A mathematical model on black market

Chenlin GU

DMA/ENS

LPSM, May 13, 2019

joint work with Alvin E. ROTH and Qingyun WU (Stanford)

Outline

1. Introduction
   - Model
   - Question
   - Heuristic analysis
2. Main results
   - Phase diagrams
3. Ideas of the proof
   - Combinations of several simple probabilistic tools
4. Generalisation
   - A model with parameter non-constant
   - Stabilisation

Motivation

- We want to study why some illegal black market exists and cannot be extinguished.

Motivation

- We want to study why some illegal black market exists and cannot be extinguished.
- Combat between three types of populations : for, against, and neutral.
- 

  Figure: Drug user

  

  Figure: Drug hater

  

  Figure: Neutral

Outline

Model

- The process $(X_t; Y_t; Z_t)_{t 2 N}$ starting from $(X_0; Y_0; Z_0)$ is discrete time N^3-valued Markov chain.
- $X_t =$ number of drug users.
- $Y_t =$ number of drug haters.
- $Z_t =$ neutral.

Model

- An outsider joins the system every system.
- He has probability p to be a drug hater (of type Y).
- - probability $(1 \quad p)$ of be a drug user or neutral (of type X or Z).
  - He tries to find drug and success with probability $q = \frac{X_t}{X_t + rY_t}$.
  - He is reasonable, so he tries to find the drug only when $\frac{X_t}{Y_t} > Kr$.

Model

- The process $(X_t, Y_t, Z_t)_{t \in \mathbb{N}}$ starting from (X_0, Y_0, Z_0) is discrete time $\mathbb{N}^3$-valued Markov chain.
- Matrix of transition

$$\begin{cases} P\left((x+1, y, z) | (x, y, z)\right) & = (1-p)\frac{x}{x+ry}\mathbf{1}_{\{\frac{x}{y} > Kr\}} \\ P\left((x, y+1, z) | (x, y, z)\right) & = p \\ P\left((x, y, z+1) | (x, y, z)\right) & = (1-p)\frac{ry}{x+ry}\mathbf{1}_{\{\frac{x}{y} > Kr\}} + (1-p)\mathbf{1}_{\{\frac{x}{y} \leq Kr\}} \end{cases}$$

- Parameters :
  - $p \in [0, 1]$ - probability to be drug hater.
  - $r \in [0, 1]$ - probability of reporting to police (for Y_t).
  - $K \in [0, \infty]$ - punishment to be caught.

Outline

Question

- Extinction := $\left\{ \lim_{t\to\infty} \frac{X_t}{X_t+Y_t+Z_t} = 0 \right\}$
- By the law of large number, $\lim_{t\to\infty} \frac{X_t}{X_t+Y_t+Z_t} = p$, thus we define $R_t := \frac{X_t}{Y_t}$ and we have

$$\text{Extinction} = \{\lim_{t\to\infty} R_t = 0\} = \{\lim_{t\to\infty} R_t \quad Kr\}.$$

- Main questions :
  1. Under which condition, the extinction of black market happens with probability 1 ?
  2. If there is no extinction (we say the market **survives** in this case), then what will be the limit (if exists) of the proportion $\left(\frac{X_t}{X_t+Y_t+Z_t}, \frac{Y_t}{X_t+Y_t+Z_t}, \frac{Z_t}{X_t+Y_t+Z_t}\right)$?

Outline

1. Introduction
   - Model
   - Question
   - Heuristic analysis
2. Main results
   - Phase diagrams
3. Ideas of the proof
   - Combinations of several simple probabilistic tools
4. Generalisation
   - A model with parameter non-constant
   - Stabilisation

Fixed point argument

- Suppose that we do not consider the utility (K plays no role) and $\lim_{t\to\infty}\frac{X_t}{Y_t}=\bar{R}$, the increment of X and Z has limit of proportion $\frac{\bar{R}}{r} \Rightarrow \lim_{t\to\infty}\frac{X_t}{Z_t}=\frac{\bar{R}}{r}$.
- This implies that $\lim_{t\to\infty}\left(\frac{X_t}{X_t+Y_t+Z_t};\frac{Y_t}{X_t+Y_t+Z_t};\frac{Z_t}{X_t+Y_t+Z_t}\right)=\left(\frac{(1-p)\bar{R}}{\bar{R}+r};p;\frac{(1-p)r}{\bar{R}+r}\right)$.
- $\boxed{\bar{R}:=\frac{1-p-pr}{p}}$.
- Compare $\bar{R}$ and Kr and take in account of the function of K.

