A mathematical model on black market

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LPSM, May 13, 2019

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Outline

Motivation

- We want to study why some illegal black market exists and cannot be extinguished.

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- Combat between three types of populations : for, against, and neutral.
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  Figure: Drug user

  

  Figure: Drug hater

  

  Figure: Neutral

Outline

Model

- The process $(X_t, Y_t, Z_t)_{t\in\mathbb{N}}$ starting from (X_0, Y_0, Z_0) is discrete time $\mathbb{N}^3$-valued Markov chain.
- $X_t =$ number of drug users.
- $Y_t =$ number of drug haters.
- $Z_t =$ neutral.

Model

- An outsider joins the system every system.
- He has probability p to be a drug hater (of type Y).
- - probability $(1-p)$ of be a drug user or neutral (of type X or Z).
  - He tries to find drug and success with probability $q = \frac{X_t}{X_t + rY_t}$.
  - He is reasonable, so he tries to find the drug only when $\frac{X_t}{Y_t} > Kr$.

Model

- The process $(X_t, Y_t, Z_t)_{t\in\mathbb{N}}$ starting from (X_0, Y_0, Z_0) is discrete time $\mathbb{N}^3$-valued Markov chain.
- Matrix of transition

$$\begin{cases} P\left((x+1,y,z)|(x,y,z)\right) & = (1-p)\frac{x}{x+ry}\mathbf{1}_{\{\frac{x}{y}>Kr\}} \\ P\left((x,y+1,z)|(x,y,z)\right) & = p \\ P\left((x,y,z+1)|(x,y,z)\right) & = (1-p)\frac{ry}{x+ry}\mathbf{1}_{\{\frac{x}{y}>Kr\}} + (1-p)\mathbf{1}_{\{\frac{x}{y}\leq Kr\}} \end{cases}$$

- Parameters :
  - $p \in [0,1]$ - probability to be drug hater.
  - $r \in [0,1]$ - probability of reporting to police (for Y_t).
  - $K \in [0,\infty]$ - punishment to be caught.

Outline

Question

- Extinction $:= \left\{ \lim_{t\to\infty} \frac{X_t}{X_t+Y_t+Z_t} = 0 \right\}$
- By the law of large number, $\lim_{t\to\infty} \frac{X_t}{X_t+Y_t+Z_t} = p$, thus we define $R_t := \frac{X_t}{Y_t}$ and we have

$$\text{Extinction} = \{\lim_{t\to\infty} R_t = 0\} = \{\lim_{t\to\infty} R_t \leq Kr\}.$$

- Main questions :
  1. Under which condition, the extinction of black market happens with probability 1 ?
  2. If there is no extinction (we say the market **survives** in this case), then what will be the limit (if exists) of the proportion $\left(\frac{X_t}{X_t+Y_t+Z_t}, \frac{Y_t}{X_t+Y_t+Z_t}, \frac{Z_t}{X_t+Y_t+Z_t}\right)$?

Outline

Fixed point argument

- Suppose that we do not consider the utility (K plays no role) and $\lim_{t\to\infty} \frac{X_t}{Y_t} = \bar{R}$, the increment of X and Z has limit of proportion $\frac{\bar{R}}{r} \Rightarrow \lim_{t\to\infty} \frac{X_t}{Z_t} = \frac{\bar{R}}{r}$.
- This implies that $\lim_{t\to\infty}\left(\frac{X_t}{X_t+Y_t+Z_t}, \frac{Y_t}{X_t+Y_t+Z_t}, \frac{Z_t}{X_t+Y_t+Z_t}\right) = \left(\frac{(1-p)\bar{R}}{\bar{R}+r}, p, \frac{(1-p)\bar{r}}{\bar{R}+r}\right)$.
- $$\boxed{\bar{R} := \frac{1-p-pr}{p}}.$$
- Compare $\bar{R}$ and Kr and take in account of the function of K.

