Lecture 1: Line Integral

Chenlin GU

DMA/ENS, PSL Research University

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Outline for section 1

Object

- What is a curve in $\mathbb{R}^d$?
- How to define a integral for along a curve γ as

$$\int_\gamma f \, d\gamma ?$$

- Similar property as Newton-Leibniz formula that

$$\int_\gamma f \, d\gamma = F(b) - F(a) ?$$

You know the answer for a special case when γ is an affine interval.

What is a curve ?

- A curve in $\mathbb{R}^d$ is a continuous parametrization $\gamma : [a, b] \to \mathbb{R}^d$.
- Thus, geometrically, two parametrizations γ_1, γ_2 may identify a same object visually. (A non-math description.)
- One can establish an equivalent relation : $\gamma_1 : [a, b] \to \mathbb{R}^d, \gamma_2 : [c, d] \to \mathbb{R}^d$, $\gamma_1 \equiv \gamma_2$ if there is a monotone bijection $\phi : [a, b] \to [c, d]$ and

$$\forall t \in [a, b], \qquad \gamma_2(\phi(t)) = \gamma_1(t).$$

Length of a Curve

One natural way to define the length of a curve $\gamma : [a, b] \to \mathbb{R}^d$ is to do at first partition $\mathcal{P}$ and then let the partition go to 0. That is: Let $\mathcal{P} = \{t_0, t_1 \cdots t_n\}$ where $a = t_0 < t_1 \cdots < t_n = b$, and $|\mathcal{P}| = \max_{0 \leqslant i \leqslant n-1} |t_{i+1} - t_i|$, then the length of the curve with the partition $L(\gamma, \mathcal{P})$ is

$$L(\gamma, \mathcal{P}) = \sum_{i=0}^{n-1} |\gamma(t_{i+1}) - \gamma(t_i)|. \tag{1.1}$$

One may define that

$$L(\gamma) := \lim_{|\mathcal{P}| \to 0} L(\gamma, \mathcal{P}). \tag{1.2}$$

Length of a Curve

Definition (Rectified Curve)

We say a curve $\gamma : [a, b] \to \mathbb{R}^d$ is rectified if eq. (1.2) is well defined and $L(\gamma) < \infty$. In this case, we say $L(\gamma)$ the length of γ.

Length of a Curve

- It is false that every curve has length, even it is continuous.
- Reason: fractal structure.
- e.x. Koch's snowflake, Brownian motion etc.

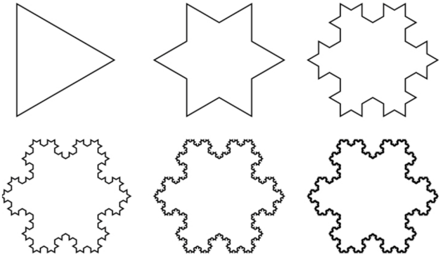


Figure: Koch's snowflake.

Length of a Curve

Remark: For these fractal objects, although the definition of length above does not work, one can also define their “length” by some other formula.

Length of a Curve

Theorem

If $\gamma : [a, b] \to \mathbb{R}^d$ is C^1, then $L(\gamma)$ exists.

Proof.

In this case $|\gamma(t_{i+1}) - \gamma(t_i)| \simeq \gamma'(t_i)|t_{i+1} - t_i|$, and one can use the uniform continuity to prove that the sum converges. □

Regular Curve

Definition (Regular Curve)

We say a curve $\gamma : [a, b] \to \mathbb{R}^d$ is regular is γ is $C^1([a, b])$ and $\gamma' \neq 0$.

Definition (Equivalent Relation for Regular Curve)

For two regular curves $\gamma_1 : [a, b] \to \mathbb{R}^d, \gamma_2 : [c, d] \to \mathbb{R}^d$, we say they are equivalent if there is a monotone C^1 bijection $\phi : [a, b] \to [c, d]$ and

$$\forall t \in [a, b], \qquad \gamma_2(\phi(t)) = \gamma_1(t).$$

Integral along Regular Curve

Theorem (Integral along Regular Curve)

For a regular curve $\gamma : [a, b] \to \mathbb{R}^d$, f continuous on γ, then we define that

$$\int_\gamma f \, d\gamma := \int_a^b f(\gamma(t))|\gamma'(t)| \, dt. \tag{1.3}$$

This integral is independent of the parametrization: for two equivalent regular curves $\gamma_1 : [a, b] \to \mathbb{R}^d, \gamma_2 : [c, d] \to \mathbb{R}^d$, we have

$$\int_a^b f(\gamma_1(t))|\gamma_1'(t)| \, dt = \int_c^d f(\gamma_2(t))|\gamma_2'(t)| \, dt. \tag{1.4}$$

Remark: One can generalize this result to a curve regular in every interval.

