Heat Kernel on the Infinite Percolation Cluster

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based on a joint work with Paul Dario

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Outline

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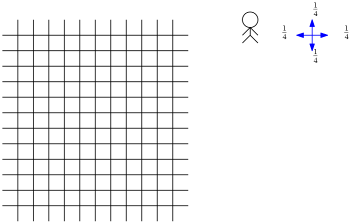
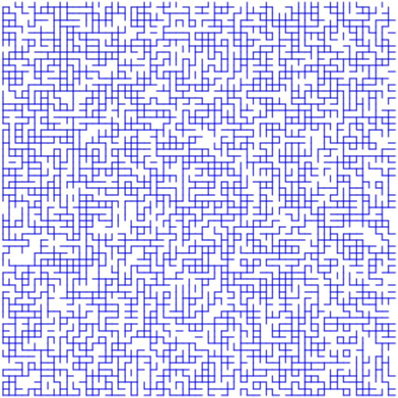
Universality of Brownian motion

- It is well-known that a centered random walk $(S_n)_{n\geqslant 1}$ on $\mathbb{Z}^d$ with variance $\bar{\sigma}^2$ converges to the Brownian motion $(\bar{\sigma}B_t)_{t\geqslant 0}$ after a scaling.



- From different viewpoints: CLT, local CLT, invariance principle.
- Question: Do these results also hold for the random walk in suitable random environment?

Random walk in the labyrinth



Question: What happens for the random walk in the labyrinth ?

Outline

Definition

Definition (Bernoulli percolation on $\mathbb{Z}^d$)

We denote by $(\mathbb{Z}^d, E_d)$ the d-dimension lattice graph. A Bernoulli percolation configuration $\{\mathbf{a}(e)\}_{e \in E_d}$ is an element of $\{0,1\}^{E_d}$, and its law is given by

$$\{\mathbf{a}(e)\}_{e \in E_d} \text{ i.i.d. }, \mathbb{P}[\mathbf{a}(e) = 1] = 1 - \mathbb{P}[\mathbf{a}(e) = 0] = \mathfrak{p}.$$

We say that the edge e is open if $\mathbf{a}(e) = 1$ and the edge e is closed if $\mathbf{a}(e) = 0$. A connected component given by $\mathbf{a}$ will be called cluster.

Example of percolation

Question: Can you feel the difference when we simulate $\mathbf{a}$ with different value of $\mathfrak{p}$?

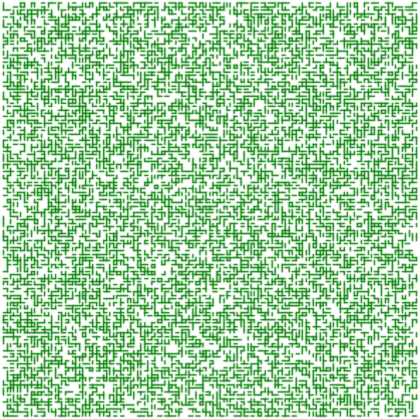


Figure: percolation in a cube of size 120×120 with $\mathfrak{p} = 0.4$.

Example of percolation

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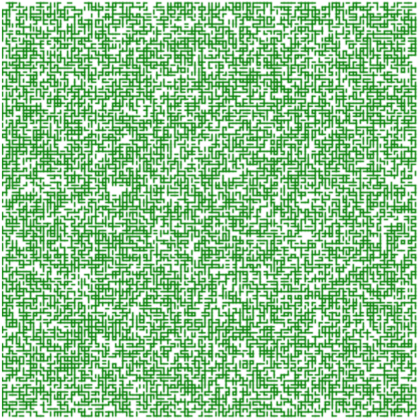

Figure: percolation in a cube of size 120×120 with $\mathfrak{p} = 0.5$.

Example of percolation

Question: Can you feel the difference when we simulate $\mathbf{a}$ with different value of $\mathfrak{p}$?

