A growth-fragmentation-isolation process on random recursive trees

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Outline for section 1

Motivation: pandemic since the beginning of 2020

Figure: Various methods are applied to stop the pandemic: social distancing, masks, lockdown, quarantine, vaccine, etc.

Motivation: pandemic since the beginning of 2020

How can the contact tracing help us in controlling the spread of epidemic ?

Model: GFI process

- GFI = growth-fragmentation-isolation process.
- Starting from a single active vertex as patient zero.
- Different states:
  - vertex: active, inactive;
  - edge: open, closed.
- Three operations: infection (growth), information decay (fragmentation), confirmation and contact-tracing (isolation).

Model: GFI process

- GFI = grow-fragmentation-isolation process.
- Starting from a single active vertex as patient zero.
- Growth (Infection): every active vertex v independently attaches a new vertex in an exponential time with parameter β. When a new vertex u is created and attached, it is active and the link between them is open.
- Fragmentation (information decay): every open edge e independently becomes closed in an exponential time with parameter γ.
- Isolation (confirmation and contact-tracing): every active vertex independently gets "confirmed" in an exponential time with parameter θ, then its associated cluster is isolated and every vertex on this cluster becomes inactive.

GFI process: growth

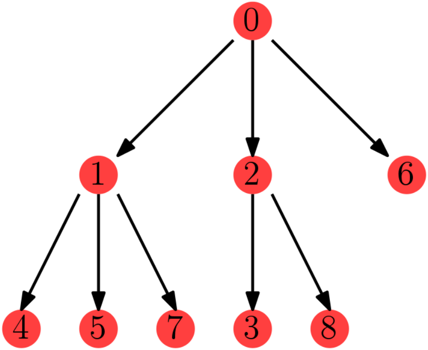


Figure: Growth: starting from vertex 0, the vertrices are attached one by one, and it forms a recursive tree.

GFI process: fragmentation

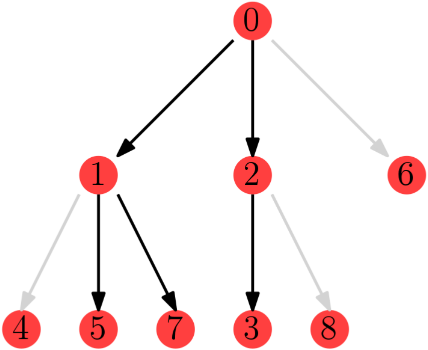


Figure: Fragmentation: the information of some links is no longer available after a while, for example the link $\{0,6\},\{1,4\},\{2,8\}$ in the image.

GFI process: isolation

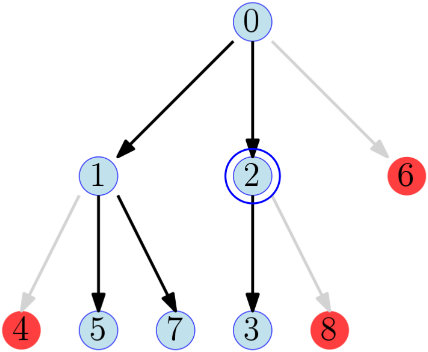


Figure: Isolation: the vertex 2 is confirmed, then all the vertices in the same clusters defined by open edges are isolated. These are the vertices in blue $\{0, 1, 2, 3, 5, 7\}$ in the image.

GFI process

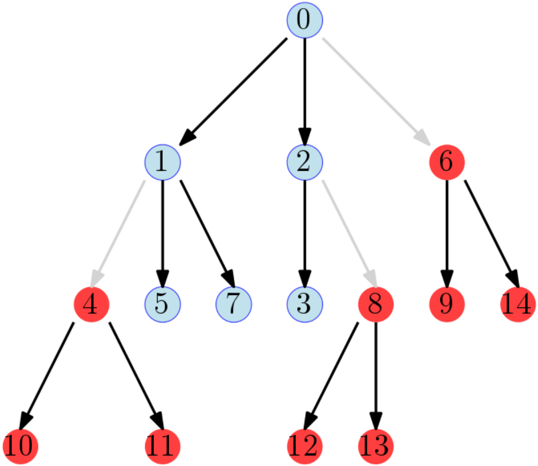


Figure: The isolated vertices are no longer active, while the other active vertices continue to attach new vertices.

