Heat Kernel on the Infinite Percolation Cluster

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based on a joint work with Paul Dario

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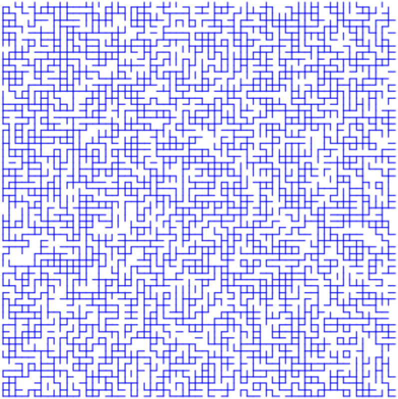
Outline



Outline

Definition

Definition (Bernoulli percolation on $\mathbb{Z}^d$)

We denote by $(\mathbb{Z}^d, E_d)$ the d-dimension lattice graph. A Bernoulli percolation configuration $\{\omega(e)\}_{e \in E_d}$ is an element of $\{0,1\}^{E_d}$, and its law is given by

$$\{\omega(e)\}_{e \in E_d} \text{ i.i.d. }, \mathbb{P}[\omega(e) = 1] = 1 - \mathbb{P}[\omega(e) = 0] = \mathfrak{p}.$$

We say that the edge e is open if $\omega(e) = 1$ and the edge e is closed if $\omega(e) = 0$. A connected component given by ω will be called cluster.

Example of percolation

Question: Can you feel the difference when we simulate ω with different value of $\mathfrak{p}$?

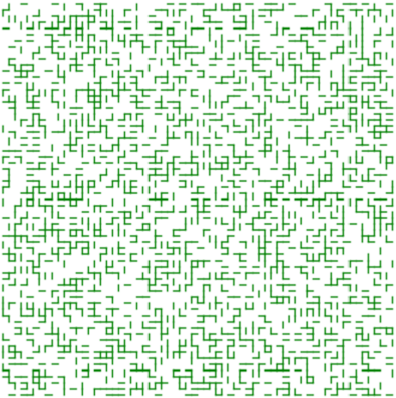


Figure: percolation in a cube of size 32×32 with $\mathfrak{p} = 0.3$.

Example of percolation

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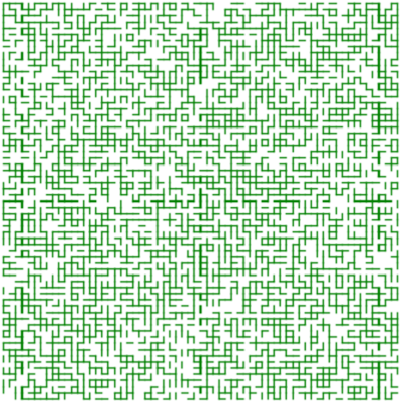

Figure: percolation in a cube of size 32×32 with $\mathfrak{p} = 0.5$.

Example of percolation

Question: Can you feel the difference when we simulate ω with different value of $\mathfrak{p}$?

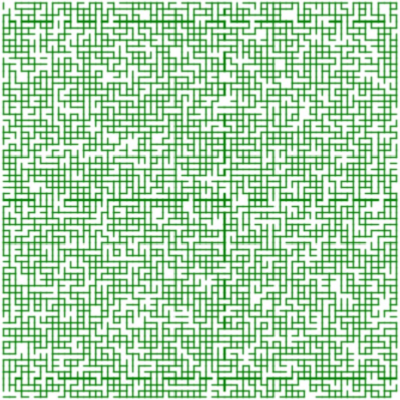

Figure: percolation in a cube of size 32×32 with $\mathfrak{p} = 0.7$.

Example of percolation

Question: Can you feel the difference when we simulate ω with different value of $\mathfrak{p}$?

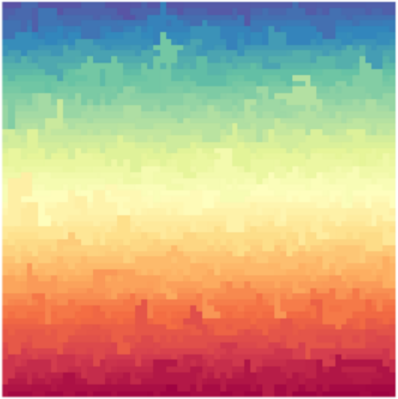

Figure: percolation colored by cluster in a cube of size 32×32 with $\mathfrak{p} = 0.3$.

Example of percolation

Question: Can you feel the difference when we simulate ω with different value of $\mathfrak{p}$?