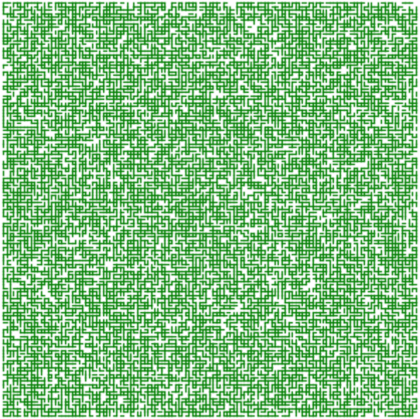

Figure: percolation in a cube of size 120×120 with $\mathfrak{p} = 0.6$.

Example of percolation

Question: Can you feel the difference when we simulate $\mathbf{a}$ with different value of $\mathfrak{p}$?

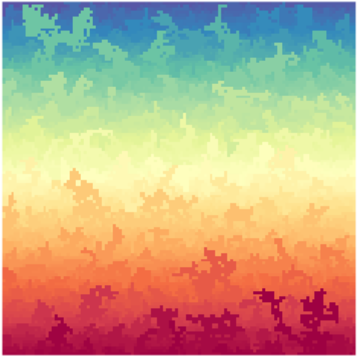

Figure: percolation colored by cluster in a cube of size 120×120 with $\mathfrak{p} = 0.4$.

Example of percolation

Question: Can you feel the difference when we simulate $\mathfrak{a}$ with different value of $\mathfrak{p}$?

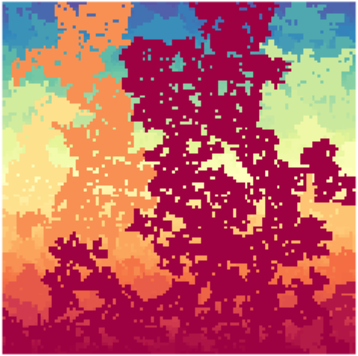

Figure: Percolation colored by clusters in a cube of size 120×120 with $\mathfrak{p} = 0.5$.

Example of percolation

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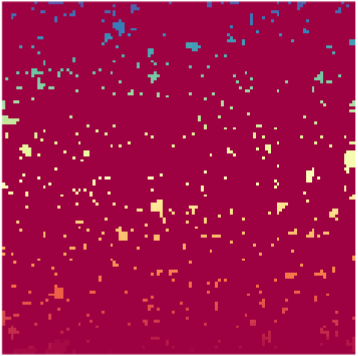


Figure: Percolation colored by clusters in a cube of size 120×120 with $\mathfrak{p} = 0.6$.

Phase transition

- $\theta(\mathfrak{p}) := \mathbb{P}[0 \text{ belongs to an infinite cluster } \mathscr{C}_\infty]$.
- It is easy to show that $\theta(\mathfrak{p})$ is monotone.
- $\mathfrak{p}_c := \inf\{\mathfrak{p} \in [0,1] : \theta(\mathfrak{p}) > 0\}$.

Theorem

For $d \geqslant 2$, we have $0 < \mathfrak{p}_c < 1$.

- We call the regime $0 \leqslant \mathfrak{p} < \mathfrak{p}_c$ subcritical, $\mathfrak{p} = \mathfrak{p}_c$ critical and $\mathfrak{p}_c < \mathfrak{p} \leqslant 1$ supercritical.
- Furthermore, by ergodicity argument, in subcritical case a.s. there is no infinite cluster. In supercritical case a.s. there exists a unique infinite cluster $\mathscr{C}_\infty$.
- Critical case: we conjecture $\theta(\mathfrak{p}_c) = 0$, but it is open for $3 \leqslant d \leqslant 10$.

Infinite cluster $\mathscr{C}_\infty$ in supercritical percolation

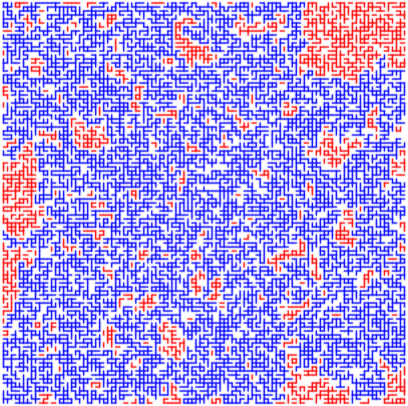


Figure: The cluster in blue is the maximal cluster in the cube.