Outline

Main theorem : part 1

Theorem (Controllable and uncontrollable black markets)

There exist three cases,

1. **Controllable:** $p(1 + r + Kr) > 1$. *In this situation, the market will become extinct with probability 1 and* $R_t \xrightarrow{a.s.} 0$.
2. **Borderline:** $p(1 + r + Kr) = 1$. *If* $1 - p \geq 2pr \Leftrightarrow K \geq 1$ *then it behaves like the controllable case; if* $1 - p < 2pr \Leftrightarrow K < 1$, *then it behaves like the uncontrollable case.*
3. **Uncontrollable:** $p(1 + r + Kr) < 1$. *The market survives with positive probability. And when it survives,* $R_t \xrightarrow{a.s.} \bar{R}$.

Main theorem : part 1

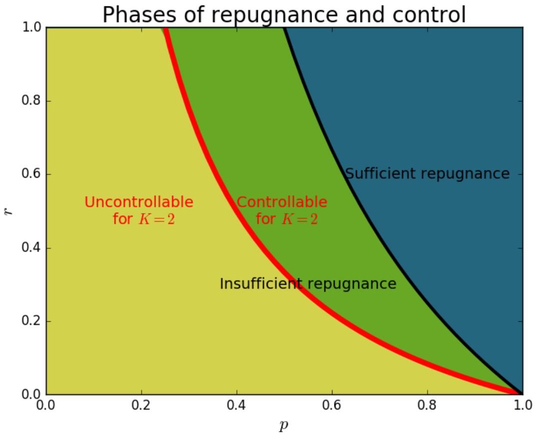

Figure: Phases of controllability

Main theorem : part 2

We can think of $I := p(1+r)$ as an index reflecting the repugnance with which the market is perceived. We define that $\tau = \inf\{t | R_t \leq Kr\}$.

Lemma (Long time behaviour)

Conditional on $\tau = \infty$, $R_t \overset{a.s.}{\to} \bar{R} \vee 0$. *Concretely,*

1. **Insufficient repugnance:** *If* $p(1+r) < 1$, *then the limit of* R_t *is almost surely* $\bar{R}$.
2. **Sufficient repugnance:** *If* $p(1+r) \geq 1$, *then the limit of* R_t *is almost surely* 0.