Questions

Notations:

- Decompose the graph into clusters by connectivity.
- $\mathscr{X}_t := \{\text{active clusters at time } t\}$,
- $\mathscr{Y}_t := \{\text{inactive clusters at time } t\}$,
- $\tau := \inf\{t \mid \mathscr{X}_t = \emptyset\}$.

Questions:

1. Is there phase transition ?
2. Is there a limit for the growth rate ?
3. What other mathematical properties can we say from this model ?

Challenges: It is quite difficult to write down the transition probability explicitly.

Phase transition

- Extinction $= \{\tau < \infty\}$,
- Survival $= \{\tau = \infty\}$.
- Recall:
  - β: growth rate;
  - γ: fragmentation rate;
  - θ: isolation rate.

Preliminary result

We fix rate of growth $\beta > 0$,

- for $\theta \geqslant \beta$, or $\theta \geqslant \gamma$, GFI process extincts almost surely.
- for $\theta < \beta$ and $\gamma \gg \theta$, GFI process has positive probability to survive.

Proof: coupling argument.

Phase transition

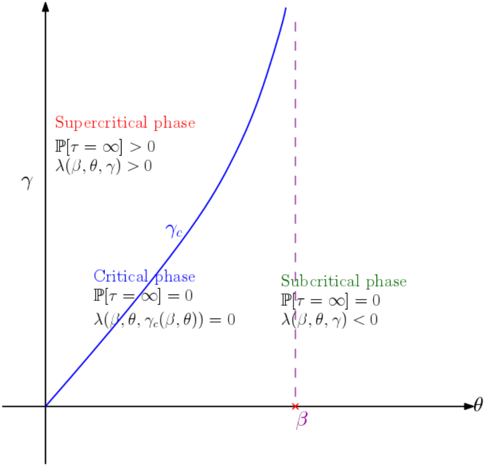


Figure: Diagrams of phases

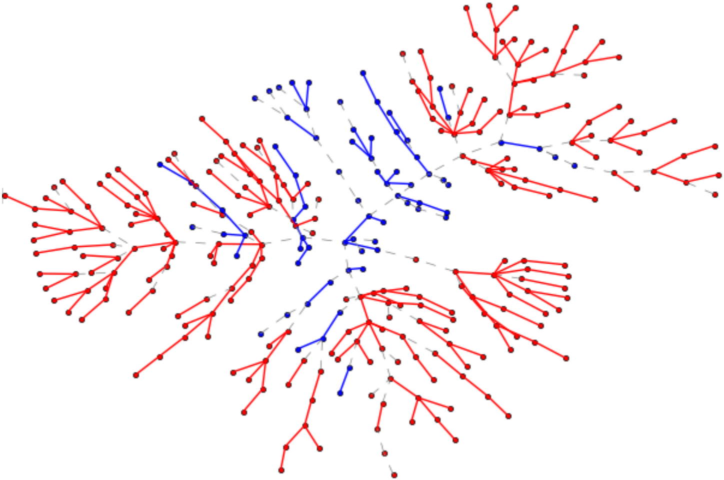


Figure: A simulation with $\beta = 0.6, \theta = 0.03, \gamma = 0.15$ with 247 active vertices and 73 inactive vertices.