Random walk on the infinite cluster

- We focus on the case supercritical percolation.
- (X_t) is a continuous-time Markov jump process starting from $y \in \mathscr{C}_\infty$, with an associated generator

$$\nabla \cdot \mathbf{a}\nabla u(x) := \sum_{z \sim x} \mathbf{a}(\{x,z\})\left(u(z) - u(x)\right).$$

- The quenched semigroup is defined as

$$p\left(t,x,y\right) = p^{\mathbf{a}}\left(t,x,y\right) := \mathbb{P}^{\mathbf{a}}_{y}\left(X_t = x\right),$$

which also solves the equation on $\mathscr{C}_\infty$ that

$$\begin{cases} \partial_t p\left(t,\cdot,y\right) - \nabla \cdot \mathbf{a}\nabla p\left(t,\cdot,y\right) = 0 & , \\ p\left(0,\cdot,y\right) = \delta_y(\cdot) & . \end{cases}$$

Random walk on the infinite cluster

- Question: for t big, does $(X_t)_{t \geqslant 0}$ looks like Brownian motion or is $p(t,x,y)$ close to a Gaussian distribution ?

Semigroup of random walk on $\mathscr{C}_\infty$

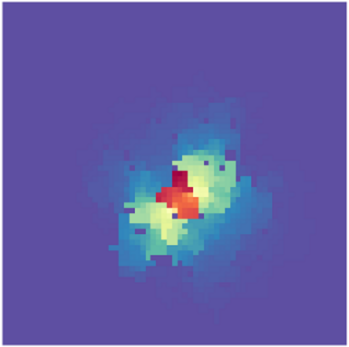


Figure: An illustration of $t^{\frac{d}{2}} p(t, \cdot, 0)$ for $t = 100$.

Semigroup of random walk on $\mathscr{C}_\infty$

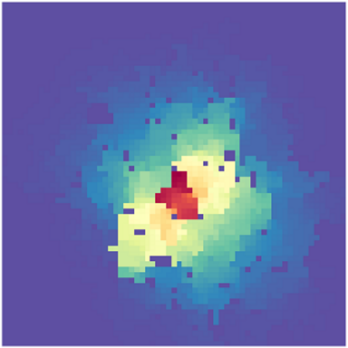


Figure: An illustration of $t^{\frac{d}{2}} p(t, \cdot, 0)$ for $t = 200$.

Semigroup of random walk on $\mathscr{C}_\infty$

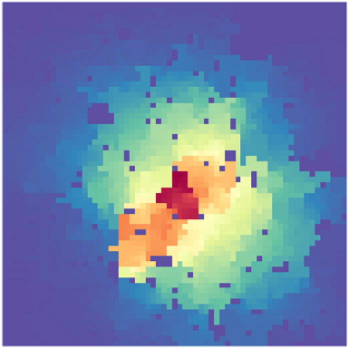


Figure: An illustration of $t^{\frac{d}{2}} p(t, \cdot, 0)$ for $t = 300$.

Semigroup of random walk on $\mathscr{C}_\infty$

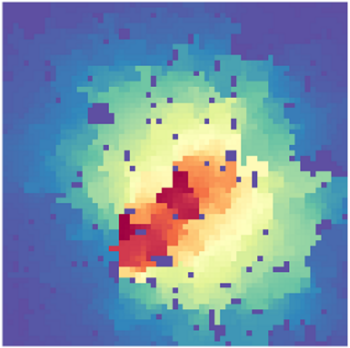


Figure: An illustration of $t^{\frac{d}{2}} p(t, \cdot, 0)$ for $t = 400$.

Semigroup of random walk on $\mathscr{C}_\infty$

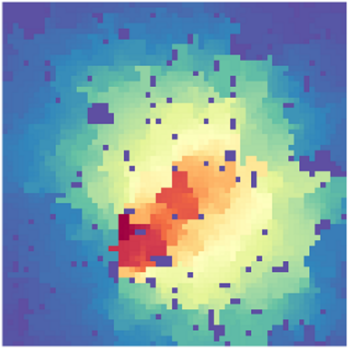


Figure: An illustration of $t^{\frac{d}{2}} p(t, \cdot, 0)$ for $t = 500$.

