MATH-UA.0224@NYU

Homework 1: Recap of one variable analysis

Due: 02/12/2020

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Exercise 1. *Prove that for a ball of radius r in $\mathbb{R}^3$, it has volume $\frac{4\pi}{3}r^3$ and area of surface $4\pi r^2$.*

Exercise 2. *Let $f : \mathbb{R}^2 \to \mathbb{R}, f(x,y) = (x^2+y^2)^{\frac{1}{2}}$. Determine if the maximum and the minimum of f exist on the following sets. If they exist, find the value of the maximum and the minimum.*

1. $S_1 = \left\{(x,y) \in \mathbb{R}^2, \frac{x^2}{4} + \frac{y^2}{9} = 1\right\}$.
2. $S_2 = \left\{(x,y) \in \mathbb{R}^2, \frac{(x-1)^2}{4} + \frac{(y-1)^2}{9} = 1\right\}$.
3. $S_3 = \{(x,y) \in \mathbb{R}^2, x^2 - y^2 = 1\}$.

Exercise 3 (Cauchy's inequality)**.** *In this question, we study the Cauchy's inequality on the inner product space. A space V is an inner product space if it is a vector space with an inner product $\langle \cdot, \cdot \rangle : V \times V \to \mathbb{R}$ satisfying*

- *Symmetric: $\forall x, y \in V, \langle x, y \rangle = \langle y, x \rangle$.*
- *Positive: $\forall x \in V, \langle x, x \rangle \geqslant 0$ and " $=$ " holds if and only if $x = 0$.*
- *Bilinear map: $\forall x, y, z \in V, \lambda \in \mathbb{R}$, we have*

$$\langle x, \lambda y + z \rangle = \lambda \langle x, y \rangle + \langle x, z \rangle . \tag{1}$$

In an inner product space $(V, \langle \cdot, \cdot \rangle)$, Cauchy's inequality inequality holds

$$\forall x, y \in V, \qquad \langle x, y \rangle^2 \leqslant \langle x, x \rangle \langle y, y \rangle , \tag{2}$$

and " $=$ " is obtained if and only if $x = \lambda y$ for some $\lambda \in \mathbb{R}$.

The following questions are about its proof and application.

1. *We prove eq. (2) step by step.*
   - *Start by proving $\langle x + \lambda y, x + \lambda y \rangle \geqslant 0$.*
   - *Do optimization on the inequality above and obtain Cauchy's inequality.*
2. *Prove that by defining $\|x\| := \langle x, x \rangle^{\frac{1}{2}}$, $(V, \|\cdot\|)$ is a normed vector space.*

3. *Prove that for any function $f, g \in C([a,b])$, we have*

$$\left(\int_a^b f(x)g(x)\,dx\right)^2 \leqslant \left(\int_a^b f^2(x)\,dx\right)\left(\int_a^b g^2(x)\,dx\right), \tag{3}$$

   and

$$\left(\int_a^b (f+g)^2(x)\,dx\right)^{\frac{1}{2}} \leqslant \left(\int_a^b f^2(x)\,dx\right)^{\frac{1}{2}} + \left(\int_a^b g^2(x)\,dx\right)^{\frac{1}{2}}. \tag{4}$$

Exercise 4 (Uniformly continuous)**.** *Let (M,d) be a metric space, a function $f : M \to \mathbb{R}$ is uniformly continuous on $S \subset M$ if and only if*

$$\forall \varepsilon > 0, \exists \delta > 0, \text{ such that } \forall x \in S, \forall y \in B_\delta(x), \text{ we have } |f(x) - f(y)| \leqslant \varepsilon. \tag{5}$$

1. *Give an example of continuous but not uniformly continuous function.*
2. *Let $K \subset M$ be a compact set and $f : K \to \mathbb{R}$ continuous. Prove that f is uniformly continuous on K.*
3. *As an application, let $f \in C([a,b])$, prove that $\int_a^b f$ is well-defined in the sense of Riemann integral.*