Name: ____________________ **NetId:** __________

Probability Limit Theorems

Midterm Exam, Fall 2021

DO NOT OPEN YET

...and wait until the proctor announces that it is time to start.

In the mean time, please write your name and NetID legibly,
and **read the instructions below carefully**.

* Please do not fold or damage the exam papers. After you finish your exam, please put the pages in correct order back into the sleeve.

* There are 6 questions in this exam, the sleeve should contain 8 pieces of paper.

* The scratch paper is included: three last papers are blank. If your solution continues on scratch paper, please clearly indicate it.

* For questions asking to prove a result, the clarity of the mathematical argument will be taken into account in the score.

* Questions formulated in terms of real functions should be answered with real functions.

Good luck!

1. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. Let $\mathcal{G} = \{A \in \mathcal{F} : \mathbb{P}(A) = 0 \text{ or } 1\}$. Show that $\mathcal{G}$ is a σ-algebra.

2. Given a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, consider the space of real-valued random variables modulo a.s. equivalence, i.e., $X_1 \sim X_2$ if $\mathbb{P}(X_1 = X_2) = 1$. Define a metric on this space by

$$d(X, Y) = \mathbb{E}(\min\{1, |X - Y|\}).$$

Show that $X_n \xrightarrow{\mathbb{P}} X$ if and only if $d(X_n, X) \xrightarrow{n\to\infty} 0$.

3. If $f \in L^1(\mathbb{R})$, show that

$$\lim_{n\to\infty} \frac{1}{2n} \int_{-n}^{n} f(x)\, dx = 0.$$

4. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, and let $\{A_n\}_{n\geq 1}$ be a sequence of measurable subsets.

(a) Suppose that

$$\lim_{n\to\infty} \mathbb{P}(A_n) = 0 \text{ and } \sum_{n=1}^{\infty} \mathbb{P}(A_n^c \cap A_{n+1}) < \infty.$$

Show, without using Borel-Cantelli lemma, that $\mathbb{P}(A_n \text{ occurs infinitely often}) = 0$.

(b) Find an example of a sequence $\{A_n\}_{n\geq 1}$ to which the result in (a) can be applied but the Borel-Cantelli lemma cannot.

5. Let U be a random variable with uniform distribution on $[0,1]$, i.e. $\mathbb{P}(U \in [a,b)) = b - a$ for all $0 \leq a \leq b \leq 1$. Consider $(X_n)_{n\geq 1}$ the random variables defined by

$$X_n = \sin(2\pi n U).$$

Show that, for every $n, m \geq 1$, $n \neq m$,

$$\mathbb{E}(X_n X_m) = \mathbb{E}(X_n)\mathbb{E}(X_m),$$

and that the random variables $(X_n)_{n\geq 1}$ are not independent.

6. (a) Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability measure, and let $\mathcal{G} \subset \mathcal{F}$ be a sub-σ-algebra of $\mathcal{F}$. Given $X \in L^2(\Omega, \mathcal{F}, \mathbb{P})$, define $Y = \mathbb{E}(X|\mathcal{G})$. Show that, if $\mathbb{E}(X^2) = \mathbb{E}(Y^2)$, then $X = Y$ a.s. .

   (b) Suppose X and Y are square-integrable random variables such that $\mathbb{E}(X|Y) = Y$ and $\mathbb{E}(Y|X) = X$. Show that $X = Y$ almost surely. (Hint: use the result in (a).)