Lecture 2: Green's Theorem

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Outline for section 1

Recap: Integral of Vector Field (1-Form)

Theorem (Integral of Vector Field)

Let $\gamma : [a, b] \to \mathbb{R}^d$ a regular curve (so $C^1([a, b])$, with d components $(\gamma_1, \gamma_2 \cdots \gamma_d)$. Let the force field $F : \mathbb{R}^d \to \mathbb{R}^d$ a continuous field, i.e.

$$F = (F_1, F_2, \cdots F_d), \forall i, F_i \in C(\mathbb{R}^d).$$

Then we define that

$$\int_\gamma F \, d\gamma := \int_a^b F(t) \cdot \gamma'(t) \, dt = \sum_{i=1}^{d} \int_a^b F_i(t) \gamma_i'(t) \, dt.$$

Moreover, for two equivalent regular curves γ, β, we have

$$\int_\gamma F \, d\gamma = \int_\beta F \, d\beta.$$

Integral of Vector Field (1-Form) in $\mathbb{R}^2$

1. $d = 2, \gamma : [0,1] \to \mathbb{R}^2$.
2. $F = (F_1, F_2)$, in language of 1-form, $F = F_1 dx_1 + F_2 dx_2$.
3. $$\int_\gamma F \, d\gamma = \int_0^1 F(\gamma(t)) \cdot \gamma'(t) \, dt = \int_\gamma F_1 dx_1 + F_2 dx_2.$$

Integral of Vector Field (1-Form) in $\mathbb{R}^2$

Let $(\gamma(t))_{t\in[0,1]} = (x_1(t), x_2(t))_{t\in[0,1]}$ and plugin in $\int_\gamma F_1 dx_1 + F_2 dx_2$ by parameterization:

$$\begin{aligned}\int_\gamma F_1 dx_1 + F_2 dx_2 &= \int_0^1 F_1(\gamma(t))dx_1(t) + F_2(\gamma(t))dx_2(t)\\ &= \int_0^1 F_1(\gamma(t))x_1'(t) + F_2(\gamma(t))x_2'(t)\,dt\\ &= \int_0^1 F(\gamma(t))\cdot\gamma'(t)\,dt.\end{aligned}$$

Thus the two define the $\int_\gamma F\,d\gamma$.

Differential of 1-Form

1. $d = 2$.
2. Differential of 0-form: $df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy$.
3. Differential of 1-form: $F = Pdx + Qdy$,

$$dF := \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dxdy.$$

4. A more formal way

$$\begin{aligned} dF &= \left(\frac{\partial P}{\partial x} dx + \frac{\partial P}{\partial y} dy \right) \wedge dx + \left(\frac{\partial Q}{\partial x} dx + \frac{\partial Q}{\partial y} dy \right) \wedge dy \\ dx \wedge dx &= dy \wedge dy = 0 \\ dx \wedge dy &= -dy \wedge dx \\ dF &= \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx \wedge dy. \end{aligned}$$

Green's Theorem

Theorem (Green's Theorem)

Let $D \subset \mathbb{R}^2$ be a region, with boundary ∂D is piece-wise smooth, positively oriented, closed and let $F = Pdx + Qdy$ a C^1 1-form on D, then we have

$$\int_D dF = \int_{\partial D} F. \tag{1.1}$$

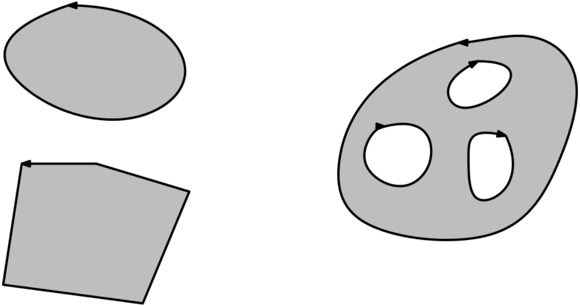
That is,

$$\int_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dxdy = \int_{\partial D} P\, dx + Q\, dy. \tag{1.2}$$

Remark: C^1 1-form means in $F = Pdx + Qdy$, P, Q are C^1.

Boundary in Green's Theorem

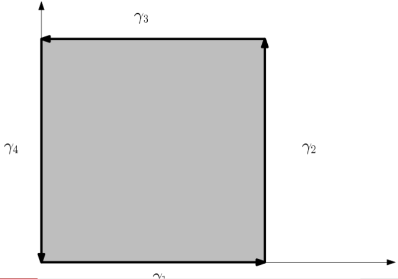
Heuristicly speaking, the interior part of the domain is always on the left hand side when we walk along the direction of the boundary.