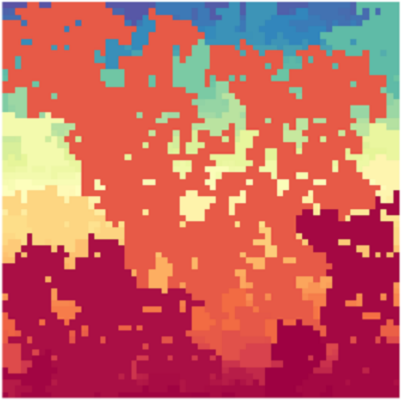

Figure: percolation colored by cluster in a cube of size 32×32 with $\mathfrak{p} = 0.5$.

Example of percolation

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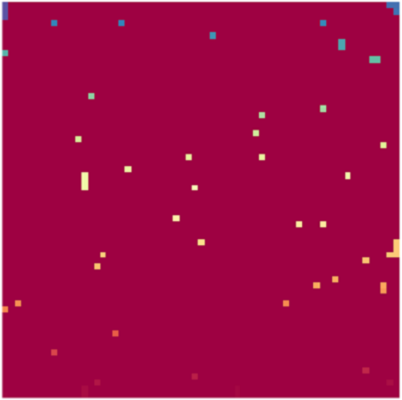


Figure: percolation colored by cluster in a cube of size 32×32 with $\mathfrak{p} = 0.7$.

Outline

Phase transition

- $\theta(\mathfrak{p}) := \mathbb{P}[0 \text{ belongs to an infinite cluster } \mathscr{C}]$.
- It is easy to show that $\theta(\mathfrak{p})$ is monotone.
- $\mathfrak{p}_c := \inf\{\mathfrak{p} \in [0,1] : \theta(\mathfrak{p}) > 0\}$.

Theorem

For $d \geqslant 2$, we have $0 < \mathfrak{p}_c < 1$.

- We call the regime $0 \leqslant \mathfrak{p} < \mathfrak{p}_c$ subcritical, $\mathfrak{p} = \mathfrak{p}_c$ critical and $\mathfrak{p}_c < \mathfrak{p} \leqslant 1$ supercritical.
- Furthermore, by ergodicity argument, in subcritical case a.s. there is no infinite cluster. In supercritical case a.s. there exists a unique infinite cluster $\mathscr{C}_\infty$.
- Critical case: we conjecture $\theta(\mathfrak{p}_c) = 0$, but it is open for $3 \leqslant d \leqslant 11$.

Outline

Random walk on the infinite cluster

- $\{\mathbf{a}(e)\}_{e \in E_d}$ is i.i.d random conductance taking value in $\{0\} \cup [\lambda, 1], 0 < \lambda < 1$ satisfying

$$\mathbb{P}[\mathbf{a}(e) > 0] = \mathfrak{p} > \mathfrak{p}_c.$$

- The subset $\{e \in E_d : \mathbf{a}(e) > 0\}$ defines a percolation (open for $\mathbf{a}(e) > 0$ and closed for $\mathbf{a}(e) = 0$) and a unique infinite cluster $\mathscr{C}_\infty$.
- (X_t) is a continuous-time Markov jump process starting from $y \in \mathscr{C}_\infty$, with an associated generator

$$\nabla \cdot \mathbf{a} \nabla u(x) := \sum_{z \sim x} \mathbf{a}(\{x, z\}) \left(u(z) - u(x)\right).$$

- The quenched semigroup (also called heat kernel or parabolic Green function) is defined as

$$p(t, x, y) = p^{\mathbf{a}}(t, x, y) := \mathbb{P}_y^{\mathbf{a}}(X_t = x).$$

Random walk on the infinite cluster

- Question: for t big, does $(X_t)_{t \geqslant 0}$ looks like Brownian motion or is $p(t, x, y)$ close to a Gaussian distribution ?