Semigroup of random walk on $\mathscr{C}_\infty$

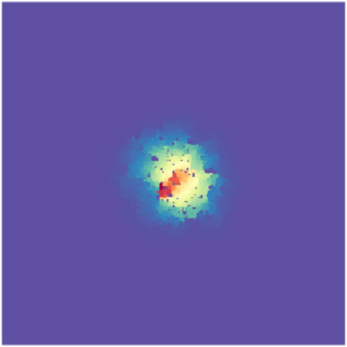


Figure: An illustration of $t^{\frac{d}{2}} p(t, \cdot, 0)$ for $t = 500$.

Semigroup of random walk on $\mathscr{C}_\infty$

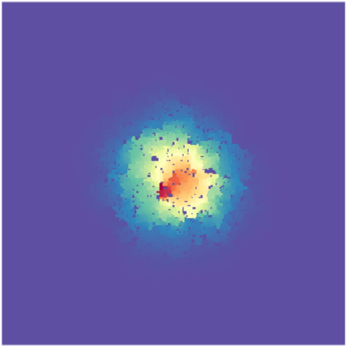


Figure: An illustration of $t^{\frac{d}{2}} p(t, \cdot, 0)$ for $t = 1000$.

Semigroup of random walk on $\mathscr{C}_\infty$

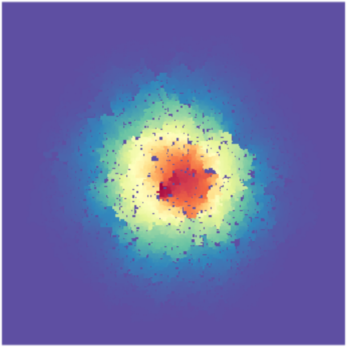


Figure: An illustration of $t^{\frac{d}{2}} p(t, \cdot, 0)$ for $t = 2000$.

Semigroup of random walk on $\mathscr{C}_\infty$

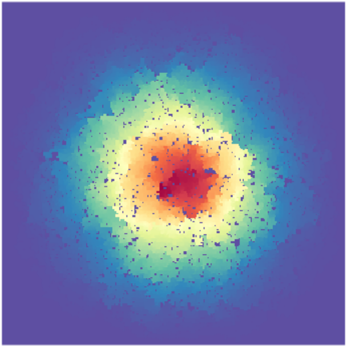


Figure: An illustration of $t^{\frac{d}{2}} p(t, \cdot, 0)$ for $t = 3000$.

Semigroup of random walk on $\mathscr{C}_\infty$

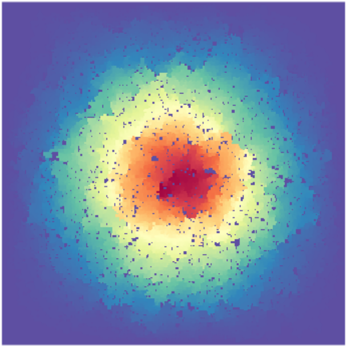


Figure: An illustration of $t^{\frac{d}{2}} p(t, \cdot, 0)$ for $t = 4000$.

Outline

Historical review: heat kernel bound

The exact Gaussian type bound on graphs and Markov chains are also well studied for long time.

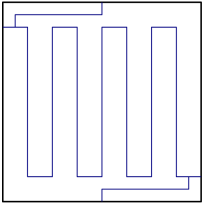
- Davies (1993) proves the Carne-Varopoulos bound for random walk on any infinite subgraph of $\mathbb{Z}^d$

$$p(t,x,y) \leqslant \begin{cases} C\exp\left(-\frac{|x-y|^2}{Ct}\right) & \text{if } |x-y| \leqslant t, \\ C\exp\left(-\frac{|x-y|}{C}\left(1+\ln\frac{|x-y|}{t}\right)\right) & \text{if } |x-y| \geqslant t. \end{cases}$$

- Delmotte (1999) proves the Gaussian bound for Markov chain on graph satisfying double volume condition and Poincaré inequality.

Challenge

Some elementary inequality is perturbed by the random geometry of the cluster.



