Homework 5: Stokes' theorem and some applications

Due: No Due

Lecturer: Chenlin GU

Exercise 1 (Can we apply Green's theorem?). *The function of winding number is defined as an integral of* 1*-form* ω *defined on* $\mathbb{R}^2\backslash\{0\}$.

$$\omega = \frac{-y}{x^2+y^2}dx + \frac{x}{x^2+y^2}dy. \tag{1}$$

Let γ *be the closed curve* ∂B_1.

1. *Calculate* $\int_{\partial B_1} \omega d\gamma$.
2. *Can we apply the Green's theorem for the integral above? Why?*

Exercise 2 (Calculus on torus). *Let* $(\alpha, \beta) \in [0, 2\pi]^2$ *and* $(\alpha, \beta) \mapsto (x, y, z)$ *be the parametrization of torus*

$$\begin{aligned} x &= (R + r\cos\alpha)\cos\beta, \\ y &= (R + r\cos\alpha)\sin\beta, \\ z &= r\sin\alpha. \end{aligned}$$

Calculate the area and volume of this torus.

Exercise 3 (Induction of Gauss's law from Coulomb's law). *We recall that the Coulomb's law: for two electric particle of charge* q_1, q_2, *and of distance* r_{12}, *the force between them is*

$$F = k_e \frac{q_1 q_2}{|r_{12}|^2}, \qquad k_e = \frac{1}{4\pi\varepsilon_0}.$$

Now we put a particle of charge q *at origin* 0, *and denote by* $E : \mathbb{R}^3 \to \mathbb{R}^3$ *the electric field generated by it, i.e. for any particle of charge* q' *at position* $r \in \mathbb{R}^3$, *it is acted a force* $F = E(r)q'$.

1. *Write down the expression* $E(r)$.
2. *Prove that for any* $r > 0$, *we have*

$$\int_{\partial B_r} E \cdot \mathbf{n}\, dS = \frac{q}{\varepsilon_0}.$$

3. *Prove that E satisfies the Gauss' law, i.e. for any domain Ω contains 0 with regular boundary*

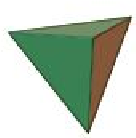
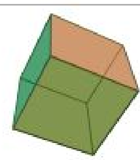
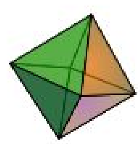
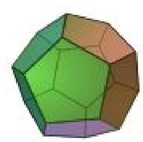
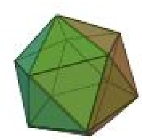
$$\int_{\partial\Omega} E \cdot \mathbf{n} \, dS = \frac{q}{\varepsilon_0}. \tag{2}$$

4. *Let ρ be the density of particle charge, and use the superposition to prove Gauss' law in general case.*

Exercise 4. *Prove that in the Maxwell equations, when $J = \rho = 0$, we can deduce the following equation for the field E, B.*

$$\mu_0\varepsilon_0 \frac{\partial^2}{\partial t^2} E = \Delta E,$$
$$\mu_0\varepsilon_0 \frac{\partial^2}{\partial t^2} B = \Delta B.$$

Exercise 5 (Platonic solid)**.** *In three-dimensional space, a Platonic solid is a regular, convex polyhedron i.e. every face is identical regular polygon (same number of edges, vertex, and same length, angles). Prove that there are 5 types of Platonic solid. (Indication: Euler formula.)*

Tetrahedron	Cube	Octahedron	Dodecahedron	Icosahedron
Four faces	Six faces	Eight faces	Twelve faces	Twenty faces
 (Animation) (3D model)	 (Animation) (3D model)	 (Animation) (3D model)	 (Animation) (3D model)	 (Animation) (3D model)