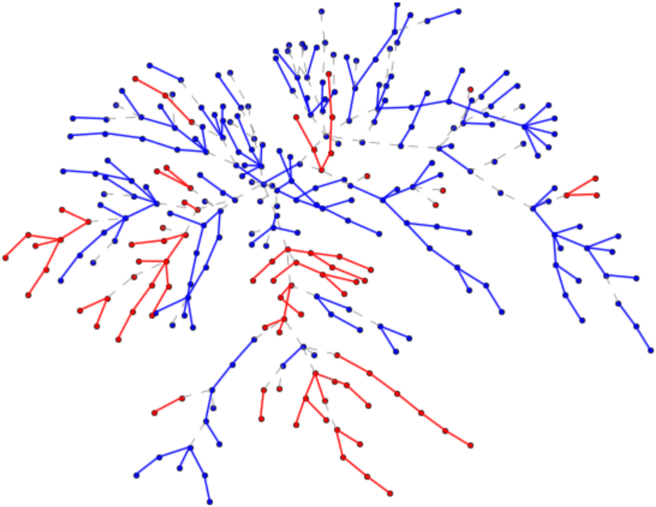
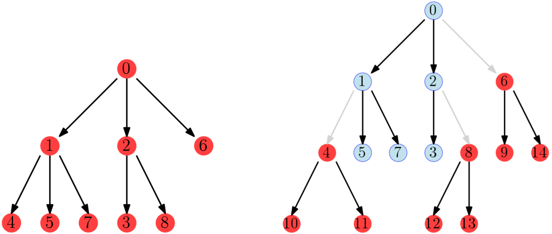


Figure: A simulation with $\beta = 0.6, \theta = 0.03, \gamma = 0.1$ with 87 active vertices and 214 inactive vertices.

Outline for section 2

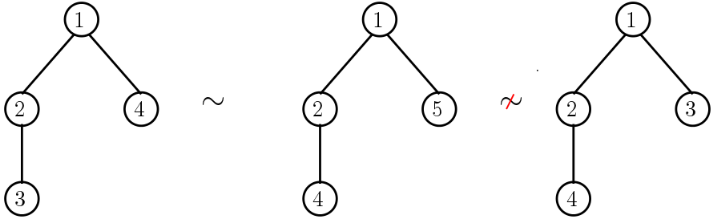
Recursive tree

- Recursive tree = labeled tree defined on finite $V \subset \mathbb{R}$, with the minimum label as its root, and for all $v \in V$, the path from root to v is increasing.
- Sometimes it is also called increasing tree.
- Label the vertices in GFI process with the birth time, it is the natural structure in clusters.



Equivalence class of recursive tree

- Equivalence class: t_1 a recursive tree on V_1 and t_2 a recursive tree on V_2, then $\mathrm{t}_1 \sim \mathrm{t}_2$ iff there exists an order-preserving function $\psi : V_1 \to V_2$, such that ψ is also a bijection between the graphs t_1 and t_2.



Equivalence class of recursive tree

- $\mathscr{T}_n =$ the set of recursive trees of size n up to the equivalence relation $\sim$.
- The recursive trees defined on $\{1, \cdots, n\}$ as a representative of the equivalence class.

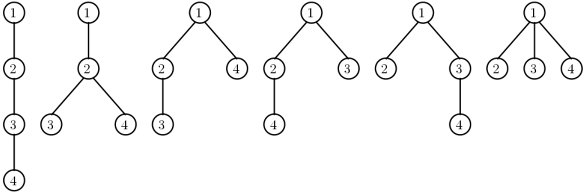


Figure: All the recursive trees (as representatives of equivalence classes) in $\mathscr{T}_4$.

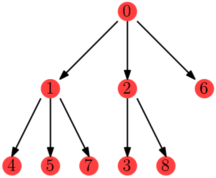
- $\mathscr{T} := \bigcup_{n=1}^{\infty} \mathscr{T}_n$, the whole space of finite recursive trees.

Random recursive tree

- RRT = (uniform) random recursive tree.
- T_n: uniformly distributed on $\mathscr{T}_n$, i.e.

$$\forall \mathbf{t} \in \mathscr{T}_n, \qquad \mathbb{P}[T_n = \mathbf{t}] = \frac{1}{(n-1)!}.$$

- Construction 1: by Yule process.



- Construction 2: by splitting property.

Splitting property of RRT

Meir and Moon (1974) discovered the following property.

