Paris-Maths

# Bertrand's postulate

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This exercise sheet aims to help high school students learn the famous Bertrand's postulate:

**Theorem 1.** *For any positive integer $n \geqslant 2$, there exits a prime $p$ such that $n < p < 2n$.*

Despites its short statement, the proof is not obvious. However, Paul Erdös gave an elementary but very elegant argument in his first paper when he was 19. In this sheet, we follow his steps to learn this magic proof. In this sheet, we denote by $\mathcal{P}$ the set of prime number.

## 1 Warm up

**Exercise 1.** *Recall that there are infinitely many prime numbers.*

**Exercise 2.** *Prove that for any positive integer $n$, we can find $n$ consecutive composite numbers.*

**Exercise 3.** *Do you think there is contradiction between Theorem 1 and Exercise 2? Remark on it.*

## 2 A first study of $\binom{2n}{n}$

Erdös' proof relies on a nice object with nice property: the combinatorial number $\binom{2n}{n}$ which is defined as

$$\binom{2n}{n} := \frac{(2n)!}{(n!)(n!)} = \frac{2n \times (2n-1)\cdots(n+1)}{n \times (n-1)\cdots 1}.$$

Erdös' argument is that: $\binom{2n}{n}$ contains the factor $\prod_{p\in\mathcal{P}, n<p<2n} p$. If there is no prime number between $n$ and $2n$, then the size of $\binom{2n}{n}$ will be too small.

In the following we explicate this paragraph and we study at first the typical size of some quantity related to $\binom{2n}{n}$.

**Exercise 4.** *Prove that $\frac{4^n}{\sqrt{2n}} \leqslant \binom{2n}{n} \leqslant 4^n$.*

**Exercise 5.** *Prove that $\prod_{p\in\mathcal{P}, n<p<2n} p \,\Big|\, \binom{2n}{n}$.*

**Exercise 6.** *Prove that $\prod_{p\in\mathcal{P}, 1<p<n} p \leqslant 4^n$. (Hint: One can prove it by induction and make use of Exercise 4 and Exercise 5.)*

# 3 Counting the prime factor in $\binom{2n}{n}$

The estimates in last section are helpful but not sufficient to finish the proof. We have to do some finer estimate for $\binom{2n}{n}$. By the canonical decomposition of $\binom{2n}{n}$, we have

$$\binom{2n}{n} = \prod_{p \in \mathcal{P}} p^{\alpha_p},$$

where $\alpha_p$ is integer and is the power of the factor $p$. In this section, we study the size of $\alpha_p$.

**Exercise 7.** *Prove that for any prime number $p > 2n$, $\alpha_p = 0$.*

**Exercise 8.** *Prove the following identity:*

$$\alpha_p = \sum_{k=1}^{\infty} \left( \left\lfloor \frac{2n}{p^k} \right\rfloor - 2 \left\lfloor \frac{2n}{p^k} \right\rfloor \right). \tag{1}$$

**Exercise 9.** *By analyzing eq. (1), prove the following three bounds for $\alpha_p$:*

1. *A general bound $\alpha_p \leqslant \lfloor \frac{\log 2n}{\log p} \rfloor$.*
2. *For the prime number $n < p < 2n$, $\alpha_p = 1$.*
3. *For the prime number $\lfloor \frac{2n}{3} \rfloor < p \leqslant n$, $\alpha_p = 0$.*

# 4 Conclusion

Now we are ready to prove Theorem 1: The main argument is to explore the equation eq. (1). On the one hand, we have obtain the lower bound in Exercise 4 for the left hand side of eq. (1); on the other hand, one can use the Exercise 9 to give an upper bound for the right hand side of eq. (1). We will see that for a big $n$, the premier numbers in $[n, 2n]$ must exist, otherwise eq. (1) cannot be true.

**Exercise 10.** *Combining all the results above and prove Theorem 1.*

**Exercise 11.** *Sometimes people think the observation for the estimate of $\alpha_p$ of $\lfloor \frac{2n}{3} \rfloor < p \leqslant n$ is essential for this problem. Explain it heuristically.*

# *Pour aller plus loin*

One can find the whole proof in [1]. Bertrand's postulate is a beginning of the study of the distribution of the premier numbers and many interesting problems still hold open in this area. The curious students can read [2] for a panorama in this direction.

# References

[1] M. Aigner, G. M. Ziegler, K. H. Hofmann, and P. Erdos. *Proofs from the Book*, volume 274. Springer, 2010.

[2] G. Tenenbaum and M. M. France. *Les nombres premiers*, volume 571. Presses universitaires de France, 1997.