Recitation 1: Measure Theory

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Exercise 1. *Let Ω be a non-empty set. If $\mathcal{F}_i$ is a σ-algebra for each $i \in I$, where I is a non-empty set of indexes, show that $\mathcal{F} = \cap_{i\in I}\mathcal{F}_i$ is also a σ-algebra.*

Exercise 2. *Let Ω be a non-empty set.*

1. *Show that if $\mathcal{F}_1 \subset \mathcal{F}_2 \cdots$ are all σ-algebra, then $\cup_{i=1}^{\infty}\mathcal{F}_i$ is an algebra.*
2. *Give an example of a sequence of σ-algebra $\mathcal{F}_1 \subset \mathcal{F}_2 \cdots$ such that $\cup_{i=1}^{\infty}\mathcal{F}_i$ is not a σ-algebra.*

Exercise 3. *A σ-algebra $\mathcal{F}$ is called countably generated if there exists a countable collection $\mathcal{G} \subset \mathcal{F}$ such that $\sigma(\mathcal{G}) = \mathcal{F}$. Show that $\mathcal{B}(\mathbb{R}^d)$ the Borel σ-algebra on $\mathbb{R}^d$ is countably generated for every $d \geq 1$.*

Exercise 4 (Cantor set)**.** *The Cantor ternary set $\mathcal{C}$ is created by iteratively deleting the open middle third from a set of line segments i.e.*

$$\mathcal{C}_0 := [0,1], \quad \mathcal{C}_1 := \left[0, \frac{1}{3}\right] \cap \left[\frac{2}{3}, 1\right], \cdots$$
$$\mathcal{C}_{n+1} := \frac{1}{3}\mathcal{C}_n \cup \left(\frac{2}{3} + \frac{\mathcal{C}_n}{3}\right),$$
$$\mathcal{C} := \cap_{n=0}^{\infty}\mathcal{C}_n.$$

In the following, let m be the Lebesgue measure on $\mathbb{R}$.

1. Calculate $m(\mathcal{C}_n)$.
2. Calculate $m(\mathcal{C})$.
3. Prove that $\mathcal{C}$ is a closed set.
4. Prove that $\mathcal{C}$ is not empty.
5. Prove that $\mathcal{C}$ is not countable.

Exercise 5. *For a sequence of events $\{A_n\}_{n\geqslant 1}$ with $\lim_{n\to\infty}\mathbb{P}[A_n] = 1$. Then for any $0 < c < 1$, show that there exists a subsequence $\{n_k\}$ with $n_k \to \infty$ such that*

$$\mathbb{P}\left[\cap_{k=1}^{\infty} A_{n_k}\right] > c.$$