MATH-UA.0224@NYU

Homework 2: Differential, Lagrange multiplier, Taylor expansion

Due: 02/26/2020

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Exercise 1. *Let* $f, g : \mathbb{R}^n \to \mathbb{R}$ *and differentiable, and we recall the definition of the gradient* $\nabla f = (\partial_{x_1} f, \partial_{x_2} f, \cdots \partial_{x_n} f)$. *Show that*

1. $\nabla(f+g) = \nabla f + \nabla g$,
2. $\nabla(fg) = f\nabla g + g\nabla f$,
3. $\nabla(f^n) = nf^{n-1}\nabla f$.

Exercise 2. *Apply Taylor expansion for* $f(x,y) = e^{2x}\sin(3y)$ *until a precision* $o((|x|+|y|)^3)$.

Exercise 3. *Calculate the eigenvalues and the associated eigenvectors for the matrix*

$$A = \begin{pmatrix} 0 & \frac{1}{2} & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & \frac{1}{2} & 0 \end{pmatrix}.$$

Exercise 4 (Arithmetic-Geometric inequality)**.** *Let* $(x_i)_{1\leqslant i\leqslant n}$ *be* n *positive number. We define the arithmetic mean and the geometric mean*

$$A(x_1, \cdots x_n) := \frac{1}{n}\sum_{i=1}^{n} x_i, \quad G(x_1, \cdots x_n) := \left(\prod_{i=1}^{n} x_i\right)^{\frac{1}{n}}.$$

1. *Prove that we have*

$$G(x_1, \cdots x_n) \leqslant A(x_1, \cdots x_n). \tag{1}$$

   (Indication: We can use Lagrange multiplier method.)

2. *Application: Prove that, the cube of edge* 10 *is the rectangular solid of volume* 1000 *which has the least total surface area. That is, we fix* $xyz = 1000$, *and the minimiser of*

$$f(x,y,z) = 2(xy + yz + zx),$$

   is attained at $x = y = z = 10$.

Exercise 5. *Find the points of the ellipsoid* $x^2 + 2y^2 + 3z^2 = 1$ *which are the closest or the farthest from the plane* $x + y + z = 10$.

Exercise 6 (d'Alembert's solution of wave equation)**.** *Let $t \in \mathbb{R}^+$ be time, $x \in \mathbb{R}$ be the position, and $f : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}$ a differentiable function which describes the displacement of a string. The physical consideration suggests that f satisfies one-dimensional wave equation*

$$\frac{\partial^2 f}{\partial x^2} = \frac{1}{a^2}\frac{\partial^2 f}{\partial t^2}, \tag{2}$$

where a is a certain positive constant. In this question, we study a solution of eq. (2)*.*

1. *We suggest a change of variable: $(u, v) \mapsto (x, y)$ such that $x = Au + Bv, y = Cu + Dv$, then prove that $g(u, v) := f(Au + Bv, Cu + Dv)$ satisfies*

$$\frac{\partial^2 g}{\partial v \partial u} = AB\frac{\partial^2 f}{\partial x^2} + (AD + BC)\frac{\partial^2 f}{\partial x \partial t} + CD\frac{\partial^2 f}{\partial t^2}. \tag{3}$$

2. *Determine the good parameters A, B, C, D so that we have $\frac{\partial^2 g}{\partial v \partial u} = 0$.*

3. *Prove that, under the parameters A, B, C, D above, there exist functions $\phi, \psi : \mathbb{R} \to \mathbb{R}$ such that*

$$g(u, v) = \phi(u) + \psi(v). \tag{4}$$

4. *Write down the expression of $f(x, t)$ with respect to the function ϕ, ψ.*

5. *To obtain the solution of f, we should also pose an initial condition. We suppose that*

$$f(0, x) = F(x), \quad \partial_t f(0, x) = G(x), \tag{5}$$

prove that under the condition eq. (5) *we have a solution for eq.* (2)

$$f(t, x) = \frac{F(x + at) + F(x - at)}{2} + \frac{1}{2a}\int_{x-at}^{x+at} G(s)\, ds. \tag{6}$$

6. *Revisit the step 1): Why we propose this change of variable? (For this question, you can write down your ideas without mathematical arguments.)*