mass: $R_t(x) = \int F_t(x,v)\mathrm{d}v$
- Momentum: $R_t(x)U_t(x) = \int F_t(x,v)\, v\, \mathrm{d}v$
- Energy : $\underbrace{\frac{1}{2}R_t(x)|U_t(x)|^2}_{\text{kinetic}} + \underbrace{\frac{3}{2}R_t(x)T_t(x)}_{\text{thermal}} = \int \underbrace{F_t(x,v)\frac{|v|^2}{2}}_{\text{kinetic energy of each particule}} \mathrm{d}v$

Different levels of description

"Mesoscopic" point of view

- 1860 : Maxwell distribution law

$$F_t(x,v) = \frac{R_t(x)}{\big(2\pi T_t(x)\big)^{3/2}} \exp\left(- \frac{|v - U_t(x)|^2}{2T_t(x)} \right) \qquad \text{(LTE)}$$

- 1872 : Boltzmann equation:

$$\partial_t F_t + v \cdot \nabla_x F_t = \begin{array}{c}\text{variation of number of} \\ \text{particules with velocity } v\end{array}$$

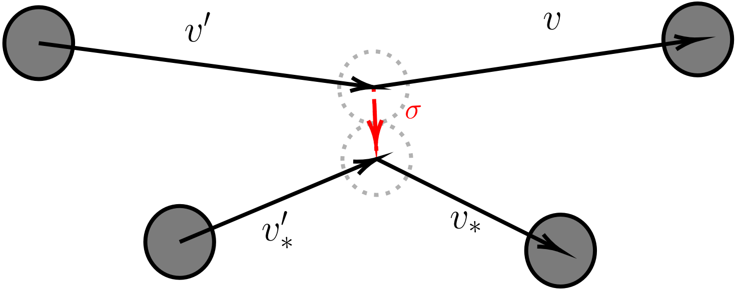
$$=: Q\Big(F_t(x,\cdot), F_t(x,\cdot)\Big)(v) \qquad \text{(BE)}$$

Different levels of description

“Mesoscopic” point of view

$$Q(f,f)(v) = \int_{\mathbb{R}^3_{v_*}\times\mathbb{S}^2_\sigma} B(v-v_*,\sigma)\big(\underbrace{f(v'_*)f(v')}_{\text{before collision}} - \underbrace{f(v)f(v_*)}_{\text{after collision}}\big)\mathrm{d}v_*\mathrm{d}\sigma$$

$$= [\text{part. that } \textbf{now} \text{ have vel. } v] - [\text{part. that } \textbf{had} \text{ vel. } v]$$



$$v' + v'_* = v + v_*, \quad |v'|^2 + |v'_*|^2 = |v|^2 + |v_*|^2$$

Different levels of description

"Mesoscopic" point of view

Mass, momentum, energy: **micro** conservation $\Rightarrow$ **macro** conservation

Theorem (Boltzmann's H-Theorem)

The **entropy**

$$H_t(x) := -\int F_t(x,v)\log F_t(x,v)\mathrm{d}v$$

is non-decreasing and maximal for LTEs:

$$\frac{R(x)}{(2\pi T(x))^{3/2}}\exp\left(-\frac{|v-U(x)|^2}{2T(x)}\right)$$

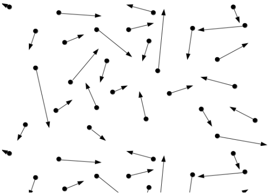
Lemma

$Q(F,F) = 0 \Leftrightarrow F$ is a LTE.

Different levels of description

Recap

Macroscopic (INS)	Mesoscopic	Microscopic
$U_t(x), R_t(x), T_t(x)$	$F_t(x, v)$	$(x_i(t), v_i(t))_{i=1}^N$
Velocity, density, temperature	Particules nb. density at position x velocity v	Exact position of particule nb. i
Fields on $\mathbb{R}_x^3$	Density on $\mathbb{R}_x^3 \times \mathbb{R}_v^3$	Vectors in $\mathbb{R}^{6N}$
$\partial_t U + U \cdot \nabla_x U = \nu \Delta_x U - \nabla_x P$	$\partial_t F + v \cdot \nabla_x F = Q(F, F)$	$\dot{v}_i =$ interactions between particules
At LTE	Tends to ETL	
Weak global solutions, incompressible initial data of finite energy uniqueness unknown	Global weak solutions, initil data with finite mass/energy/entropy, uniqueness unknown	Uniqueness