Semigroup of random walk on cluster

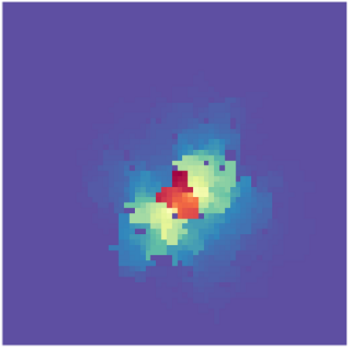


Figure: T=100

Semigroup of random walk on cluster

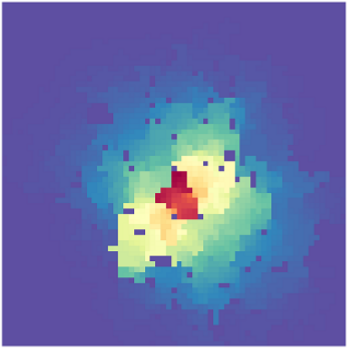


Figure: T=200

Semigroup of random walk on cluster

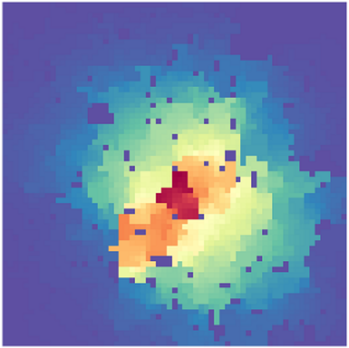


Figure: T=300

Semigroup of random walk on cluster

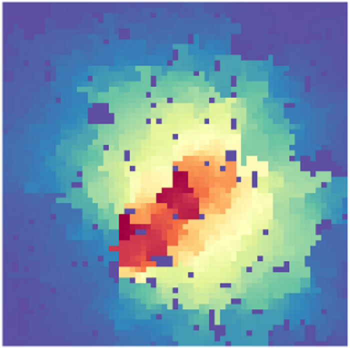


Figure: T=400

Semigroup of random walk on cluster

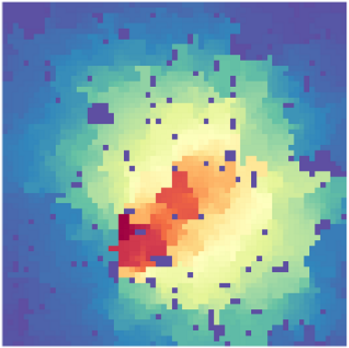


Figure: T=500

Semigroup of random walk on cluster

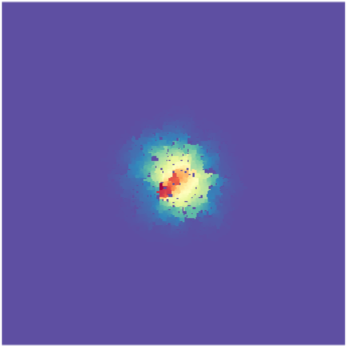


Figure: T=500

Semigroup of random walk on cluster

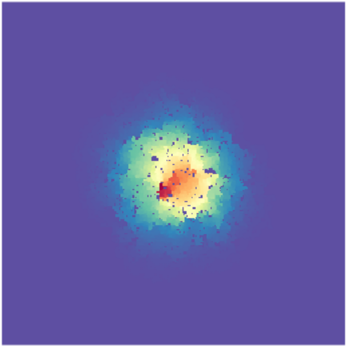


Figure: T=1000

Semigroup of random walk on cluster

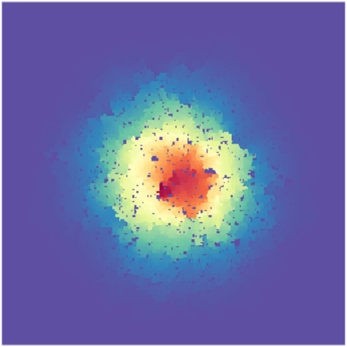


Figure: T=2000

Semigroup of random walk on cluster

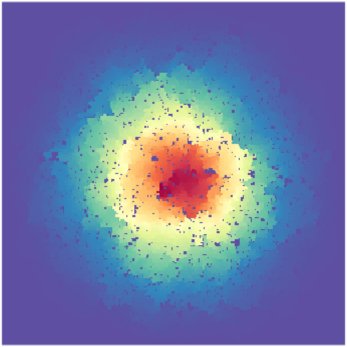


Figure: T=3000

Semigroup of random walk on cluster

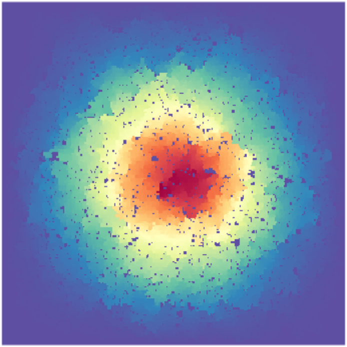


Figure: T=4000

Outline

Historical review: convergence in law

The convergence of the heat kernel is a very classical topic.

