Paris-Maths

Coloring

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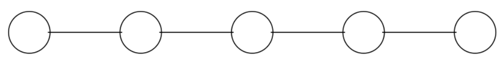
Definition 1 (Graph)**.** A graph is an ordered pair $G = (V, E)$ comprising:

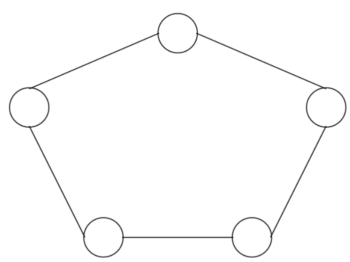
- V, a set of vertices;
- $E \subset \{\{x, y\} | x, y \in V, x \neq y\}$ a set of edges.

1 Enumerations

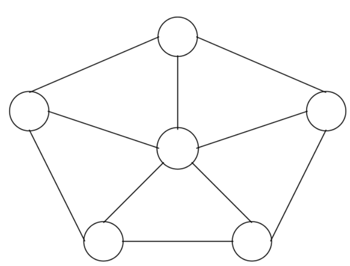
Exercise 1. *Counter the number of different coloring for the following graphs with 5 colors, such that every connected vertices have different colors.*



1.



2.

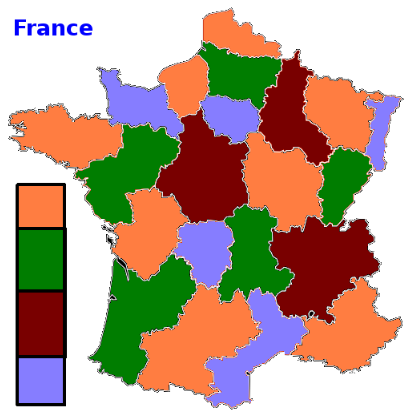


3.

2 Theorem of 6 Colors

The famous theorem of 4 colors states that

"No more than four colors are required to color the regions of any map so that no two adjacent regions have the same color."



In this part, we prove an easier version with 6 colors.

Exercise 2. *Find a counter example that "the theorem of 3 colors" is not true.*

Definition 2 (Planar Map)**.** A planar map is a graph that can be embedded in the plane, i.e., it can be drawn on the plane in such a way that its edges intersect only at their endpoints.

Definition 3 (Faces, Degrees)**.** In a planar map, we denote by V, F, E respectively the set of vertices, faces and edges. For every $v \in V$, we define $deg(v)$ the number of vertices connected to v. Similarly, for every $f \in F$, we define $deg(f)$ the number of edges contained in this face.

Exercise 3. *Prove that, in a planar map, we have that*

$$2\#E = \sum_{v \in V} deg(v) = \sum_{f \in F} deg(f).$$

Exercise 4 (Euler's formula)**.** *For a finite planar map, Euler's formula establishes that*

$$\#V + \#F - \#E = 2.$$

Exercise 5. *Prove that, in a planar map, there exists at least one vertex $v \in V$ such that $deg(v) \leqslant 5$. Similarly, there exists at least one face $f \in F$ such that $deg(f) \leqslant 5$.*

Exercise 6. *Prove that, we can color any planar map with 6 colors, such that the adjacent regions do not have the same color.*