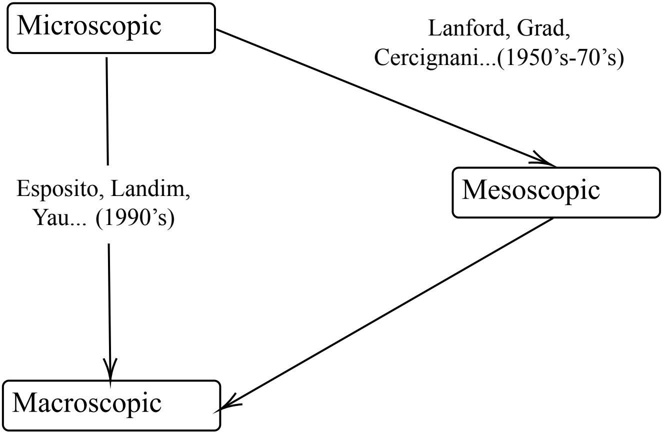
The problem of hydrodynamic limits

Hilbert's sixth problem

Le livre de M. Boltzmann sur les Principes de la Mécanique nous incite à établir et à discuter du point de vue mathématique d'une manière complète et rigoureuse les méthodes basées sur l'idée de ***passage à la limite****, et qui de la conception atomique nous conduisent aux lois du mouvement des continua.*

D. Hilbert at the second ICM, Paris, 1900

The problem of hydrodynamic limits
Hilbert's sixth problem



The problem of hydrodynamic limits

Why?

- Axiomatization of physics
- Approximate Boltzmann with hydrodynamic model
- Develop numerical schemes

The problem of hydrodynamic limits

Hydrodynamic limit of Boltzmann

$$\partial_t F + v\cdot\nabla_x F = Q(F,F)$$

$$\Big\downarrow \text{ write with macro variables}$$

$$\varepsilon\partial_t F^\varepsilon + v\cdot\nabla_x F^\varepsilon = \frac{1}{\varepsilon} Q(F^\varepsilon, F^\varepsilon)$$

Average time between collisions$= \varepsilon \ll 1$

$$F_t^\varepsilon(x,v) \xrightarrow{\varepsilon\to 0} F_t^0(x,v) = R_t(x)\exp\left(-\frac{|v-U_t(x)|^2}{2T_t(x)}\right)$$

The problem of hydrodynamic limits

Hydrodynamic limit of Boltzmann

Gas at thermodynamic equilibrium (constant heat, mass density, at rest) :

$$M = (2\pi)^{-d/2} \exp(-|v|^2/2),\ Q(M,M) = 0,$$

Statistical fluctuation of order ε around M:

$$F^\varepsilon_{|t=0} = M + \varepsilon f_{\text{in}},\ F^\varepsilon = M + \varepsilon f^\varepsilon,$$

Macroscopic fluctuations of order ε:

- Mass: $R^\varepsilon(t,x) = \int (M + \varepsilon f^\varepsilon)\,\mathrm{d}v = 1 + \varepsilon \rho^\varepsilon(t,x)$
- Velocity: $U^\varepsilon(t,x) = (\dots) = 0 + \varepsilon u^\varepsilon(t,x)$
- Temperature: $T^\varepsilon(t,x) = (\dots) = 1 + \varepsilon \theta^\varepsilon(t,x)$

The problem of hydrodynamic limits

Hydrodynamic limit of Boltzmann

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$$M = (2\pi)^{-d/2} \exp(-|v|^2/2),\ Q(M,M) = 0,$$

Statistical fluctuation of order ε around M:

$$F^\varepsilon_{|t=0} = M + \varepsilon f_{\text{in}},\ F^\varepsilon = M + \varepsilon f^\varepsilon,$$