Splitting property of RRT

Let $n \geqslant 2$ and T_n the canonical random recursive tree of size n. We choose uniformly one edge in T_n and remove it. Then T_n is split into two subtrees T_n^0 and T_n^*, corresponding to two connected components, where T_n^0 contains the root of T_n and T_n^* does not. Then we have

$$\mathbb{P}\left[|T_n^*| = j\right] = \frac{n}{n-1}\frac{1}{j(j+1)}, \qquad j = 1, 2, \cdots, n-1.$$

Furthermore, conditionally on $|T_n^*| = j$, T_n^0 and T_n^* are two independent RRT's of size respectively $(n-j)$ and j.

Size process

- Empirical measure: let $\mathcal{M}$ be punctual measure on $\mathbb{N}_+$,

$$X_t = \sum_{\mathcal{C} \in \mathscr{X}_t} \delta_{|\mathcal{C}|}, \qquad Y_t = \sum_{\mathcal{C} \in \mathscr{Y}_t} \delta_{|\mathcal{C}|},$$

  and we call $(X_t, Y_t)_{t \geqslant 0}$ size process of GFI process.
- Key observation: for every $t \geqslant 0$, conditioned on the size of clusters, every cluster (active or inactive) is a RRT and they are independent.
- Consequence: $(\mathcal{F}_t)_{t \geqslant 0}$ natural filtration for $(X_t, Y_t)_{t \geqslant 0}$, then $(X_t, Y_t)_{t \geqslant 0}$ is a $\mathcal{M}^2$-valued Markov process under $(\mathcal{M}^2, (\mathcal{F}_t)_{t \geqslant 0}, \mathbb{P})$.

Branching process

- $(X_t)_{t\geqslant 0}$ is an infinite-type branching process.
- Transitions rates: for a cluster of size n, it
  - i) becomes an isolated cluster of size n at rate θn;
  - ii) becomes a RRT of size $(n+1)$ at rate βn;
  - iii) splits into two RRTs of size $(n-j,j)$ at rate $\gamma n \frac{1}{j(j+1)}$, for $n \geqslant 2, 1 \leqslant j \leqslant n-1$.

Generator

Let $F : \mathbb{R}^2 \to \mathbb{R}$ a bounded Borel function. We set

$$F_{f,g} : (\mu, \nu) \in \mathcal{M}^2 \to F(\langle \mu, f \rangle, \langle \nu, g \rangle) \in \mathbb{R},$$

then we have

$$\begin{aligned}
&\mathcal{A}F_{f,g}(\mu, \nu) \\
&= \sum_{n=1}^{\infty} \mu(\{n\})\beta n \left(F(\langle \mu + \delta_{n+1} - \delta_n, f \rangle, \langle \nu, g \rangle) - F(\langle \mu, f \rangle, \langle \nu, g \rangle)\right) \\
&\quad + \sum_{n=1}^{\infty} \mu(\{n\})\theta n \left(F(\langle \mu - \delta_n, f \rangle, \langle \nu + \delta_n, g \rangle) - F(\langle \mu, f \rangle, \langle \nu, g \rangle)\right) \\
&\quad + \sum_{n=1}^{\infty} \mu(\{n\})\gamma(n-1) \sum_{j=1}^{n-1} \left(\frac{n}{n-1} \frac{1}{j(j+1)}\right) \times \\
&\qquad \left(F(\langle \mu + \delta_j + \delta_{n-j} - \delta_n, f \rangle, \langle \nu, g \rangle) - F(\langle \mu, f \rangle, \langle \nu, g \rangle)\right).
\end{aligned}$$

Main result 1: Malthusian exponent

Theorem (Malthusian exponent)

The following limits exist and coincide and are finite

$$\lambda := \lim_{t\to\infty} \frac{1}{t} \log(\mathbb{E}[|\mathscr{X}_t|]) = \lim_{t\to\infty} \frac{1}{t} \log(\mathbb{E}[|\mathscr{Y}_t|]) \in (-\infty, \infty).$$

Here $|\mathscr{X}_t|$ (resp. $|\mathscr{Y}_t|$) is the number of active (resp. inactive) clusters at time t. If $\lambda \leqslant 0$, then extinction occurs a.s. : $\mathbb{P}[\tau < \infty] = 1$. Otherwise, survival occurs with positive probability $\mathbb{P}[\tau = \infty] > 0$.