- In simple random walk setting, we have CLT, local CLT, Donsker theorem.
- To treat the convergence of random process, we have Dubins-Schwartz theorem for continuous martingale, and Stroock and Varadhan theorem for general Markov process.
- Berger, Biskup, Mathieu and Piatnitski prove that ω almost surely, the random walk on the infinite percolation cluster converges to the Brownian motion.

Historical review: heat kernel bound

The exact Gaussian type bound on graphs and Markov chains are also well studied for long time.

- Davies proves the Carne-Varopoulos bound for random walk on any infinite subgraph of $\mathbb{Z}^d$

$$p(t,x,y) \leqslant \begin{cases} C \exp\left(-\frac{|x-y|^2}{Ct}\right) & \text{if } |x-y| \leqslant t, \\ C \exp\left(-\frac{|x-y|}{C}\left(1 + \ln\frac{|x-y|}{t}\right)\right) & \text{if } |x-y| \geqslant t. \end{cases}$$

- Delmotte proves the Gaussian bound for Markov chain on graph satisfying double volume condition and Poincaré inequality.

Historical review: heat kernel bound

- Barlow prove the Gaussian bound for the heat kernel p with big t. That is for $t > \mathcal{T}_{NA}(y)$

$$p(t,x,y) \leqslant \begin{cases} Ct^{-d/2}\exp\left(-\frac{|x-y|^2}{Ct}\right) & |x-y| \leqslant t, \\ Ct^{-d/2}\exp\left(-\frac{|x-y|}{C}\left(1+\ln\frac{|x-y|}{t}\right)\right) & |x-y| \geqslant t, \end{cases}$$

and with Hambly they also prove the local CLT.

Outline

Main result: Quantitative local CLT

Theorem (Dario, Gu 19+)

For each exponent $\delta > 0$*, there exist a positive constant* $C < \infty$ *and an exponent* $s > 0$*, depending only on the parameters* $d, \lambda, \mathfrak{p}$ *and* δ*, such that for every* $y \in \mathbb{Z}^d$*, there exists a non-negative random time* $\mathcal{T}_{\mathrm{par},\delta}(y)$ *satisfying the stochastic integrability estimate*

$$\forall T \geqslant 0, \ \mathbb{P}\left(\mathcal{T}_{\mathrm{par},\delta}(y) \geqslant T\right) \leqslant C \exp\left(-\frac{T^s}{C}\right),$$

such that, on the event $\{y \in \mathscr{C}_\infty\}$*, for every* $x \in \mathscr{C}_\infty$ *and every* $t \geqslant \max\left(\mathcal{T}_{\mathrm{par},\delta}(y), |x-y|\right)$*,*

$$\left|p(t,x,y) - \theta(\mathfrak{p})^{-1}\bar{p}(t,x-y)\right| \leqslant C t^{-\frac{d}{2}-\left(\frac{1}{2}-\delta\right)} \exp\left(-\frac{|x-y|^2}{Ct}\right).$$