Outline for section 2

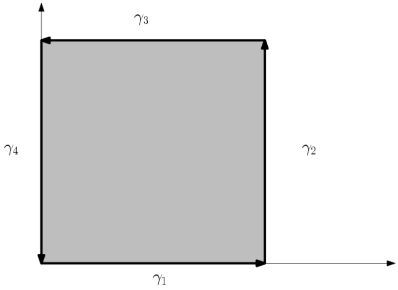
Step 1: An Easy Case in $I^2 = [0,1]^2$

$$\begin{aligned}\int_{I^2} dF &= \int_0^1 \int_0^1 \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dxdy \\ &= \int_0^1 \left(\int_0^1 \frac{\partial Q}{\partial x}\, dx\right) dy - \int_0^1 \left(\int_0^1 \frac{\partial P}{\partial y}\, dy\right) dx \\ &= \int_0^1 Q(1,y)\, dy - \int_0^1 Q(0,y)\, dy - \int_0^1 P(x,1)\, dx + \int_0^1 P(x,0)\, dx.\end{aligned}$$



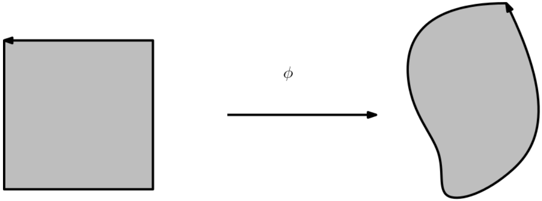
Step 1: An Easy Case in $\mathrm{I}^2 = [0,1]^2$

$$\int_{\partial \mathrm{I}^2} \mathrm{F} = \int_{\gamma_1} \mathrm{F} + \int_{\gamma_2} \mathrm{F} + \int_{\gamma_3} \mathrm{F} + \int_{\gamma_4} \mathrm{F}$$
$$= \int_0^1 \mathrm{P}(x,0)\,dx + \int_0^1 \mathrm{Q}(1,y)\,dy - \int_0^1 \mathrm{P}(x,1)\,dx - \int_0^1 \mathrm{Q}(0,y)\,dy.$$



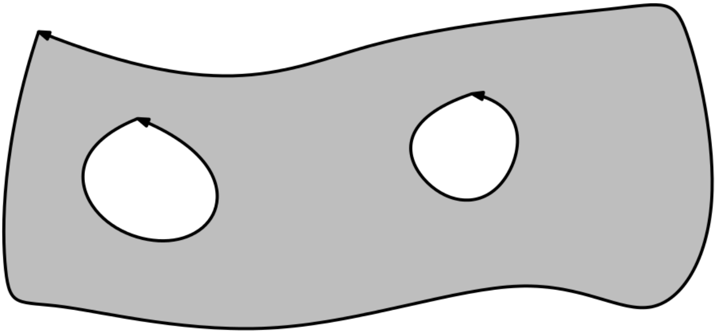
Step 2: Result in Simple Connected Domain

$\phi : \mathrm{I}^2 \to \mathrm{D}$ and the result is a detailed change of variable.



Step 3: In the Case with Genus

We do decomposition of domain and apply the result of simply connected domain one by one.



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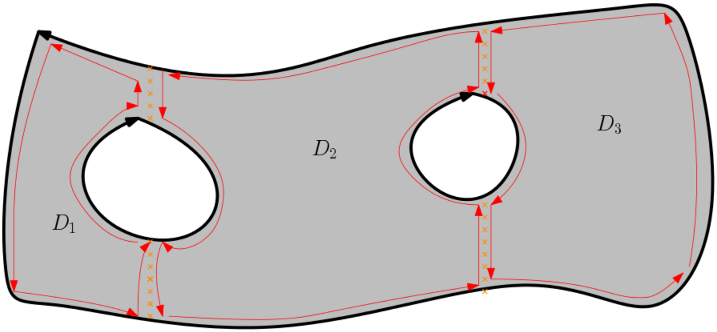

Outline for section 3

Characterization of Exact 1-Form

Theorem

Let $F = Pdx + Qdy$ be a C^1 1-form, then the following conditions are equivalent

1. It is exact.
2. There exits a potential function f such that $F = df$. (F is gradient field)
3. $\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y}$.
4. $\int_\gamma F$ are equal for all the regular curve γ connecting a and b.

Characterization of Exact 1-Form

Proof.

- (1) and (2) are equivalent by definition.
- (2) $\Rightarrow$ (3), in this case we have $P = \frac{\partial f}{\partial x}, Q = \frac{\partial f}{\partial y}$. Since they are C^1, we have

$$\frac{\partial Q}{\partial x} = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial P}{\partial y}.$$

□

Characterization of Exact 1-Form

- (3) $\Rightarrow$ (4), let γ_1, γ_2 two regular curves connecting A and B, we make two together as a closed curve $\gamma_3 = \gamma_1 \cup \bar{\gamma}_2$, and it suffices to prove $\int_{\gamma_3} F = 0$. We use Green's theorem: let D be the domain with boundary γ_3

$$\int_{\gamma_3} F \, d\gamma_3 = \int_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dxdy = 0.$$

  This concludes that $\int_{\gamma_1} F = \int_{\gamma_2} F$.
- (4) $\Rightarrow$ (1), we can construct explicitly a potential function: set $f(0) = 0$ and $f(a) = \int_{\gamma} F$ with a curve connecting $0, a$. This definition defines a potential field.