Outline

Main theorem : part 1

Theorem (Controllable and uncontrollable black markets)

There exist three cases,

1. **Controllable:** $p(1 + r + Kr) > 1$. *In this situation, the market will become extinct with probability 1 and* $R_t \xrightarrow{a.s.} 0$.
2. **Borderline:** $p(1 + r + Kr) = 1$. *If* $1 \quad p \quad 2pr$, $K \quad 1$ *then it behaves like the controllable case; if* $1 \quad p < 2pr$, $K < 1$, *then it behaves like the uncontrollable case.*
3. **Uncontrollable:** $p(1 + r + Kr) < 1$. *The market survives with positive probability. And when it survives,* $R_t \xrightarrow{a.s.} \bar{R}$.

Main theorem : part 1

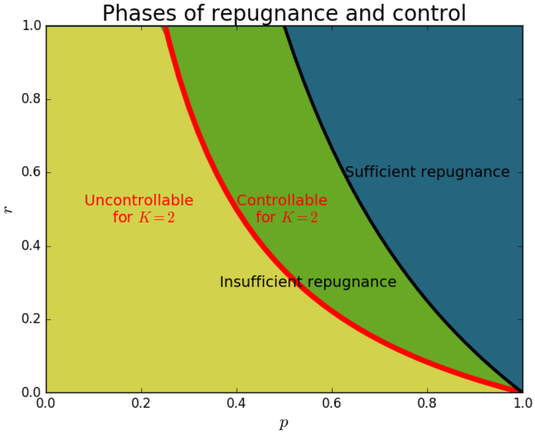

Figure: Phases of controllability

Main theorem : part 2

We can think of $I := p(1 + r)$ as an index reflecting the repugnance with which the market is perceived. We define that $= \inf\{t \mid R_t \quad Kr\}$.

Lemma (Long time behaviour)

Conditional on $= \infty$, $R_t \xrightarrow{a.s.} R \vee 0$. Concretely,

- Insufficient repugnance: If $p(1 + r) < 1$, then the limit of R_t is almost surely R.
- Sufficient repugnance: If $p(1 + r) \quad 1$, then the limit of R_t is almost surely 0.

Main theorem : part 2

Figure: Phases of repugnance

Main theorem : Some simulations

Figure: Phases of su cient repugnance phase

Main theorem : Some simulations

Figure: Phases of controllable phase

Main theorem : Some simulations

Figure: Phases of uncontrollable phase

Main theorem : Some simulations

Figure: Phases of controllable, borderline phase

Main theorem : Some simulations

Figure: Phases of uncontrollable, borderline phase

Outline

Embedding to a continuous time stochastic process

Figure: We can embed the process $(X_n; Y_n; Z_n)_{n \in \mathbb{N}}$ into a Markov jump process $(X_t; Y_t; Z_t)_{t \geq 0}$.

We can do it even better to define all the process $((X_t; Y_t; Z_t)_{t \geq 0}; p; r)$ in a same probability space $(\Omega; \mathcal{F}; \mathbb{P})$.

Analysis of martingale 1

The key is to analyse $W_t = Y_t \quad (R)^{1} X_t$.

$E[X_t] = X_0 e^{(1 \quad p)t}$; $E[W_t] = W_0 e^{prt}$.

$(X_t)_{t \quad 0}$ process of Yule and $(e^{(1 \quad p)t} X_t)_{t \quad 0}$ converges to a limit of law exp(1).

$\frac{X_t}{Y_t} = \frac{e^{(1 \quad p)t} X_t}{e^{(1 \quad p)t} W_t + (R)^{1} e^{(1 \quad p)t} X_t}$ converges to di erent limit.

This proves the long time behaviours and also the controllable case $p(1 + r + Kr) > 1$.

Coupling with a random walk

- Uncontrollable case $p(1 + r + Kr) < 1$.
- $S_t = X_t - KrY_t$.
- $S_{t+1} = S_t + 1_{\{p \le U_{t+1} < 1\}} 1_{\{0 \le V_{t+1} < \frac{X_t}{X_t + rY_t}\}} - Kr 1_{\{0 \le U_{t+1} < p\}}$.
- $S'_{t+1} = S'_t + 1_{\{p \le U_{t+1} < 1\}} 1_{\{0 \le V_{t+1} < \frac{K}{1+K}\}} - Kr 1_{\{0 \le U_{t+1} < p\}}$ defines a simple random walk.
- $E[S'_{t+1} - S'_t] > 0$, thus $(S'_t)_{t \in N}$ has a positive drift and is transient.
- $S_t \ge S'_t$ before $\frac{X_t}{Y_t} < Kr$.

Critical case $p(1 + r + Kr) = 1$.