Equation on f^ε:

$$\varepsilon \partial_t F^\varepsilon + v \cdot \nabla_x F^\varepsilon = \frac{1}{\varepsilon} Q(F^\varepsilon, F^\varepsilon),$$

$$\Downarrow$$

$$\varepsilon \partial_t \left(M + \varepsilon f^\varepsilon\right) + v \cdot \nabla_x \Big(M + \varepsilon f^\varepsilon\Big) = \frac{1}{\varepsilon} Q(M + \varepsilon f^\varepsilon, M + \varepsilon f^\varepsilon),$$

The problem of hydrodynamic limits

Hydrodynamic limit of Boltzmann

Gas at thermodynamic equilibrium (constant heat, mass density, at rest) :

$$M = (2\pi)^{-d/2} \exp(-|v|^2/2),\ Q(M, M) = 0,$$

Statistical fluctuation of order ε around M:

$$F^\varepsilon_{|t=0} = M + \varepsilon f_{\rm in},\ F^\varepsilon = M + \varepsilon f^\varepsilon,$$

Equation on f^ε:

$$\varepsilon \partial_t F^\varepsilon + v \cdot \nabla_x F^\varepsilon = \frac{1}{\varepsilon} Q(F^\varepsilon, F^\varepsilon),$$

$$\Downarrow \quad Q(M, M) = 0$$

$$\partial_t f^\varepsilon = \frac{1}{\varepsilon^2} (\mathcal{L} + \varepsilon v \cdot \nabla_x) f^\varepsilon + \frac{1}{\varepsilon} Q(f^\varepsilon, f^\varepsilon)$$

where

$$\mathcal{L} := Q(M, \cdot) + Q(\cdot, M)$$

The problem of hydrodynamic limits

Hydrodynamic limit of Boltzmann

Definition (microscopic, macroscopic)

- f is **macroscopic**
  $\stackrel{\text{def}}{\equiv} f(x,v) = \Big(1\rho(x) + u(x)\cdot v + \frac{1}{2}(|v|^2 - d)\theta(x)\Big)M(v)$
- f is **well-prepared** $\stackrel{\text{def}}{\equiv} \begin{cases} f \text{ is macroscopic,} \\ \nabla_x \cdot u(x) = 0, \\ \rho(x) + \theta(x) = 0 \end{cases}$
- f is **microscopic** $\stackrel{\text{def}}{\equiv} \int f(x,v)\varphi(v)\mathrm{d}v = 0,\ \varphi(v) = 1, v, |v|^2$

Remark: Unique decomposition $f = f_{\text{macro}} + f_{\text{micro}}$

$$= f_{\text{well-prepared}} + f_{\text{ill-prepared}} + f_{\text{micro}}$$

The problem of hydrodynamic limits

Hydrodynamic limit of Boltzmann

Theorem (Bardos,Golse,Levermore,Saint-Raymond ('91-'03))

If $M + \varepsilon f_t^\varepsilon(x,v)$ **weak solution** of Boltzmann, then

$$f^\varepsilon \rightharpoonup f_t^0(x,v) = \left(\rho_t(x) + u_t(x)\cdot v + \frac{1}{2}(|v|^2 - 3)\theta_t(x)\right) M,$$

$$f_0^0(x,v) = f_{\text{in,well-p.}}(x,v)$$

where ρ, u, θ are **weak solutions** of incompressible Navier-Stokes:

$$\begin{cases} \partial_t u + u\cdot\nabla_x u = \nu\Delta_x u - \nabla_x p, \\ \partial_t\theta + u\cdot\nabla_x\theta = \kappa\Delta_x\theta, \\ \text{div}_x u = 0, \quad \nabla_x(\rho+\theta) = 0, \end{cases} \qquad \text{(NSFI)}$$

and ν, κ depend only on Q and M.

The problem of hydrodynamic limits

Hydrodynamic limit of Boltzmann

Question: Strong solutions ? Strong convergence ? What functional space ?