Main theorem : part 2

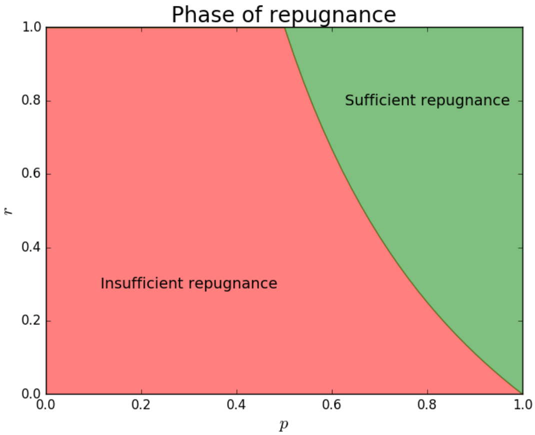

Figure: Phases of repugnance

Main theorem : Some simulations

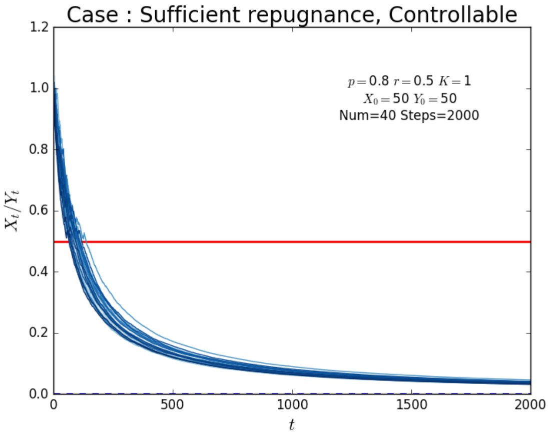

Figure: Phases of sufficient repugnance phase

Main theorem : Some simulations

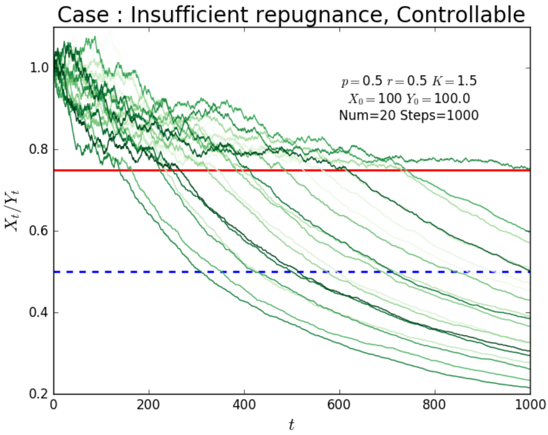

Figure: Phases of controllable phase

Main theorem : Some simulations

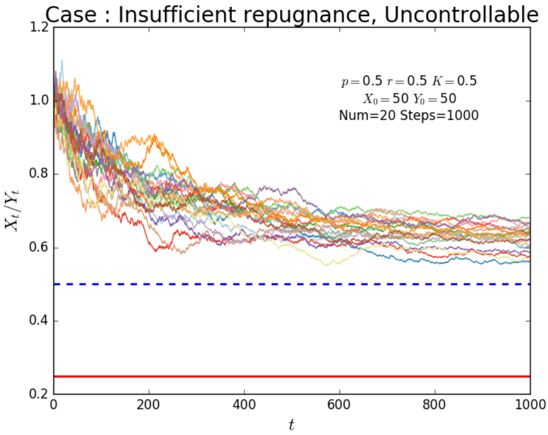

Figure: Phases of uncontrollable phase

Main theorem : Some simulations

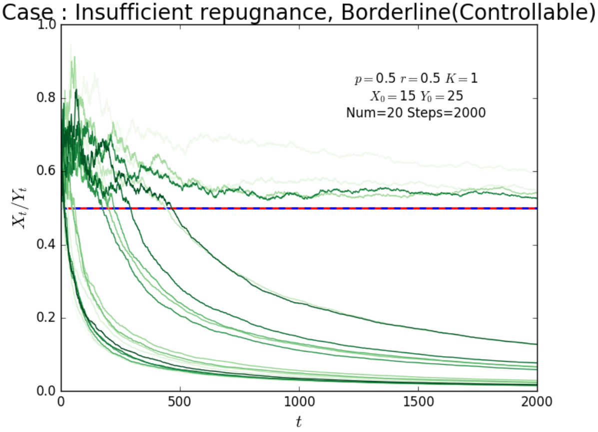

Figure: Phases of controllable, borderline phase

Main theorem : Some simulations

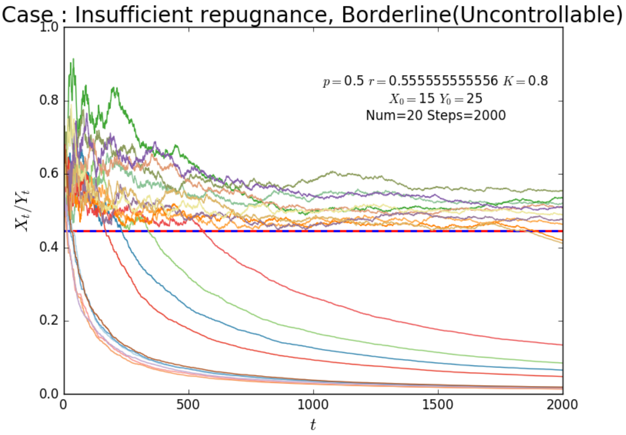

Figure: Phases of uncontrollable, borderline phase

Outline

Embedding to a continuous time stochastic process

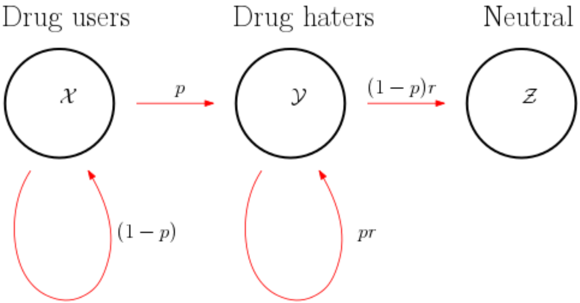

Figure: We can embed the process $(X_n, Y_n, Z_n)_{n\in\mathbb{N}}$ into a Markov jump process $(\mathcal{X}_t, \mathcal{Y}_t, \mathcal{Z}_t)_{t\geq 0}$.