Lemma

In the borderline case, i.e. when $R = Kr$:

- Small variance: $1 - p < 2pr \iff K < 1$. Then T has a positive probability to be infinite and $(X_t; Y_t)$ has a positive probability to always stay above the slope R, in other words, the market has a positive probability of surviving.
- Big variance: $1 - p \geq 2pr \iff K \geq 1$. Then T is almost surely finite and $(X_t; Y_t)$ will finally pass the slope, meaning the market always becomes extinct.

Analysis of martingale 2

$1 - p - 2pr > 0$:

$$E[W_t^2] = W_0^2 e^{2prt} + W_0(e^{2prt} - e^{prt}) + \frac{A}{1 - p - 2pr} X_0(e^{(1-p)t} - e^{2prt});$$

and the typical size of $E[W_t^2]$ is at the order of $e^{(1-p)t}$.

$1 - p - 2pr = 0$:

$$E[W_t^2] = W_0^2 e^{2prt} + W_0(e^{2prt} - e^{prt}) + AX_0 t e^{2prt};$$

and the typical size is of te^{2prt}.

$1 - p - 2pr < 0$:

$$E[W_t^2] = W_0^2 e^{2prt} + W_0(e^{2prt} - e^{prt}) + \frac{A}{2pr - (1 - p)} X_0(e^{2prt} - e^{(1-p)t}):$$

This expression is the same as in the first case, but its typical size is of e^{2prt}.

Outline

A model with parameter non-constant

Same model and same question about long time behaviors.

$$
\begin{cases}
P((x+1;y;z)|(x;y;z)) &= (1-p)\frac{x}{x+ry}1_{\{\frac{x}{y}>Kr\}} \\
P((x;y+1;z)|(x;y;z)) &= p \\
P((x;y;z+1)|(x;y;z)) &= (1-p)\frac{ry}{x+ry}1_{\{\frac{x}{y}>Kr\}} + (1-p)1_{\{\frac{x}{y}\leq Kr\}}
\end{cases}
$$

What happens if $(p;r;K)$ are non-constant ?

One varied model: $p := p\left(\frac{y}{x+y+z}\right);\ r := r\left(\frac{y}{x+y+z}\right)$.

A model with parameter non-constant

One varied model:

$p \quad \frac{y}{x+y+z} \quad := 0{:}5 + 0{:}3 \sin \quad \frac{10y}{x+y+z} \quad ; r := 0{:}1; K = 0{:}5.$

Several di erent possible limits ?

Figure: We believe that the limit should be the xed point of= p( ).

Outline

Numerical simulations

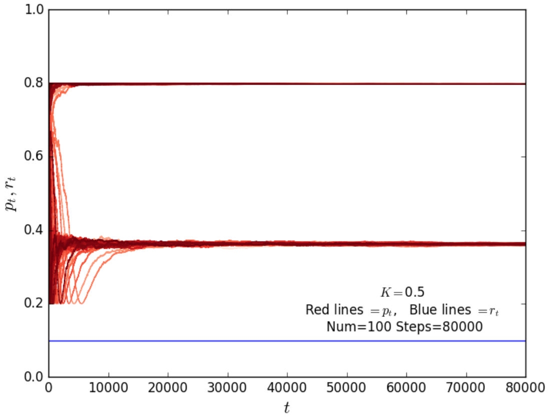

Figure: Stabilisation of p.

Numerical simulations

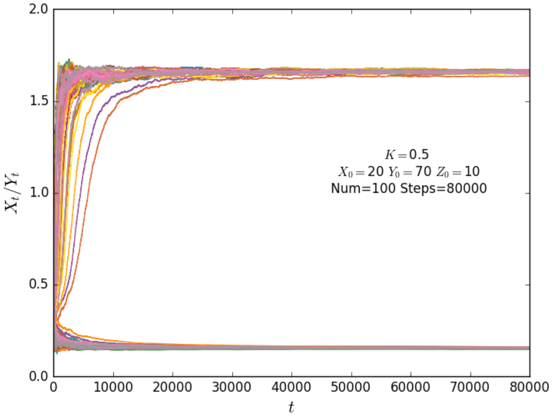

Figure: Stabilisation of X_t=Y_t.

Stochastic approximation

- $N_t = X_t + Y_t + Z_t; \ \ _t = \frac{X_t}{N_t}$.
- $\ _{t+1} = \ \ _t \quad \frac{1}{N_t+1} \quad \ _t \quad \mathbf{1}_{fU_{t+1} \ \ p(\ _t)g}$.
- $\ _t$ converges to the fixed points of $\quad = p(\quad)$. (Robbins–Monro)
- Unstable points are not possible. (i.e Saddle points in SGD are not stable. - Proof non trivial.)

Thanks for your attentions.