Classification of phases:

- Subcritical phase: $\lambda < 0$;
- Critical phase: $\lambda = 0$;
- Supercritical phase: $\lambda > 0$.

Main result 2: limit of size

Theorem (Law of large numbers for $(X_t)_{t\geqslant 0}$)

Assume that $\lambda > 0$. Then there exists a probability distribution π on $\mathbb{N}_+$ and a random variable $W \geqslant 0$, such that for any function $f : \mathbb{N}_+ \to \mathbb{R}$ of at most polynomial growth, we have

$$e^{-\lambda t}\langle X_t, f\rangle \xrightarrow{t\to\infty} W\langle \pi, f\rangle, \qquad \text{a.s. and in } L^2.$$

Besides, $\{\tau = \infty\} = \{W > 0\}$ a.s. and on this event

$$\frac{\langle X_t, f\rangle}{\langle X_t, 1\rangle} \xrightarrow{t\to\infty} \langle \pi, f\rangle \quad \text{a.s.}.$$

Outline for section 3

Classical method: Perron-Frobinius theorem

Perron-Frobinius theorem

$(A)_{1\leqslant i,j\leqslant n}$ positive matrix with $A_{i,j} > 0$ for all $1 \leqslant i, j \leqslant n$. Then there exits a leading positive eigenvalue λ called Perron's root, such that

- any other eigenvalue λ_i (possibly complex) in absolute value is strictly smaller than λ, i.e. $|\lambda_i| < \lambda$;
- it has associated left and right eigenvectors π, h such that

$$\pi A = \lambda \pi, \qquad Ah = \lambda h.$$

- Consequence: $\mu A^n = \lambda^n \pi + o(\lambda^n)$.
- Interpretation: in multi-type branching, A as the production matrix and π is the limit distribution of types.
- Question: How can we generalize it to infinite dimension ?

First moment semigroup

- $\mathbb{P}_{\delta_n}$ and $\mathbb{E}_{\delta_n}$ for initial condition $(X_0, Y_0) = (\delta_n, 0)$.
- $M_t f(n) := \mathbb{E}_{\delta_n}[\langle X_t, f\rangle]$
- Its generator is

$$\begin{aligned}
&\mathcal{L}f(n) \\
&= \underbrace{\beta n(f(n+1) - f(n))}_{\mathbf{I}} \underbrace{-\theta n f(n)}_{\mathbf{II}} \\
&\qquad \underbrace{+ \gamma(n-1)\sum_{j=1}^{n-1} \frac{n}{n-1}\frac{1}{j(j+1)}\left(f(j) + f(n-j) - f(n)\right).}_{\mathbf{III}}
\end{aligned}$$

$\mathbf{I}, \mathbf{II}, \mathbf{III}$ are respectively *the growth*, *the isolation* and *the fragmentation*.

Existence of Perron's root in size process

- Method: Bansaye, Cloez, Gabriel, and Marguet (2019) - a non-conservative Harri's method.
- A sufficient condition: we need to find a couple of functions (ψ, V) and $a < b, \xi > 0$ such that
  - $\mathcal{L}V \leqslant aV + \zeta\psi$, and $b\psi \leqslant \mathcal{L}\psi \leqslant \xi\psi$.
  - for any R large enough, the set $K = \{x \in \mathbb{N}_+ : \psi(x) \geqslant V(x)/R\}$ is a non-empty finite set and for any $x, y \in K$ and $t_0 > 0$,

$$M_{t_0}(x, y) > 0.$$

- It ensures the existence of Perron's root for $\mathcal{L}$ and (ψ, V) also controls the size of (π, h), i.e. $h \lesssim V, \pi \lesssim V^{-1}$.

