Lecture 5: Stokes' Theorem on Manifold

Chenlin GU

DMA/ENS, PSL Research University

April 20, 2020

Recap

- Green's theorem: transform an integral of 1-form to 2-form

$$\int_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dxdy = \int_{\partial D} P\, dx + Q\, dy. \tag{0.1}$$

- Language of differential form: $\varphi = Pdx + Qdy$, then we have $\int_{\partial D} \varphi = \int_D d\varphi$ because

$$\begin{aligned} d\varphi &= dP \wedge dx + dQ \wedge dy \\ &= \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx \wedge dy. \end{aligned}$$

- Question: Does it make sense for a general k-form φ and a general domain D?

Outline for section 1

Outline

Manifold

A manifold is an object that locally looks like that of $\mathbb{R}^n$.

Definition (Manifold)

A n-dimensional manifold is an object $(M, (f_\alpha)_{\alpha \in I})$ such that a family of injective functions $f_\alpha : U_\alpha \subset \mathbb{R}^n \to M$:

1. $\bigcup_{\alpha \in I} f_\alpha(U_\alpha) = M$.
2. For any $V = f_\alpha(U_\alpha) \bigcap f_\beta(U_\beta)$, that

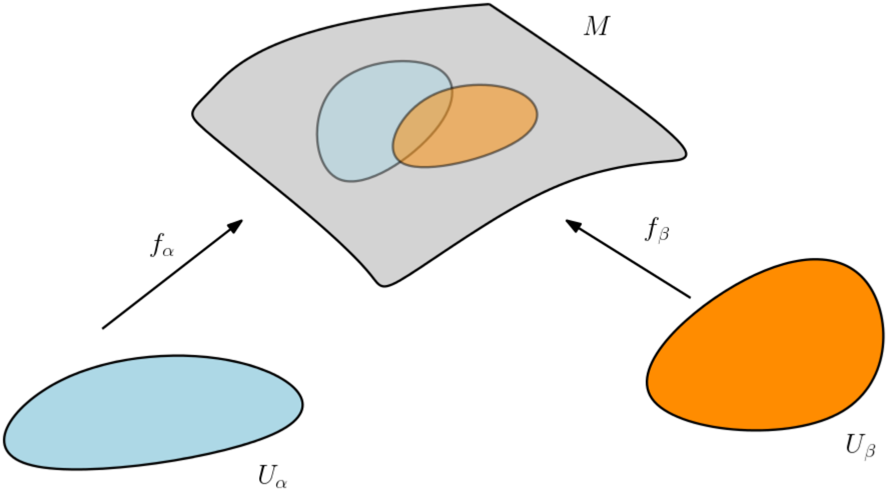
$$f_\beta^{-1} \circ f_\alpha : f_\alpha^{-1}(V) \to f_\beta^{-1}(V)$$
$$f_\alpha^{-1} \circ f_\beta : f_\beta^{-1}(V) \to f_\alpha^{-1}(V)$$

   are differentiable.
3. $(f_\alpha)_{\alpha \in I}$ is a maximal set.

We also call $(f_\alpha)_{\alpha \in I}$ atlas and one function f_α local chart.

Remark: Usually, we admit that the manifold has countable basis.

Manifold



Manifold - Exploration of Maps

Manifold

Examples of manifold:

- $\mathbb{R}^n$.
- n-dimensional unit ball B_1.
- $(n-1)$-dimensional unit sphere $\mathbb{S}^{n-1}$.
- Torus.
- Projective space: $P^2 = (\mathbb{R}^3 \backslash 0)/\sim$.

Manifold

A non-trivial example of manifold: Projective space $\mathrm{P}^2 = (\mathbb{R}^3 \backslash 0)/\sim$ that

$$(x, y, z) \sim (\lambda x, \lambda y, \lambda z), \lambda \neq 0.$$

One can use three maps to cover it and then add all the compatible maps

- $f_1 : \mathbb{R}^2 \to \mathrm{P}^2, f_1(u, v) = [1, u, v]$.
- $f_2 : \mathbb{R}^2 \to \mathrm{P}^2, f_2(u, v) = [u, 1, v]$.
- $f_3 : \mathbb{R}^2 \to \mathrm{P}^2, f_3(u, v) = [u, v, 1]$.