- We can do it even better to define all the process $((\mathcal{X}_t, \mathcal{Y}_t, \mathcal{Z}_t)_{t\geq 0}, p, r)$ in a same probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

Analysis of martingale 1

- The key is to analyse $\mathcal{W}_t = \mathcal{Y}_t - (\bar{R})^{-1}\mathcal{X}_t$.
- $\mathbb{E}[\mathcal{X}_t] = \mathcal{X}_0 e^{(1-p)t}, \mathbb{E}[\mathcal{W}_t] = \mathcal{W}_0 e^{prt}$.
- $(\mathcal{X}_t)_{t\geq 0}$ process of Yule and $(e^{-(1-p)t}\mathcal{X}_t)_{t\geq 0}$ converges to a limit of law $exp(1)$.
- $\frac{\mathcal{X}_t}{\mathcal{Y}_t} = \frac{e^{-(1-p)t}\mathcal{X}_t}{e^{-(1-p)t}\mathcal{W}_t + (\bar{R})^{-1}e^{-(1-p)t}\mathcal{X}_t}$ converges to different limit.
- This proves the long time behaviours and also the controllable case $p(1 + r + Kr) > 1$.

Coupling with a random walk

- Uncontrollable case $p(1 + r + Kr) < 1$.
- $S_t = X_t - KrY_t$.
- $S_{t+1} = S_t + \mathbf{1}_{\{p \le U_{t+1} < 1\}} \mathbf{1}_{\{0 \le V_{t+1} < \frac{X_t}{X_t + rY_t}\}} - Kr\mathbf{1}_{\{0 \le U_{t+1} < p\}}$.
- $\bar{S}_{t+1} = \bar{S}_t + \mathbf{1}_{\{p \le U_{t+1} < 1\}} \mathbf{1}_{\{0 \le V_{t+1} < \frac{K}{1+K}\}} - Kr\mathbf{1}_{\{0 \le U_{t+1} < p\}}$ defines a simple random walk.
- $\mathbb{E}[\bar{S}_{t+1} - \bar{S}_t] > 0$, thus $(\bar{S}_t)_{t \in \mathbb{N}}$ has a positive drift and is transient.
- $S_t \ge \bar{S}_t$ before $\frac{X_t}{Y_t} < Kr$.

Critical case $p(1 + r + Kr) = 1$.

Lemma

In the borderline case, i.e. when $\bar{R} = Kr$*:*

1. **Small variance:** $1 - p < 2pr \iff K < 1$. *Then* T *has a positive probability to be infinite and* $(\mathcal{X}_t, \mathcal{Y}_t)$ *has a positive probability to always stay above the slope* $\bar{R}$*, in other words, the market has a positive probability of surviving.*
2. **Big variance:** $1 - p \geq 2pr \iff K \geq 1$. *Then* T *is almost surely finite and* $(\mathcal{X}_t, \mathcal{Y}_t)$ *will finally pass the slope, meaning the market always becomes extinct.*

Analysis of martingale 2

- $1 - p - 2pr > 0$:

$$\mathbb{E}[\mathcal{W}_t^2] = \mathcal{W}_0^2 e^{2prt} + \mathcal{W}_0(e^{2prt} - e^{prt}) + \frac{A}{1-p-2pr}\mathcal{X}_0(e^{(1-p)t} - e^{2prt}),$$

and the typical size of $\mathbb{E}[\mathcal{W}_t^2]$ is at the order of $e^{(1-p)t}$.

- $1 - p - 2pr = 0$:

$$\mathbb{E}[\mathcal{W}_t^2] = \mathcal{W}_0^2 e^{2prt} + \mathcal{W}_0(e^{2prt} - e^{prt}) + A\mathcal{X}_0 t e^{2prt},$$

and the typical size is of te^{2prt}.