Historical review: heat kernel bound

- Barlow (2004) proves the Gaussian bound for the heat kernel p with big t. That is for $t > \mathcal{T}_{NA}(y)$

$$p\left(t,x,y\right) \leqslant \begin{cases} Ct^{-d/2}\exp\left(-\frac{|x-y|^2}{Ct}\right) & |x-y| \leqslant t, \\ Ct^{-d/2}\exp\left(-\frac{|x-y|}{C}\left(1+\ln\frac{|x-y|}{t}\right)\right) & |x-y| \geqslant t. \end{cases}$$

- Barlow and Hambly (2009) also prove the local CLT: there exists $\bar{p}$ Gaussian such that for any $T > 0$

$$\limsup_{n\to\infty} \sup_{x\in\mathscr{C}_\infty} \sup_{t\geqslant T} \left|n^{\frac{d}{2}} p\left(nt,x,y\right) - \theta^{-1}(\mathfrak{p})\bar{p}(t,|x-y|)\right| = 0.$$

Historical review: convergence in law

Berger and Biskup (2007), Mathieu and Piatnitski (2007) prove that $\mathbf{a}$ almost surely, the random walk on the infinite percolation cluster converges to the Brownian motion in the Skorokhod topology

$$\left(\frac{1}{\sqrt{n}} X_{nt}\right)_{t \geqslant 0} \overset{n \to \infty}{\Longrightarrow} (\bar{\sigma} B_t)_{t \geqslant 0}\,.$$

Proof: corrector method

1. Tightness of $\left(\frac{1}{\sqrt{n}} X_{n\cdot}\right)_{n \geqslant 1}$.
2. Identify the limit: the corrector ϕ_{e_i} such that $-\nabla \cdot \mathbf{a}(e_i + \nabla \phi_{e_i}) = 0$. Then we have

$$M_t = \left(X_t \cdot e_1 + \phi_{e_1}(X_t), \cdots, X_t \cdot e_d + \phi_{e_d}(X_t)\right),$$

is a martingale and the martingale convergence theorem applies

$$\left(\frac{1}{\sqrt{n}} M_{nt}\right)_{t \geqslant 0} \overset{n \to \infty}{\Longrightarrow} (\bar{\sigma} B_t)_{t \geqslant 0}.$$

3. Corrector is sublinear: $\limsup_{x \to \infty} \frac{\phi_{e_i}(x)}{|x|} = 0$, $|X_{nt}| \simeq \sqrt{nt}$ implies

$$\frac{1}{\sqrt{n}} \phi_{e_i}(X_{nt}) \overset{n \to \infty}{\longrightarrow} 0.$$

Outline

Main result: Quantitative local CLT

Theorem (Dario, Gu, AOP 2021)

For each exponent $\delta > 0$, there exist a positive constant $C(d, \mathfrak{p}, \delta) < \infty$ and an exponent $s(d, \mathfrak{p}, \delta) > 0$, such that for every $y \in \mathbb{Z}^d$, there exists a non-negative random time $\mathcal{T}_{\mathrm{par},\delta}(y)$ satisfying the stochastic integrability estimate

$$\forall T \geqslant 0,\ \mathbb{P}\left(\mathcal{T}_{\mathrm{par},\delta}(y) \geqslant T\right) \leqslant C \exp\left(-\frac{T^s}{C}\right),$$

such that, on the event $\{y \in \mathscr{C}_\infty\}$, for every $x \in \mathscr{C}_\infty$ and every $t \geqslant \max\left(\mathcal{T}_{\mathrm{par},\delta}(y), |x-y|\right)$,

$$\left|p(t,x,y) - \theta(\mathfrak{p})^{-1}\bar{p}(t,x-y)\right| \leqslant C t^{-\frac{d}{2}-\left(\frac{1}{2}-\delta\right)} \exp\left(-\frac{|x-y|^2}{Ct}\right).$$

Remark: $\theta(\mathfrak{p}) = \mathbb{P}[0 \in \mathscr{C}_\infty]$ is the factor of the density normalization. $(\bar{p}(t, \cdot - y))_{t \geqslant 0}$ is the semigroup of the limit Brownian motion $(\bar{\sigma} B_t)_{t \geqslant 0}$.