1. Construct strong solutions for $f_{\text{in}} \in \mathbf{X}_{x,v}$
2. Prove strong convergence in $\mathbf{X}_{x,v}$

Existence and convergence of strong solutions

Initial data with Gaussian decay (Bardos–Ukai/Gallagher–Tristani)

Functional space: $\mathbf{G} = \left\{ f \, : \, |f(x,v)| \leq C e^{-|v|^2} \right\}$,

$$\partial_t f^\varepsilon = \frac{1}{\varepsilon^2}\big(\mathcal{L} + \varepsilon v \cdot \nabla_x\big) f^\varepsilon + \frac{1}{\varepsilon} Q(f^\varepsilon, f^\varepsilon),$$

$$\Big\downarrow \text{ Duhamel with } f^\varepsilon_{|t=0} = f_{\text{in}}$$

$$f^\varepsilon(t) = U^\varepsilon(t) f_{\text{in}} + \Psi^\varepsilon(t)(f^\varepsilon, f^\varepsilon),$$

Where we denote

$$U^\varepsilon(t) := \exp\left(\frac{1}{\varepsilon^2} \left(\mathcal{L} + \varepsilon v \cdot \nabla_x \right) \right),$$

$$\Psi^\varepsilon(t)(f^\varepsilon, f^\varepsilon) := \frac{1}{\varepsilon} \int_0^t U^\varepsilon(t - t') Q\left(f^\varepsilon(t'), f^\varepsilon(t') \right) \mathrm{d}t'$$

Existence and convergence of strong solutions

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$$f^\varepsilon(t) = U^\varepsilon(t) f_{\text{in}} + \Psi^\varepsilon(t)(f^\varepsilon, f^\varepsilon),$$

- Spectral study by Ellis, Pinsky, Ukai (1975-’86) + Fourier
  - $\|U^\varepsilon(t) f_{\text{in}}\| \leq C \|f_{\text{in}}\|$ and $\|\Psi^\varepsilon(f,g)\| \leq C\|f\|\|g\|$
  - prove $U^\varepsilon \to U^0$, $\Psi^\varepsilon \to \Psi^0$
- Existence of f^ε: Banach’s fixed point theorem
  - on f^ε (Bardos-Ukai, 1991) if $\|f_{\text{in}}\|_{\mathbf{G}} \ll 1$
  - on $f^\varepsilon - f^0$ (Gallagher-Tristani, 2019) if $\varepsilon \ll 1$
- Let $\varepsilon \to 0$ in the equation

Existence and convergence of strong solutions

Initial data with polynomial decay (Briant–Merino–Mouhot/G.)

If F^ε =physical gas:

- Mass: $\displaystyle\int F^\varepsilon \,\mathrm{d}v = \|F^\varepsilon\|_{L^1_v} < \infty$
- Energy: $\displaystyle\int F^\varepsilon \,|v|^2 \mathrm{d}v = \left\||v|^2 F^\varepsilon\right\|_{L^1_v} < \infty$

i.e. $\displaystyle\int f_{\text{in}}(x,v)\left(1+|v|^2\right)\mathrm{d}v < \infty$

Question: Is it enough for existence/uniqueness ? Convergence ?

Existence and convergence of strong solutions

Initial data with polynomial decay (Briant–Merino–Mouhot/G.)

If F^ε =physical gas:

- Mass: $\int F^\varepsilon \,\mathrm{d}v = \|F^\varepsilon\|_{L^1_v} < \infty$
- Energy: $\int F^\varepsilon \,|v|^2 \mathrm{d}v = \big\||v|^2 F^\varepsilon\big\|_{L^1_v} < \infty$

i.e. $\int f_{\text{in}}(x,v)\left(1+|v|^2\right)\mathrm{d}v < \infty$

Question: Is it enough for existence/uniqueness ? Convergence ?

Partial answer: $\mathbf{P} := \left\{ f \ : \ \int f(x,v)\left(1+|v|^{3+\delta}\right)\mathrm{d}v < \infty \right\}$

Theorem (G. 2021)

Let $f_{\text{in}} \in \mathbf{P}$, Boltzmann has a **unique strong solution** f^ε on $[0,T]$ for $\varepsilon \ll 1$

T = lifespan of Navier-Stokes solution f^0 with $f^0_{|t=0} = f_{\text{in,well-p.}}$

Existence and convergence of strong solutions

Initial data with polynomial decay (Briant–Merino–Mouhot/G.)