Integral along Regular Curve

Proof.

Let $\phi : [a, b] \to [c, d]$ the function such that $\gamma_1 = \gamma_2 \circ \phi$, and let $s = \phi(t)$, then we have

$$\begin{aligned}
\int_a^b f(\gamma_1(t))|\gamma_1'(t)|\,dt &= \int_a^b f(\gamma_2 \circ \phi(t))|(\gamma_2 \circ \phi(t))'|\,dt \\
&= \int_a^b f(\gamma_2(\phi(t)))|\gamma_2'(\phi(t))|\phi'(t)\,dt \\
&= \int_c^d f(\gamma_2(s))|\gamma_2'(s)|\,ds.
\end{aligned}$$

□

Integral along Regular Curve

Interpretation 1 of the equation

$$\int_a^b f(\gamma_1(t))|\gamma_1'(t)|\,dt = \int_c^d f(\gamma_2(t))|\gamma_2'(t)|\,dt.$$

: Alice and Bob finish a riding tour and they count the number of the audience, which has density f. Alice has trace γ_1 during time $[a, b]$ while Bob with trace γ_2 during time $[c, b]$. If we suppose they finish the same tour, with the same audience, the total number does not depend on how and when they finish.

Integral along Regular Curve

Interpretation 2 of the equation

$$\int_a^b f(\gamma_1(t))|\gamma_1'(t)|\, dt = \int_c^d f(\gamma_2(t))|\gamma_2'(t)|\, dt.$$

Let f be the mass density of a string, then this equation calculate the total mass of this string, which is

$$\lim_{|\mathcal{P}|\to 0} \sum_{i=0}^{n-1} f(t_i)|\gamma(t_{i+1}) - \gamma(t_i)|.$$

Outline for section 2

Work done by the force

Let a particle moving along the curve $\gamma : [a, b] \to \mathbb{R}^d$, drive by a force field $F : \mathbb{R}^d \to \mathbb{R}^d$, then what is the work done by the force ?

- Recall the formula: $W = F \cdot \Delta S$.
- We do a partition $\mathcal{P}$ of the curve, $\mathcal{P} = \{t_0, t_1 \cdots t_n\}$ where $a = t_0 < t_1 \cdots < t_n = b$, and $|\mathcal{P}| = \max_{0 \leqslant i \leqslant n-1} |t_{i+1} - t_i|$
- We use linear interpolation and suppose the force constant on every interval, then

$$W(\gamma, F, \mathcal{P}) = \sum_{i=0}^{n-1} F(t_i) \cdot (\gamma(t_{i+1}) - \gamma(t_i)) \,.$$

- We take the limit

$$W(\gamma, F) = \lim_{|\mathcal{P}| \to 0} W(\gamma, F, \mathcal{P}).$$

Work done by the force

Question: How to make the procedure above rigorous ?

Integral of Vector Field

Theorem (Integral of Vector Field)

Let $\gamma : [a, b] \to \mathbb{R}^d$ a regular curve (so $C^1([a, b])$, with d components $(\gamma_1, \gamma_2 \cdots \gamma_d)$. Let the force field $F : \mathbb{R}^d \to \mathbb{R}^d$ a continuous field, i.e.

$$F = (F_1, F_2, \cdots F_d), \forall i, F_i \in C(\mathbb{R}^d).$$

Then we define that

$$\int_\gamma F \, d\gamma := \int_a^b F(t) \cdot \gamma'(t) \, dt = \sum_{i=1}^d \int_a^b F_i(t)\gamma_i'(t) \, dt.$$

Moreover, for two equivalent regular curves γ, β, we have

$$\int_\gamma F \, d\gamma = \int_\beta F \, d\beta.$$

Gradient Field

Definition (Gradient Field)

$F : \mathbb{R}^d \to \mathbb{R}^d$ is a gradient field if and only if there exists $f : \mathbb{R}^d \to \mathbb{R}$ differentiable such that

$$F = \nabla f, \qquad \text{i.e.} \qquad F_i = \partial_i f.$$

Integral of Gradient Field

Lemma

Let $\gamma : [0,1] \to \mathbb{R}^d$ a regular curve, F a continuous gradient field that $F = \nabla f$, then

$$\int_\gamma F \, d\gamma = f(\gamma(1)) - f(\gamma(0)).$$

In this case, we call f potential function.

Integral of Gradient Field

Proof.