Errors between the semigroups

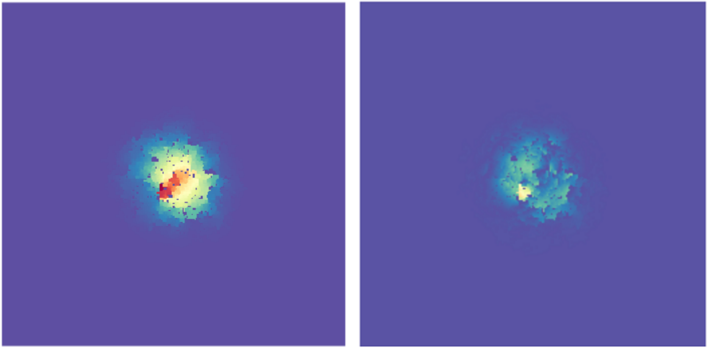


Figure: $t = 500$, the image on the left: $t^{\frac{d}{2}} p(t, \cdot, 0)$;
the image on the right: $t^{\frac{d}{2}} \left| p(t, \cdot, 0) - \theta(\mathfrak{p})^{-1} \bar{p}(t, \cdot) \right|$.

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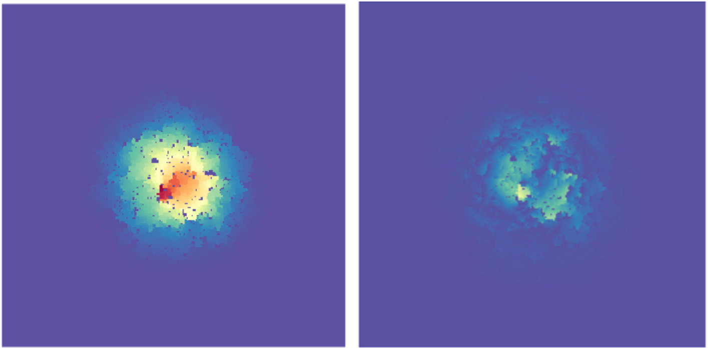

Figure: $t = 1000$, the image on the left: $t^{\frac{d}{2}} p(t, \cdot, 0)$;
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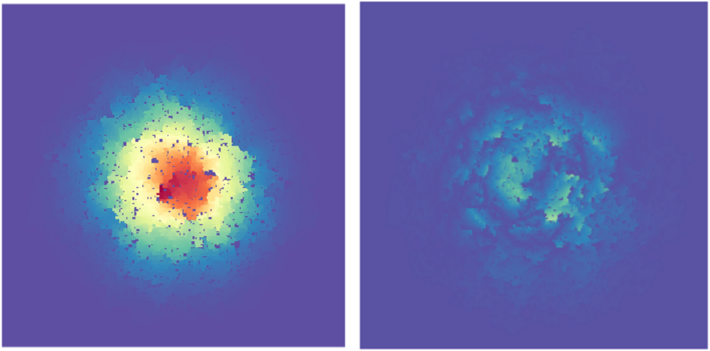

Figure: $t = 2000$, the image on the left: $t^{\frac{d}{2}} p(t, \cdot, 0)$;
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Errors between the semigroups

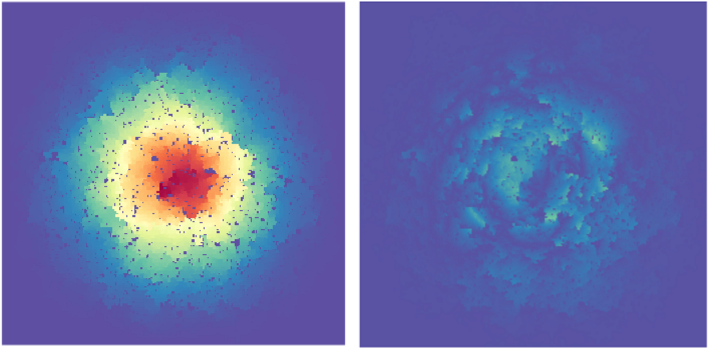

Figure: $t = 3000$, the image on the left: $t^{\frac{d}{2}} p(t, \cdot, 0)$;
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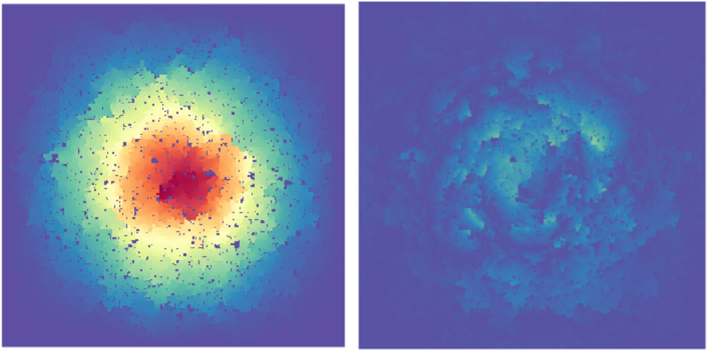