Theorem (G. 2021)

$$f^\varepsilon = f^0 + u^\varepsilon + f^\varepsilon_{\text{mic}} + u^\varepsilon_{\text{ac}}$$

- Navier-Stokes: f^0 with $f^0_{|t=0} = f_{\text{in,well-p.}}$
- Error term: $\sup_t \|u^\varepsilon(t)\| \to 0, \ (\varepsilon \to 0)$
- Initial layer: $\|f^\varepsilon_{\text{mic}}(t)\| \lesssim e^{-\gamma t/\varepsilon^2} \|f_{\text{in,mic}}\|$
- Acoustic waves: $u^\varepsilon_{\text{ac}} \rightharpoonup 0$ and $f_{\text{in,ill-p.}} = 0 \Leftrightarrow u^\varepsilon_{\text{ac}} = 0$

Remark: $\Omega = \mathbb{R}^d \Rightarrow \int \|u^\varepsilon_{\text{ac}}(t)\|^q \mathrm{d}t \to 0$, with $0 < q < \frac{2}{d-1}$

Existence and convergence of strong solutions

Initial data with polynomial decay (Briant–Merino–Mouhot/G.)

First results in polynomial spaces

- [Mouhot, '05]: Enlargement Theory
- [Gualdani, Mischler, Mouhot, '17]: strong solution for Boltzmann and $\|f_{\mathrm{in}}\| \ll 1$
- [Briant, Merino, Mouhot, '19]: weak hydrodynamic limit

Proof of the theorem
Strategy

$$f_{\text{in}} = f_{\text{in,well-p.}} + f_{\text{in,ill-p.}} + f_{\text{in,mic}} \rightsquigarrow f^\varepsilon = f^0 + u^\varepsilon_{\text{ac}} + f^\varepsilon_{\text{mic}} + u^\varepsilon$$

- [Fujita-Kato] $f_{\text{in,well-p.}} \rightsquigarrow f^0$ on $[0,T]$
- [Ellis-Pinsky] Spectral analysis of $\mathcal{L} + v \cdot \nabla_x$: $f_{\text{in,ill-p.}} \rightsquigarrow u^\varepsilon_{\text{ac}}$

[Gualdani, Mischler, Mouhot] Decomposition $\mathcal{L} = \mathcal{B} + \mathcal{A}$:

$$\text{Boltzmann} \longrightarrow \text{Coupled system on } f^\varepsilon_{\text{mic}} \text{ and } u^\varepsilon$$

- Energy method : $f_{\text{in,mic}} \rightsquigarrow f^\varepsilon_{\text{mic}}$
- Fixed point $\rightsquigarrow u^\varepsilon$: source term $f^\varepsilon_{\text{mic}} \Rightarrow$ need to extends results of Gallagher-Tristani from **G** to **P**

Proof of the theorem

Splitting of the equation

- GMM splitting $\mathcal{L} = \mathcal{B} + \mathcal{A}$

$$\mathcal{B}f \approx -(1+|v|)f, \quad \mathcal{A} : \mathbf{P} \to \mathbf{G} \text{ bounded}$$

- $f^\varepsilon(t) = h^\varepsilon(t) + g^\varepsilon(t) \in \mathbf{P} + \mathbf{G}$,
  $h^\varepsilon_{|t=0} = f_{\text{in,mic}} \in \mathbf{P}, \quad g^\varepsilon_{|t=0} = f_{\text{in,mac}} \in \mathbf{G}$

$$\partial_t f^\varepsilon = \frac{1}{\varepsilon^2}\big(\mathcal{L} + \varepsilon v \cdot \nabla_x\big) f^\varepsilon + \frac{1}{\varepsilon} Q(f^\varepsilon, f^\varepsilon), \; f^\varepsilon_{|t=0} = f_{\text{in}}$$

$$\Uparrow$$

$$\partial_t h^\varepsilon = \frac{1}{\varepsilon^2}\left(\mathcal{B} + \varepsilon v \cdot \nabla_x\right) h^\varepsilon + \frac{1}{\varepsilon} Q(h^\varepsilon, h^\varepsilon + 2g^\varepsilon),$$

$$\partial_t g^\varepsilon = \frac{1}{\varepsilon^2}\left(\mathcal{L} + \varepsilon v \cdot \nabla_x\right) g^\varepsilon \underbrace{+ \frac{1}{\varepsilon^2}\mathcal{A} h^\varepsilon}_{\in \mathbf{G}} + \frac{1}{\varepsilon} Q(g^\varepsilon, g^\varepsilon),$$