- $1 - p - 2pr < 0$:

$$\mathbb{E}[\mathcal{W}_t^2] = \mathcal{W}_0^2 e^{2prt} + \mathcal{W}_0(e^{2prt} - e^{prt}) + \frac{A}{2pr-(1-p)}\mathcal{X}_0(e^{2prt} - e^{(1-p)t}).$$

This expression is the same as in the first case, but its typical size is of e^{2prt}.

Outline

A model with parameter non-constant

- Same model and same question about long time behaviors.

$$\begin{cases} P\left((x+1,y,z)|(x,y,z)\right) & = (1-p)\frac{x}{x+ry}\mathbf{1}_{\{\frac{x}{y}>Kr\}} \\ P\left((x,y+1,z)|(x,y,z)\right) & = p \\ P\left((x,y,z+1)|(x,y,z)\right) & = (1-p)\frac{ry}{x+ry}\mathbf{1}_{\{\frac{x}{y}>Kr\}} + (1-p)\mathbf{1}_{\{\frac{x}{y}\le Kr\}} \end{cases}$$

- What happens if (p, r, K) are non-constant ?
- One varied model: $p := p\left(\frac{y}{x+y+z}\right), r := r\left(\frac{y}{x+y+z}\right)$.

A model with parameter non-constant

- One varied model:
$p\left(\frac{y}{x+y+z}\right) := 0.5 + 0.3\sin\left(\frac{10y}{x+y+z}\right), r := 0.1, K = 0.5.$
- Several different possible limits ?

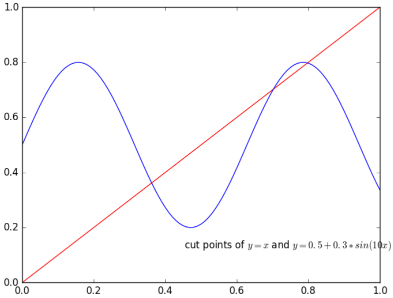

Figure: We believe that the limit should be the fixed point of $\theta = p(\theta)$.

Outline

Numerical simulations

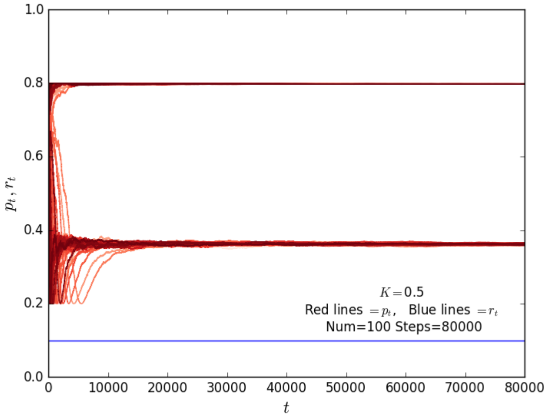

Figure: Stabilisation of p.

Numerical simulations

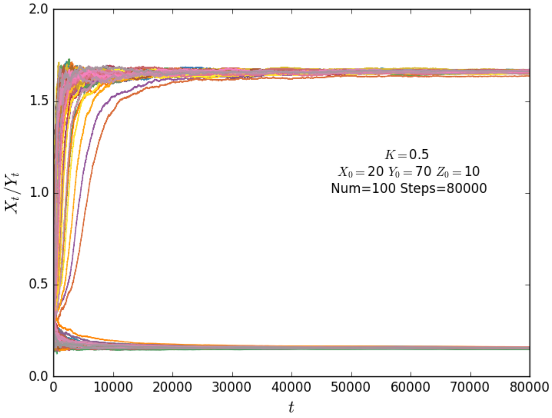

Figure: Stabilisation of X_t/Y_t.

Stochastic approximation

- $N_t = X_t + Y_t + Z_t, \theta_t = \frac{X_t}{N_t}$.
- $\theta_{t+1} = \theta_t - \frac{1}{N_t+1}\left(\theta_t - \mathbf{1}_{\{U_{t+1} \leq p(\theta_t)\}}\right)$.
- θ_t converges to the fixed points of $\theta^* = p(\theta^*)$. (Robbins–Monro)
- Unstable points are not possible. (i.e Saddle points in SGD are not stable. - Proof non trivial.)

Thanks for your attentions.