MATH-SHU 362

Recitation 9: Higher Order ODE (2)

Lecturer: Chenlin Gu

Exercise 1. *In this question, we study the long time behavior of higher order system and the norm of matrix. Recall that the norm of a vector $\mathbf{x} \in \mathbb{R}^d$ is $\|\mathbf{x}\| = (\sum_{i=1}^n (x_i)^2)^{\frac{1}{2}}$ and the norm of a matrix $\mathbf{A} = (\mathbf{A}_{i,j})_{1 \leqslant i,j \leqslant d} \in \mathbb{R}^{d\times d}$ is defined as*

$$\|\mathbf{A}\| := \sup_{\mathbf{x}\in\mathbb{R}^d\setminus\{0\}} \frac{\|\mathbf{A}\mathbf{x}\|}{\|\mathbf{x}\|}.$$

1. *Some elementary properties of norm.*
   - *(a) Prove that $\|\mathbf{A}\mathbf{x}\| \leqslant \|\mathbf{A}\|\|\mathbf{x}\|$.*
   - *(b) Prove that, for $n \in \mathbb{N}$, $\|\mathbf{A}^n\| \leqslant \|\mathbf{A}\|^n$.*
   - *(c) Justify the triangle inequality that for any two matrix $\mathbf{A}, \mathbf{B} \in \mathbb{R}^{d\times d}$*

     $$\|\mathbf{A} + \mathbf{B}\| \leqslant \|\mathbf{A}\| + \|\mathbf{B}\|.$$

2. *Comparison with the norm and elements in matrix.*
   - *(a) Prove that $\max_{i,j} |\mathbf{A}_{i,j}| \leqslant \|\mathbf{A}\| \leqslant \sqrt{d} \max_{i,j} |\mathbf{A}_{i,j}|$.*
   - *(b) In the course, we define that $\exp(\mathbf{A}) = \sum_{n=0}^{\infty} \frac{\mathbf{A}^n}{n!}$, which is well-defined because $\sum_{n=0}^{\infty} \frac{\|\mathbf{A}\|^n}{n!}$ converges. Prove that in the sum, the element $\sum_{n=0}^{\infty} \frac{(\mathbf{A}^n)_{i,j}}{n!}$ also converges.*

3. *In this question, we suppose that $\mathbf{A}$ is symmetric matrix with d different real eigenvalues $\lambda_1 > \lambda_2 > \cdots > \lambda_d$, and associated eigenvector $\mathbf{A}\xi_i = \lambda_i \xi_i$. We would like to study the long time behavior of the higher order system*

   $$\mathbf{x}'(t) = \mathbf{A}\mathbf{x}(t), \qquad \mathbf{x}(0) = \mathbf{x}_0.$$

   - *(a) Prove that $(\xi_i)_{1\leqslant i \leqslant d}$ are orthogonal, i.e. $\xi_i \cdot \xi_j = 0$ for $i \neq j$.*
   - *(b) Prove that $\|\mathbf{A}\| = \max_{1\leqslant i\leqslant d} |\lambda_i|$.*
   - *(c) Write down the expression of $\mathbf{x}(t)$ in function of $\mathbf{A}$ and $\mathbf{x}_0$.*
   - *(d) Suppose that $\mathbf{x}_0 = \sum_{i=1}^d c_i \xi_i$, then prove that*

     $$\lim_{t\to\infty} e^{-\lambda_1 t}\mathbf{x}(t) = c_1 \xi_1. \tag{1}$$

   - *(e) Is (1) valid if $\mathbf{A}$ is a general matrix of real eigenvalues? If it is true, prove it. Otherwise, find a counter example.*