Errors between the heat kernels

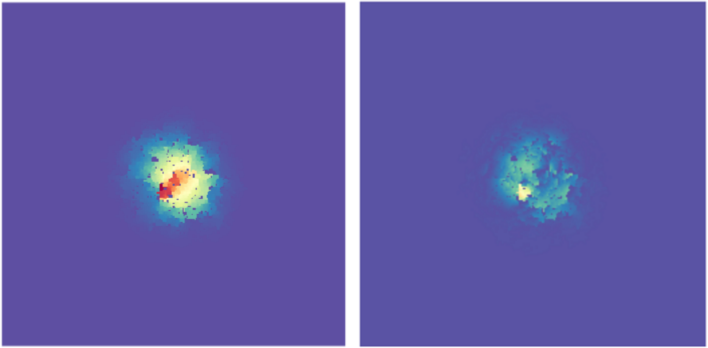


Figure: T=500

Errors between the heat kernels

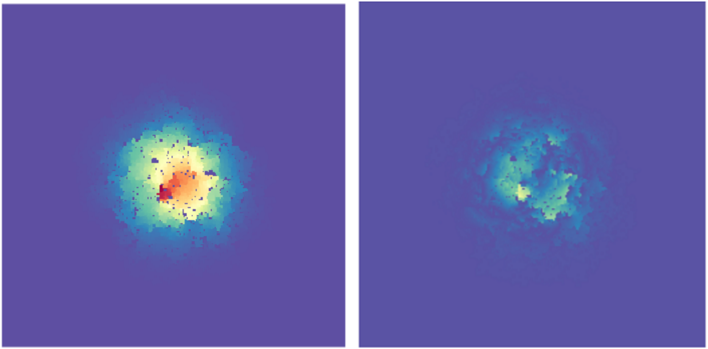


Figure: T=1000

Errors between the heat kernels

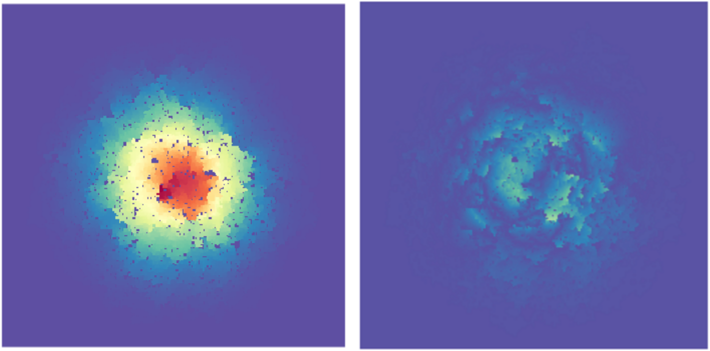


Figure: T=2000

Errors between the heat kernels

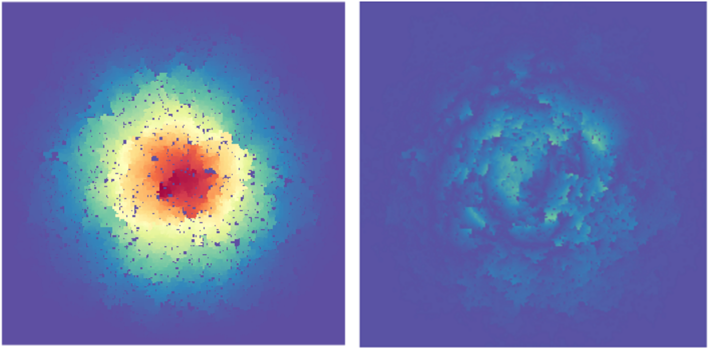


Figure: T=3000

Errors between the heat kernels

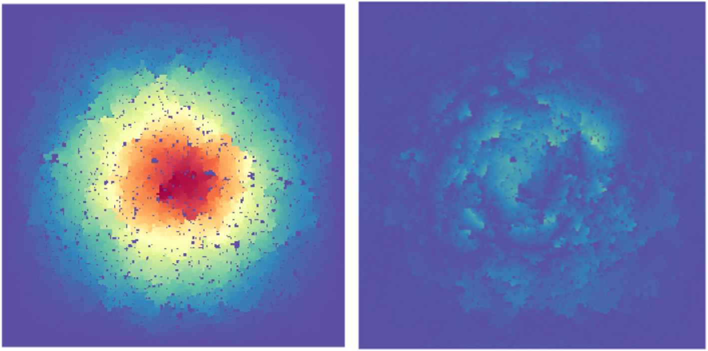
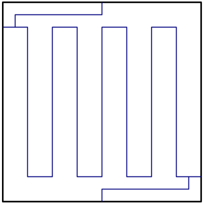


Figure: T=4000

Challenge

Some elementary inequality is perturbed by the random geometry of the cluster.



Outline

Homogenization theory

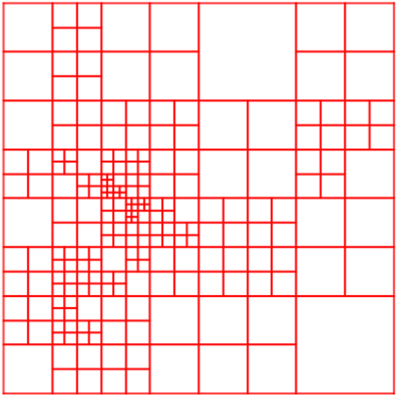
- Homogenization theory studies the errors between the equation and its homogenized solution, for example $(\partial_t - \nabla \cdot \mathbf{a}\nabla)p = (\partial_t - \frac{\sigma^2}{2}\Delta)\bar{p}$, intuitively we have

$$p(t,x,y) \simeq \bar{p}(t,x,y) + \sum_{k=1}^{d} \partial_k \bar{p}(t,x,y)\phi_k(x),$$

  where $\{\phi_k\}_{1 \leqslant k \leqslant d}$ is the collection of corrector.
- Early classical work in homogenization: Bensoussan, Lions, Papanicolaou, Jikov, Kozlov, Oleunik, Yurinskii, Naddaf, Spencer, Allaire, Kenig, Lin, Shen etc.
- Quantitative analysis in stochastic homogenization setting: Armstrong, Kuusi, Mourrat, Smart, Gloria, Neukamm and Otto etc.