Proof of the theorem

Splitting of the equation

- GMM splitting $\mathcal{L} = \mathcal{B} + \mathcal{A}$
- $f^\varepsilon(t) = h^\varepsilon(t) + g^\varepsilon(t) \in \mathbf{P} + \mathbf{G}$,
  $h^\varepsilon_{|t=0} = f_{\text{in,mic}} \in \mathbf{P}$, $\quad g^\varepsilon_{|t=0} = f_{\text{in,mac}} \in \mathbf{G}$

$$\partial_t f^\varepsilon = \frac{1}{\varepsilon^2}\big(\mathcal{L} + \varepsilon v \cdot \nabla_x\big) f^\varepsilon + \frac{1}{\varepsilon} Q(f^\varepsilon, f^\varepsilon), \; f^\varepsilon_{|t=0} = f_{\text{in}}$$

$$\Uparrow$$

$$\text{Fixed point of } \Xi : (h^\varepsilon, g^\varepsilon) \mapsto \left(\overline{h}^\varepsilon, \overline{g}^\varepsilon\right),$$

defined by

$$\partial_t \overline{h}^\varepsilon = \frac{1}{\varepsilon^2}\left(\mathcal{B} + \varepsilon v \cdot \nabla_x\right) \overline{h}^\varepsilon + \frac{1}{\varepsilon} Q(h^\varepsilon, h^\varepsilon + 2g^\varepsilon),$$

$$\partial_t \overline{g}^\varepsilon = \frac{1}{\varepsilon^2}\left(\mathcal{L} + \varepsilon v \cdot \nabla_x\right) \overline{g}^\varepsilon + \underbrace{\frac{1}{\varepsilon^2}\mathcal{A} h^\varepsilon}_{\in \mathbf{G}} + \frac{1}{\varepsilon} Q(g^\varepsilon, g^\varepsilon)$$

- Ξ well defined+contraction $\Leftrightarrow$ a priori estimates+stability

Proof of the theorem

Control of the polynomial part

$$\partial_t \overline{h}^\varepsilon = \frac{1}{\varepsilon^2} \left(\mathcal{B} + \varepsilon v \cdot \nabla_x\right) \overline{h}^\varepsilon + \frac{1}{\varepsilon} Q(h^\varepsilon, h^\varepsilon + 2g^\varepsilon)$$

Energy inequality + Grönwall lemma $\Rightarrow$

- Exists unique solution
- Stability estimate (same initial condition $\overline{h}_1^\varepsilon(t=0) = \overline{h}_2^\varepsilon(t=0)$)

$$\|\overline{h}_1^\varepsilon - \overline{h}_2^\varepsilon\| \lesssim \varepsilon \, \|h_1^\varepsilon - h_2^\varepsilon, g_1^\varepsilon - g_2^\varepsilon\| \times (\dots)$$

Great contraction estimate!

Proof of the theorem

Study of the Gaussian part

$$\partial_t g^\varepsilon = \frac{1}{\varepsilon^2}(\mathcal{L} + \varepsilon v \cdot \nabla_x) g^\varepsilon + \frac{1}{\varepsilon} Q(g^\varepsilon, g^\varepsilon) + \frac{1}{\varepsilon^2} \mathcal{A} h^\varepsilon,$$

$$\Big\downarrow \text{ Duhamel with } g^\varepsilon_{|t=0} = f_{\text{in,mac}}$$

$$g^\varepsilon = U^\varepsilon f_{\text{in,mac}} + \Psi^\varepsilon(g^\varepsilon, g^\varepsilon) + \frac{1}{\varepsilon^2} \mathcal{A} U^\varepsilon * h^\varepsilon,$$

- Already well understood by BU/GT
- ...but convolution term not "small"

$$U^\varepsilon(t) := \exp\left(\frac{1}{\varepsilon^2}(\mathcal{L} + \varepsilon v \cdot \nabla_x)\right),$$

$$\Psi^\varepsilon(t)(f^\varepsilon, f^\varepsilon) := \frac{1}{\varepsilon} \int_0^t U^\varepsilon(t - t') Q\left(f^\varepsilon(t'), f^\varepsilon(t')\right) \mathrm{d}t'$$

