Homework 3: Fixed point, inverse function and implicit function

Due: 03/11/2020

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Exercise 1 (Equivalence between two norms). *For a vector space X, we say two norms $\|\cdot\|_A, \|\cdot\|_B$ are equivalent if there exist two positive constants C_1, C_2 such that*

$$\forall x \in X, \quad C_1\|x\|_B \leqslant \|x\|_A \leqslant C_2\|x\|_B.$$

In $\mathbb{R}^n$, we define that

$$1 \leqslant p < \infty, \quad \|x\|_p := \left(\sum_{i=1}^{n} |x_i|^p\right)^{\frac{1}{p}}, \quad \|x\|_\infty := \max_{1 \leqslant i \leqslant n} |x_i|. \tag{1}$$

1. *Verify that $\|\cdot\|_1, \|\cdot\|_2, \|\cdot\|_\infty$ defines a norm.*
2. $\star$ *Verify that $\|\cdot\|_p$ defines a norm for a general $1 \leqslant p < \infty$.*
3. *Prove that in $\mathbb{R}^n$, all the norms $\|\cdot\|_p, 1 \leqslant p \leqslant \infty$ are equivalent.*

(The question with $\star$ is not obligatory. You can skip it if you find it difficult and proceed the next question. Check Hölder's inequality and Chebychev inequality in Wiki if you want an indication.)

Exercise 2. *Show that $f(x, y) = \left(\frac{x}{x^2+y^2}, \frac{y}{x^2+y^2}\right)$ is locally invertible in every point $\mathbb{R}^2 \backslash \{0\}$. Compute the inverse explicitly.*

Exercise 3. *Consider the function $f(x) = x^3$.*

1. *Is it invertible at the neighbor of origin ? If yes, calculate f^{-1}.*
2. *Calculate $f'(0)$. Recall that the inverse function theorem at a requires $f'(a) \neq 0$, so what do you think about it?*

Exercise 4 (Gradient descent). *In this question, we study the algorithm of gradient descent to obtain the minimum of a function f. We suppose the following condition for the function f: we use $\|\cdot\|$ for the standard norm $\|\cdot\|_2$*

- *$f : \mathbb{R}^n \to \mathbb{R}$ is $C^2(\mathbb{R}^n)$.*
- *∇f is M-Liptchitz, that is*

$$\forall x, y \in \mathbb{R}^n, \quad \|\nabla f(x) - \nabla f(y)\| \leqslant M\|x - y\|. \tag{2}$$

- *f is m-convex, that is*

$$\forall \xi \in \mathbb{R}^n, \forall x \in \mathbb{R}^n, \quad \xi \cdot D^2 f(x) \xi \geqslant m \|\xi\|^2. \tag{3}$$

- *The minimum x_* for f exists.*

The gradient algorithm is

$$x_{n+1} = x_n - \eta \nabla f(x_n), \tag{4}$$

by a good choice of step size $\eta > 0$ and we hope to know how it converges to x_*.

1. *Write down the Euler equation for the minimum.*

2. *Prove the uniqueness of the minimum. (Indication: By the m-convex property.)*

3. *By Newton-Leibniz formula, prove that*

$$\forall x, y \in \mathbb{R}^n, \quad f(y) - f(x) = \int_0^1 \nabla f((1-t)x + ty) \cdot (y - x) \, dt. \tag{5}$$

4. *Using the M-Liptchitz property of ∇f and prove*

$$\forall x, y \in \mathbb{R}^n, \quad f(y) \leqslant f(x) + \nabla f(x) \cdot (y - x) + \frac{M}{2} \|y - x\|^2. \tag{6}$$

5. *Put eq. (4) into eq. (6) and deduce*

$$f(x_{n+1}) \leqslant f(x_n) - \eta \left(1 - \frac{M}{2} \eta\right) \|\nabla f(x_n)\|^2. \tag{7}$$

   What is a good choice of η from eq. (7)?

6. *Prove that m-convex property implies that*

$$\forall x, y \in \mathbb{R}^n, \quad f(y) \geqslant f(x) + \nabla f(x) \cdot (y - x) + \frac{m}{2} \|y - x\|^2. \tag{8}$$

7. *Put eq. (4) into eq. (8) and obtain that*

$$\left(\eta - \frac{m}{2} \eta^2\right) \|\nabla f(x_n)\|^2 \geqslant f(x_n) - f(x_{n+1}). \tag{9}$$

8. *Combine eq. (7) and eq. (9) and prove that there exists $0 < \theta < 1$ depending on m, M and a proper choice η such that*

$$f(x_{n+1}) - f(x_*) \leqslant \theta \left(f(x_n) - f(x_*) \right). \tag{10}$$