Outline

Calderón-Zygmund decomposition



An example: Pisztora's good cube

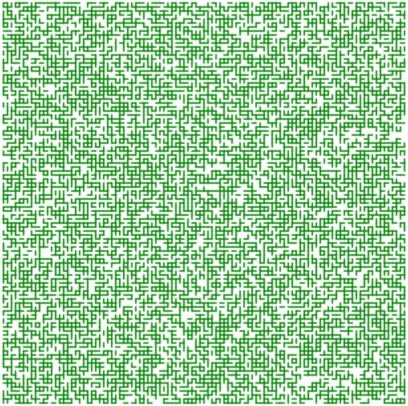


Figure: Can you tell all the connected components in the graph ?

An example: Pisztora's good cube

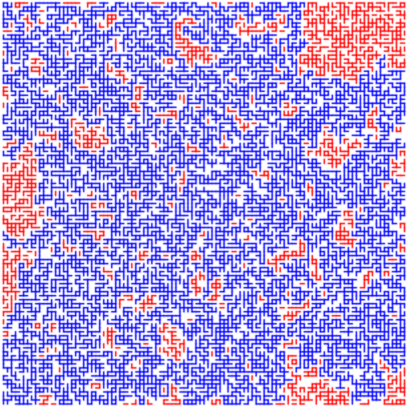


Figure: The cluster in blue is the maximal cluster in the cube.

An example: Pisztora's good cube

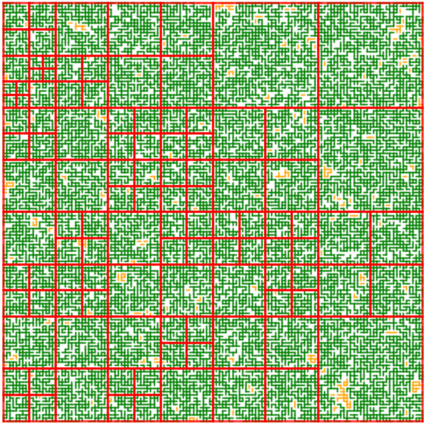


Figure: Decomposition of a big cube into of disjoint small cubes with good properties.

Theorem (Armstrong, Dario 18)

Let $\mathcal{G} \subset \mathcal{T}$ a sub-collection of triadic cubes satisfying the following: for every $\square = z + \square_n \in \mathcal{T}$, $\{\square \notin \mathcal{G}\} \in \mathcal{F}(z + \square_{n+1})$, and there exist two positive constants K, s we have $\sup_{z \in 3^n\mathbb{Z}^d} \mathbb{P}[z + \square_n \notin \mathcal{G}] \leqslant K \exp(-K^{-1}3^{ns})$. Then, $\mathbb{P}$-almost surely there exists $\mathcal{S} \subset \mathcal{T}$ a partition of $\mathbb{Z}^d$ with the following properties:

1. *Cubes containing elements of $\mathcal{S}$ are good: for every $\square, \square' \in \mathcal{T}$,*

$$\square \subset \square', \square \in \mathcal{S} \Longrightarrow \square' \in \mathcal{G}.$$

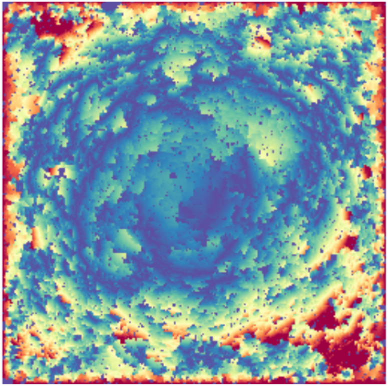
2. *Neighbors of elements of $\mathcal{S}$ are comparable: for every $\square, \square' \in \mathcal{S}$ such that $\operatorname{dist}(\square, \square') \leqslant 1$, we have*

$$\frac{1}{3} \leqslant \frac{\operatorname{size}(\square)}{\operatorname{size}(\square')} \leqslant 3.$$

3. *Estimate for the coarseness: we use $\square_{\mathcal{S}}(x)$ to represent the unique element in $\mathcal{S}$ containing a point $x \in \mathbb{Z}^d$, then its size has exponential tail*

Exercise

Question: Give a concentration inequality estimate for $\left|\frac{|\mathscr{C}_\infty \cap \square|}{|\square|} - \theta(p)\right|$?



Thank you for your attention.