Existence of Perron's root in size process

Perron's root for $(X_t)_{t\geqslant 0}$

There exists a unique triplet (λ, π, h) where $\lambda \in \mathbb{R}$ and $\pi = (\pi(n))_{n\in\mathbb{N}_+}$ is a positive vector and $h : \mathbb{N}_+ \to (0, \infty)$ is a positive function, s.t. for all $t \geqslant 0$,

$$\pi M_t = e^{\lambda t}\pi, \qquad M_t h = e^{\lambda t} h, \qquad \sum_{n\geqslant 1} \pi(n) = \sum_{n\geqslant 1} \pi(n)h(n) = 1.$$

Moreover, we have

- h is bounded: $0 < \inf_{n\geqslant 1} h(n) \leqslant \sup_{n\geqslant 1} h(n) < \infty$;
- π decays fast: for all $p > 0$, $\sum_{n\geqslant 1} \pi(n)n^p < \infty$;
- for every $p > 0$ there exists $C, \omega > 0$ s.t. for any $n, m \geqslant 1$, $t \geqslant 0$,

$$\left|e^{-\lambda t} M_t(n,m) - h(n)\pi(m)\right| \leqslant C n^p m^{-p} e^{-\omega t}.$$

Many-to-two formula

- Many-to-two formula:

$$\mathbb{E}_{\delta_x}\left[\langle X_t, f\rangle^2\right] = M_t(f^2)(x)$$
$$+2\int_0^t \sum_{n\geqslant 1} M_s(x,n)\left(\sum_{1\leqslant j\leqslant n-1}\kappa(n,j)M_{t-s}f(j)M_{t-s}f(n-j)\right)\mathrm{d}s.$$

Idea: write down the genealogy of active clusters and find the common ancestor.

- Application 1: $\mathscr{M}_t = e^{-\lambda t}\langle X_t, h\rangle$ is a L^2 positive martingale converging to r.v. W.
- Application 2: L^2 bound: define $\| f \|_p := \sum_{m\geqslant 1}|f(m)|m^{-(p+2)} \in (-\infty,\infty)$, then

$$\mathbb{E}\left[\langle X_t, f\rangle^2\right] \leqslant C_0 e^{2\lambda t}\left(|\langle \pi, f\rangle|^2 + \| f \|_p e^{-\sigma t}\right).$$

Outline for section 4

Main result 2: limit of size

Theorem (Law of large numbers for $(X_t)_{t\geqslant 0}$)

Assume that $\lambda > 0$. Then there exists a probability distribution π on $\mathbb{N}_+$ and a random variable $W \geqslant 0$, such that for any function $f : \mathbb{N}_+ \to \mathbb{R}$ of at most polynomial growth, we have

$$e^{-\lambda t}\langle X_t, f\rangle \xrightarrow{t\to\infty} W\langle \pi, f\rangle, \qquad \text{a.s. and in } L^2.$$

Besides, $\{\tau = \infty\} = \{W > 0\}$ a.s. and on this event

$$\frac{\langle X_t, f\rangle}{\langle X_t, 1\rangle} \xrightarrow{t\to\infty} \langle \pi, f\rangle \quad \text{a.s.}.$$

Law of large number for $(X_t)_{t\geqslant 0}$

- Martingale $\mathscr{M}_t$ + L^2 estimate + Borel-Cantelli $\Longrightarrow e^{-\lambda t}\langle X_t, f\rangle$ converges in L^2 and $a.s.$ along any discrete time $\{k\Delta\}_{k\geqslant 1}$.
- Control of fluctuation in interval $[k\Delta, (k+1)\Delta)$.
- Argument of Athreya (1968): same argument applies to both multi-type branching and countable-type branching for the convergence of one type.