Figure: $t = 4000$, the image on the left: $t^{\frac{d}{2}} p(t, \cdot, 0)$;
the image on the right: $t^{\frac{d}{2}} \left| p(t, \cdot, 0) - \theta(\mathfrak{p})^{-1} \bar{p}(t, \cdot) \right|$.

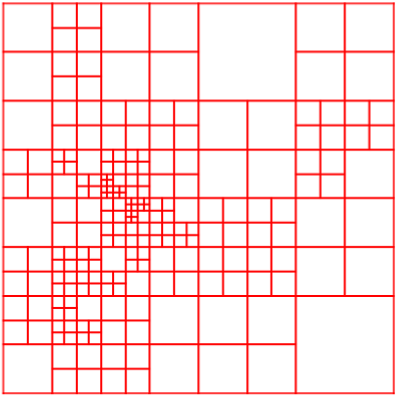
Ingredient 1: Homogenization theory

- Homogenization theory studies the errors between the equation and its homogenized solution, for example $(\partial_t - \nabla \cdot \mathbf{a}\nabla)p = (\partial_t - \frac{\bar{\sigma}^2}{2}\Delta)\bar{p}$, intuitively we have

$$p(t,x,y) \simeq \bar{p}(t,x,y) + \sum_{k=1}^{d} \partial_k \bar{p}(t,x,y)\phi_{e_k}(x),$$

  where $\{\phi_{e_k}\}_{1 \leqslant k \leqslant d}$ is the collection of corrector.
- Early classical work in homogenization: Bensoussan, Lions, Papanicolaou, Jikov, Kozlov, Oleunik, Yurinskii, Naddaf, Spencer, Allaire, Kenig, Lin, Shen etc.
- Quantitative analysis in stochastic homogenization setting: Armstrong, Kuusi, Mourrat, Smart, Gloria, Neukamm and Otto etc.

Ingredient 2: Calderón-Zygmund decomposition



Partition of good cube

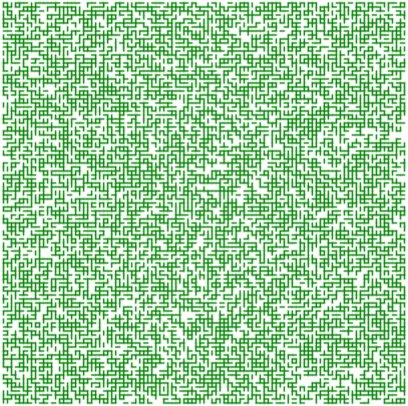


Figure: Can you tell all the connected components in the graph ?

Partition of good cube

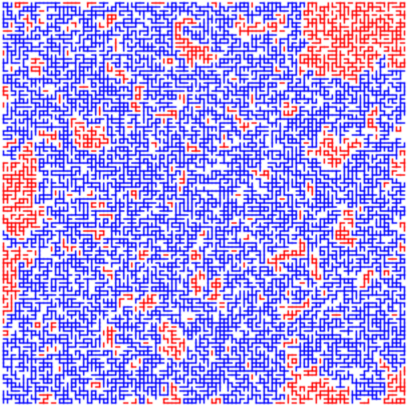


Figure: The cluster in blue is the maximal cluster in the cube.

