MATH-UA.0224@NYU

Homework 4: $\mathbb{R}^d$ Integral, line integral and Green theorem

Due: 04/08/2020

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Exercise 1 (Value of Gamma function)**.** *In this question, we study the some typical numerical value of the Gamma function. We recall the definition that*

$$\Gamma(\alpha) = \int_0^\infty x^{\alpha-1} e^{-x}\, dx, \tag{1}$$

and its value is well defined for $\alpha \in (0, \infty)$.

1. *Prove the recurrence equation that* $\forall \alpha \in (0, \infty)$
$$\Gamma(\alpha+1) = \alpha\Gamma(\alpha). \tag{2}$$
2. *Use eq.* (2) *to prove that for* $n \in \mathbb{N}$, $\Gamma(n) = (n-1)!$.
3. *Prove that* $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$. *(Indication: you can try a change of variable in eq.* (1) *that* $x = y^2$.*)*

Exercise 2 (Area of n-ball and area of $\mathbb{S}^{n-1}$)**.** *In this question, we study the volume of* n*-dimensional unit ball*

$$B_1^n = \left\{ (x_1, \cdots x_n) \,\middle|\, \sum_{i=1}^n (x_i)^2 \leqslant 1 \right\},$$

and the area of the surface area that

$$\mathbb{S}^{n-1} = \left\{ (x_1, \cdots x_n) \,\middle|\, \sum_{i=1}^n (x_i)^2 = 1 \right\}.$$

1. *The main idea is a recurrence equation: let the volume* $V_n = V(B_1^n)$, *we have the equation*
$$V_n = \frac{2\pi V_{n-2}}{n}. \tag{3}$$
Indication: you can make a change of variable $x_1 = r\cos\theta, x_2 = r\sin\theta$, *and make use of the volume* V_{n-2}.
2. *Establish the formula* $V_n = \frac{\pi^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2}+1\right)}$.

3. *Deduce from it* ω_{n-1} *the area of surface* $\mathbb{S}^{n-1}$ *that* $\omega_{n-1} = \frac{2\pi^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2}\right)}$.

Exercise 3. *Let* C *be the unit circle* $x^2+y^2=1$*, oriented counterclockwise. Evaluate the following integral by using Green's theorem to convert to a double integral over the unit disk* D*:*

1. $\int_C (3x^2 - y)\, dx + (x + 4y^3)\, dy$.

2. $\int_C (x^2 + y^2)\, dy$.

Exercise 4. *Find a potential function* ϕ *such that* $F = \nabla\phi$ *for the following vector field* $F : \mathbb{R}^2 \to \mathbb{R}^2$.

1. $F(x, y) = (2xy^3, 3x^2y^2)$.

2. $F(x, y) = (\sin 2x \cos^2 y, -\sin^2 x \sin 2y)$.