Argument of Athreya (1968)

- $X_t(n) :=$ number of clusters of size n.
- A sufficient and necessary condition:
  $\varliminf_{t\to\infty} e^{-\lambda t} X_t(n) \geqslant W\pi(n), \qquad$ almost surely for all $n \geqslant 1$.
- $$\begin{aligned}
&\varlimsup_{t\to\infty} e^{-\lambda t} X_t(n) h(n) \\
&= \lim_{k\to\infty} \left(\sum_{i\geqslant 1} e^{-\lambda t_k} X_{t_k}(i) h(i) - \sum_{i\geqslant 1, i\neq n} e^{-\lambda t_k} X_{t_k}(i) h(i) \right) \\
&\leqslant W - \sum_{i\geqslant 1, i\neq n} \varliminf_{k\to\infty} e^{-\lambda t_k} X_{t_k}(i) h(i) \\
&\leqslant W - \sum_{i\geqslant 1, i\neq n} W\pi(i) h(i) \\
&= W\pi(n) h(n).
\end{aligned}$$

Argument of Athreya (1968)

- An observation:

$$\forall t \in [k\Delta, (k+1)\Delta), \qquad X_t(n) \geqslant X_{k\Delta}(n) - N_{k,\Delta}(n),$$

  where $N_{k,\Delta}(n)$ is the number of active clusters of size n at time $k\Delta$ that will encounter at least one event within $(k\Delta, (k+1)\Delta)$.
- Thus it only involves the jump rate of one type.

Law of large number for $(X_t)_{t \geqslant 0}$

- Martingale $\mathscr{M}_t$ + L^2 estimate + Borel-Cantelli $\Longrightarrow e^{-\lambda t} \langle X_t, f \rangle$ converges in L^2 and $a.s.$ along any discrete time $\{k\Delta\}_{k \geqslant 1}$.
- Control of fluctuation in interval $[k\Delta, (k+1)\Delta)$.
- Argument of Athreya (1968): applies to the convergence of one type $e^{-\lambda t} X_t(n) \to \pi(n)$.
- Cutoff and coupling argument wit an increasing process $(\widetilde{X}_t)_{t \geqslant 0}$ improve the result to arbitrary f with polynomial increment.

Main result 2: limit of size

Bias of the limit distribution $\widetilde{\pi}(n) := \frac{\pi(n)n}{\sum_{j=1}^{\infty} \pi(j)j}$.

Corollary (Law of large number for $(Y_t)_{t\geqslant 0}$)

For any function $f : \mathbb{N}_+ \to \mathbb{R}$ of at most polynomial growth, we have that

$$e^{-\lambda t}\langle Y_t, f\rangle \xrightarrow{t\to\infty} W\left(\frac{\theta}{\lambda}\right)\left(\sum_{j=1}^{\infty}\pi(j)j\right)\langle\widetilde{\pi}, f\rangle, \qquad \textit{almost surely and in } L^2,$$

and

$$\frac{\langle Y_t, f\rangle}{\langle Y_t, 1\rangle} \xrightarrow{t\to\infty} \langle\widetilde{\pi}, f\rangle, \qquad \textit{almost surely on } \{\tau = \infty\}.$$

Interpretation: there are unobserved small active clusters.

Law of large number for $(Y_t)_{t\geqslant 0}$

- Heuristic argument:

$$\lim_{s\searrow t}\frac{\mathbb{E}[\langle Y_s, f\rangle - \langle Y_t, f\rangle|\mathcal{F}_t]}{s-t} = \theta\langle X_t, [x]f\rangle \sim_{t\to\infty} \theta e^{\lambda t} W\langle \pi, [x]\rangle\langle\widetilde{\pi}, f\rangle,$$

- Polynomial function $[x^p](n) := n^p$.
- Observation: $H_t := \langle X_t, h\rangle - \left(\frac{\lambda}{\theta}\right)\langle Y_t, h/[x]\rangle$ is a martingale.
- General function by decomposition

$$\begin{aligned} H_t^f &:= \langle X_t, f\rangle - \left(\frac{\lambda}{\theta}\right)\langle Y_t, f/[x]\rangle \\ &= \langle \pi, f\rangle H_t + A_t + B_t \\ A_t &= \langle X_t, f - \langle \pi, f\rangle h\rangle \\ B_t &= \left(\frac{\lambda}{\theta}\right)\langle Y_t, (f - \langle \pi, f\rangle h)/[x]\rangle, \end{aligned}$$

A_t and B_t are small as they remove the principle eigenvector.