Partition of good cube

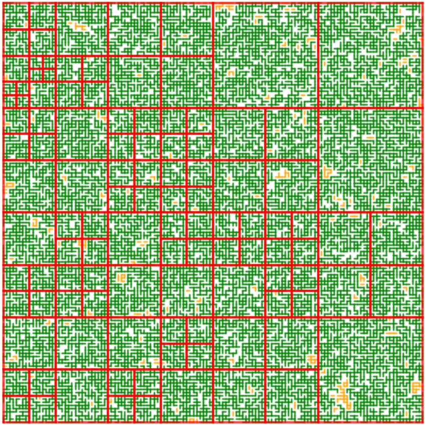


Figure: Decomposition of a big cube into of disjoint small cubes with good properties.

Partition of good cube

Theorem (Armstrong, Dario 2018)

Let $\mathcal{G} \subset \mathcal{T}$ a sub-collection of triadic cubes satisfying the following: for every $\square = z + \square_n \in \mathcal{T}$, $\{\square \notin \mathcal{G}\} \in \mathcal{F}(z + \square_{n+1})$, and there exist two positive constants K, s we have $\sup_{z \in 3^n\mathbb{Z}^d} \mathbb{P}[z + \square_n \notin \mathcal{G}] \leqslant K \exp(-K^{-1}3^{ns})$. Then, $\mathbb{P}$-almost surely there exists $\mathcal{S} \subset \mathcal{T}$ a partition of $\mathbb{Z}^d$ with the following properties:

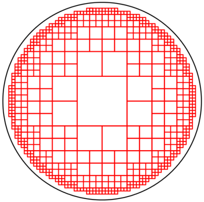
1. *Cubes containing elements of $\mathcal{S}$ are good.*
2. *Neighbors of elements of $\mathcal{S}$ are comparable.*
3. *Estimate for the coarseness: we use $\square_{\mathcal{S}}(x)$ to represent the unique element in $\mathcal{S}$ containing a point $x \in \mathbb{Z}^d$, then its size has exponential tail.*

Ingredient 3: Whitney decomposition

We treat

$$(\partial_t - \nabla \cdot \mathbf{a} \nabla) u = 0 \qquad (0,\infty) \times \mathscr{C}_\infty,$$
$$\left(\partial_t - \frac{1}{2}\bar{\sigma}^2 \Delta\right) \bar{u} = 0 \qquad (0,\infty) \times \mathbb{R}^d,$$

with suitable coherent boundary condition. Here the first equation is defined on $\mathscr{C}_\infty$ and $-\nabla \cdot \mathbf{a}\nabla$ is a finite difference operator; the second equation is defined on $\mathbb{R}^d$ and Δ is the standard Laplace operator. We also need the Whitney decomposition to overcome some technical obstacles here.



For Further Reading I

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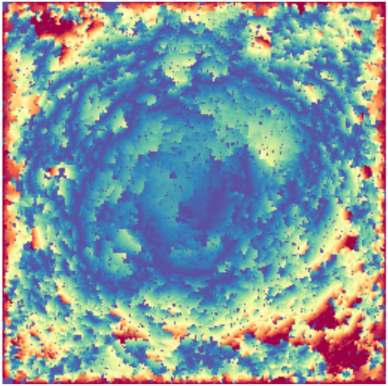

Thank you for your attention.

Easter eggs

热传导方程发展简介

顾陈琳

学号：10302010022

专业：数学类

摘要：热传导方程是数学物理方程中基本方程之一，和实际生活联系紧密。本文旨在概括热传导方程发展历史和在现代社会的应用。

关键字：热传导方程 傅里叶 数学物理方程 适定性 Tikhonov 方法 期权定价

第 4 节 总结

热传导方程和实际联系紧密，诸如粒子扩散、物种迁徙、热量传播、随机运动都可以看作是这个模型的变形，因此运用广泛。在 20 世纪 70 年代，Fisher Black、Myron Scholes 和 Robert Merton 在期权定价领域取得了巨大突破，发展了“Black-Scholes”方程，对近 30 年金融工程的发展起到了决定性作用。(赫尔 2009,)他们在 1997 年获得了诺贝尔经济学奖。其中“Black-Scholes”方程正是一个抛物线方程，关于热传导方程最大值原理也可以类似地运用到这里。

另一点启示是：数学的发展和其他学科一直是紧密联系着的。现实问题可以为数学提供研究课题、基本模型，数学可以为现实问题提供严谨、系统解法，而现实又可以对数学理论进行初步的检验。这种在实践中学习的方法应当是我们追求的。