Proof of the theorem

Study of the Gaussian part

Lemma - Convolution splitting (G. 2021)

$$\begin{cases} \partial_t \overline{h}^\varepsilon = \frac{1}{\varepsilon^2}\left(\mathcal{B} + \varepsilon v \cdot \nabla_x\right)\overline{h}^\varepsilon + \frac{1}{\varepsilon} Q(h^\varepsilon, h^\varepsilon + 2g^\varepsilon), \\ \overline{h}^\varepsilon_{|t=0} = f_{\text{in,mic}}, \end{cases}$$

$$\Downarrow$$

$$\frac{1}{\varepsilon^2} U^\varepsilon * \mathcal{A}\overline{h}^\varepsilon(t) = o(1) + \mathcal{O}\left(e^{-t/\varepsilon^2}\right) \in \mathbf{G}$$

Idea of proof: Duhamel on $\mathcal{B} = \mathcal{L} - \mathcal{A}$ and equation for $\overline{h}^\varepsilon$

$$\left.\begin{aligned} \frac{1}{\varepsilon^2} U^\varepsilon * \mathcal{A} V^\varepsilon &= U^\varepsilon - V^\varepsilon \\ \overline{h}^\varepsilon &= V^\varepsilon f_{\text{in,mic}} + \dots \end{aligned}\right\} \Rightarrow \frac{1}{\varepsilon^2} U^\varepsilon * \mathcal{A}\overline{h}^\varepsilon(t) = U^\varepsilon(t) f_{\text{in,mic}} + \dots$$

$$V^\varepsilon = \exp\left(t\varepsilon^{-2}\left(\mathcal{B} + \varepsilon v \cdot \nabla_x\right)\right)$$

Requirements: generalize behavior of U^ε on micro. fluctuations $\in \mathbf{P}$

Proof of the theorem

Study of the Gaussian part

$$g^\varepsilon = U^\varepsilon f_{\text{in,mac}} + \Psi^\varepsilon(g^\varepsilon, g^\varepsilon) + \frac{1}{\varepsilon^2} U^\varepsilon * \mathcal{A} h^\varepsilon,$$

$$\Big\downarrow\; u^\varepsilon := g^\varepsilon - f^0 - u^\varepsilon_{\text{ac}} - \mathcal{O}\left(e^{-t/\varepsilon^2}\right)$$

$$u^\varepsilon = o(1) + \underbrace{\{\text{Linear}\}}_{\text{contraction}}(u^\varepsilon) + \underbrace{\Psi^\varepsilon}_{\text{bounded}}(u^\varepsilon, u^\varepsilon),$$

Problem: {Linear} depends on f^0 **which may be large**
Solution: equivalent norm $\to$ {Linear} is a contraction

Proof of the theorem

Recap

$f_{\text{in}} = f_{\text{in,well-p.}} + f_{\text{in,ill-p.}} + f_{\text{in,mic}} \rightsquigarrow f^{\varepsilon} = f^0 + u^{\varepsilon}_{\text{ac}} + f^{\varepsilon}_{\text{mic}} + u^{\varepsilon}$

- [Fujita-Kato] $f_{\text{in,well-p.}} \rightsquigarrow f^0$ on $[0, T]$
- [Ellis-Pinsky] Spectral analysis of $\mathcal{L} + v \cdot \nabla_x$: $f_{\text{in,ill-p.}} \rightsquigarrow u^{\varepsilon}_{\text{ac}}$

[Gualdani, Mischler, Mouhot] Decomposition $\mathcal{L} = \mathcal{B} + \mathcal{A}$:

Boltzmann $\longrightarrow$ Coupled system on $f^{\varepsilon}_{\text{mic}}$ and u^{ε}

- Energy method : $f_{\text{in,mic}} \rightsquigarrow f^{\varepsilon}_{\text{mic}}$
- Fixed point $\rightsquigarrow u^{\varepsilon}$: source term $f^{\varepsilon}_{\text{mic}} \Rightarrow$ need to extends results of Gallagher-Tristani from **G** to **P**

Possible extensions

- Other kinetic models: (Landau, Boltzmann with hard potentials...)
- Other limiting hydrodynamic models

Thanks for your attention!