Main result 3: limit on $\mathscr{T}$

Theorem (Limit of empirical measure of clusters)

Consider any $p > 0$ *and* $f : \mathscr{T} \to \mathbb{R}$ *such that*

$$\sup_{\mathbf{t} \in \mathscr{T}} \frac{|f(\mathbf{t})|}{|\mathbf{t}|^p} < \infty.$$

Then on the event $\{\tau = \infty\}$

$$\frac{1}{|\mathscr{X}_t|} \sum_{\mathcal{C} \in \mathscr{X}_t} f(\mathcal{C}) \xrightarrow{t \to \infty} \mathbb{E}[f(T_\pi)], \quad \frac{1}{|\mathscr{Y}_t|} \sum_{\mathcal{C} \in \mathscr{Y}_t} f(\mathcal{C}) \xrightarrow{t \to \infty} \mathbb{E}[f(T_{\widetilde{\pi}})] \quad \textit{a.s.}.$$

Law of large number on $\mathscr{T}$

- Once again: Cutoff argument + argument of Athreya.
- It suffices $\forall n \in \mathbb{N}_+, \forall \mathbf{t} \in \mathscr{T}_n, \varliminf_{t\to\infty} e^{-\lambda t} X_t(\mathbf{t}) \geqslant W \frac{\pi(n)}{(n-1)!}$, because

$$\begin{aligned}
\varlimsup_{t\to\infty} e^{-\lambda t} X_t(\mathbf{t}) &= \lim_{k\to\infty} \left(\sum_{\mathbf{t}'\in\mathscr{T}_n} e^{-\lambda t_k} X_{t_k}(\mathbf{t}') - \sum_{\mathbf{t}'\in\mathscr{T}_n, \mathbf{t}'\neq\mathbf{t}} e^{-\lambda t_k} X_{t_k}(\mathbf{t}') \right) \\
&\leqslant W\pi(n) - \sum_{\mathbf{t}'\in\mathscr{T}_n, \mathbf{t}'\neq\mathbf{t}} \varliminf_{k\to\infty} e^{-\lambda t_k} X_{t_k}(\mathbf{t}') \\
&\leqslant W\pi(n) - \sum_{\mathbf{t}'\in\mathscr{T}_n, \mathbf{t}'\neq\mathbf{t}} W \frac{\pi(n)}{(n-1)!} \\
&= W \frac{\pi(n)}{(n-1)!}.
\end{aligned}$$

- The control of fluctuation is like that of $X_t(n)$.

Outline for section 5

Existence of phases

- Continuity of $(\beta, \gamma, \theta) \mapsto \lambda(\beta, \gamma, \theta)$.
- Monotonicity.
- Test function to show the existence of $\mathcal{L}f < 0$ and $\mathcal{L}f > 0$.

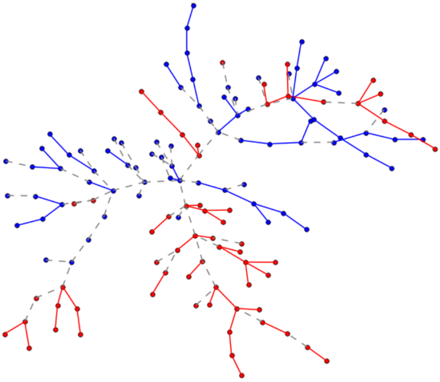
General initial condition

We go back to GFI model. Same results apply to a deterministic initial condition $G_0 = (V_0, E_0)$. We can randomize the initial condition with a RRT T_{V_0}, and then the absolute continuity helps apply previous results

$$\mathbb{P}_{G_0} \stackrel{d}{=} \mathbb{P}_{T_{V_0}}[\cdot \mid T_{V_0} = G_0].$$


疫情不退
我们不退


常州
接您回家



Thank you for your attention.