F continuous implies that $f \in C^1(\mathbb{R}^d)$, then we have $t \mapsto f(\gamma(t)) \in C^1([0,1])$, we use Newton-Leibniz formula that

$$\begin{aligned} f(\gamma(1)) - f(\gamma(0)) &= \int_0^1 \frac{d}{dt} f(\gamma(t))\, dt \\ &= \int_0^1 \nabla f(\gamma(t)) \cdot \gamma'(t)\, dt \\ &= \int_0^1 F(\gamma(t)) \cdot \gamma'(t)\, dt. \end{aligned}$$

It is the lemma. □

Integral of Gradient Field

As a corollary, for F continuous field, then $\int_\gamma \mathrm{F}\, d\gamma$ only depends on the end points rather than how they are connected.

Outline for section 3

The integral of vector field is in fact an integral of 1-form, which is a very intuitive example of the general integral of k-form.

Objective of this section: start to be familiar with the terminology of differential form.

Dual Space of $\mathbb{R}^d$

- We define the dual space of $\mathbb{R}^d$

$$(\mathbb{R}^d)^* := \{T | T : \mathbb{R}^d \to \mathbb{R} \text{ linear function } \}.$$

- $(\mathbb{R}^d)^*$ is itself a d dimension linear space. Since every linear map has form

$$T = \sum_{i=1}^{d} \alpha_i e_i^*, \qquad e_i^*(e_j) = \delta_{ij}, \qquad T(\sum_{i=1}^{d} x_i e_i) = \sum_{i=1}^{d} \alpha_i x_i.$$

Dual Space of $\mathbb{R}^d$

Proof.

We test T with $\{e_1 \cdots e_d\}$ that

$$\forall 1 \leqslant i \leqslant d, \quad \alpha_i := T(e_i).$$

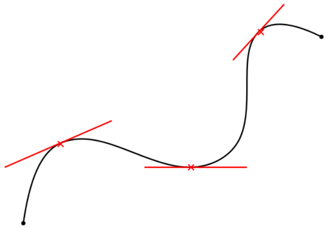
Then, for any $x = \sum_{i=1}^d x_i e_i$, we have

$$T(x) = T(\sum_{i=1}^d x_i e_i) = \sum_{i=1}^d x_i T(e_i) = \sum_{i=1}^d \alpha_i x_i.$$

Let e_i^* such that $e_i^*(e_j) = \delta_{ij}$, then we have obviously $T = \sum_{i=1}^d \alpha_i e_i^*$. □

Tangent Space of Regular Curve

For a regular space $\gamma : [a, b] \to \mathbb{R}^d$, for every $p \in (a, b)$, it is associated to a tangent space $T_p\gamma$ of dimension 1 and also a dual space $T_p^*\gamma$ (= cotangent space) of dimension 1.



Remark: $\gamma' \neq 0$ is important here.

Integral of 1-Form

- We revisit the vector field $\mathrm{F} : \mathbb{R}^{\mathrm{d}} \to \mathbb{R}^{\mathrm{d}}$.
- F is considered as 1-form:

$$\mathrm{F} = \mathrm{F}_1 \mathrm{dx}_1 + \mathrm{F}_2 \mathrm{dx}_1 \cdots \mathrm{F}_{\mathrm{d}} \mathrm{dx}_{\mathrm{d}}.$$

- For any $\mathrm{p} \in \gamma$, $\mathrm{F}(\mathrm{p}) \in \mathrm{T}^*_{\mathrm{p}}\gamma$ that

$$\forall \mathrm{h} \in \mathrm{T}_{\mathrm{p}}\gamma, \qquad \mathrm{F}(\mathrm{p})(\mathrm{h}) = \mathrm{F}(\mathrm{p}) \cdot \mathrm{h}.$$

- $\int_\gamma \mathrm{F}$ is understood as the integral of this linear functional - that is the integral of 1-form.

Gradient as Exterior Differential of 0-form

- A function $g : \mathbb{R}^d \to \mathbb{R}$ is 0-form.
- We define the exterior differential of a 0-form

$$dg = \sum_{i=1}^{d} \frac{\partial g}{\partial x_i} dx_i.$$

- A gradient vector field F is an 1-form such that $F = dg$, it is also said exact.

Homotopy

Definition (Homotopic Function)

Two continuous curves $\gamma_0, \gamma_1 : [a, b] \to U \subset \mathbb{R}^d$ with the same end point

$$\gamma_0(a) = \gamma_1(a), \qquad \gamma_0(b) = \gamma_1(b).$$

They are homotopic if there exists a continuous function $H : [0, 1] \times [a, b] \to U$ s.t.

$$\begin{aligned} H(0, \cdot) = \gamma_0(\cdot), H(1, \cdot) = \gamma_1(\cdot), \\ \forall s \in [0, 1], H(s, a) = \gamma_0(a) = \gamma_1(a), \\ H(s, b) = \gamma_0(b) = \gamma_1(b). \end{aligned}$$

H is called homotopy function for γ_0, γ_1.

Homotopy

- The theorem for the gradient field can be stated as: the integral is equal for homotopic curves.