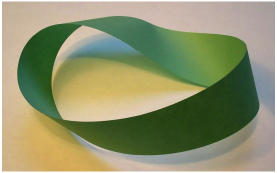
Then we see that $f_2^{-1} \circ f_1(u, v) = f_2^{-1}([1, u, v]) = \left(\frac{1}{u}, \frac{v}{u}\right)$.

Manifold

Definition (Orientable Manifold)

For a manifold $(\mathrm{M}, (\mathrm{f}_\alpha)_{\alpha \in \mathrm{I}})$ is said orientable iff for any two local charts $\mathrm{f}_\alpha, \mathrm{f}_\beta$ with common image, $\mathrm{d}(\mathrm{f}_\beta^{-1} \circ \mathrm{f}_\alpha)$ has positive determinant.

A famous counter example of non-orientable manifold: Möbius band.



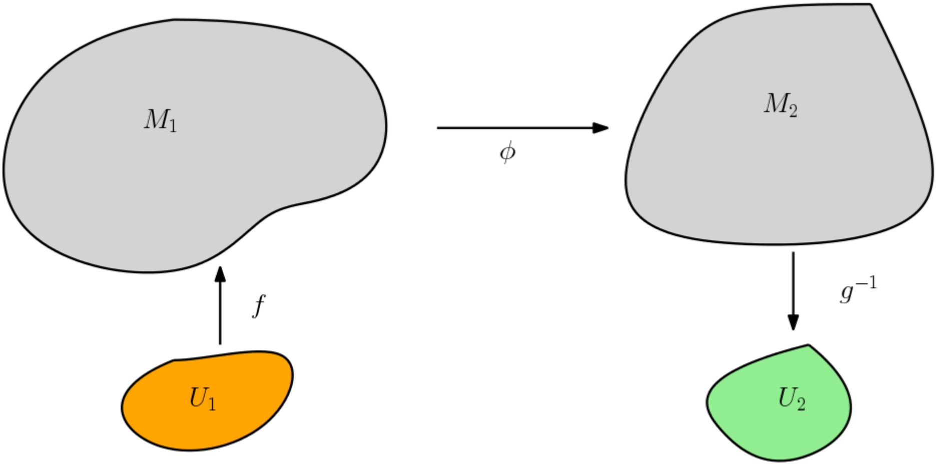
Outline

Differential Map on Manifold

Definition (Differential Map)

Let M_1 and M_2 be two differential manifolds, then a function (map) $\phi : \mathrm{M}_1 \to \mathrm{M}_2$ is said differentiable if for any local chart $\mathrm{f} : \mathrm{U}_1 \to \mathrm{M}_1, \mathrm{g} : \mathrm{U}_2 \to \mathrm{M}_2$ such that $\phi \circ \mathrm{f}(\mathrm{U}_1) \subset \mathrm{g}(\mathrm{U}_2)$, we have $\mathrm{g}^{-1} \circ \phi \circ \mathrm{f} : \mathrm{U}_1 \to \mathrm{U}_2$ is differentiable.
In the case $\mathrm{M}_2 = \mathbb{R}$, we call ϕ a differentiable function on M_1.

Differential Map on Manifold



Differential Map on Manifold

Remark: The definition does not depend on the choice of local charts. For two pairs of charts in the definition,

$$\mathrm{f}_1, \mathrm{f}_2 : \mathrm{U}_1 \to \mathrm{M}_1,$$
$$\mathrm{g}_1, \mathrm{g}_2 : \mathrm{U}_2 \to \mathrm{M}_2.$$

Then we have

$$\mathrm{g}_2^{-1} \circ \phi \circ \mathrm{f}_2 = \underbrace{\left(\mathrm{g}_2^{-1} \circ \mathrm{g}_1\right)}_{\text{differentiable}} \circ \left(\mathrm{g}_1^{-1} \circ \phi \circ \mathrm{f}_1\right) \circ \underbrace{\left(\mathrm{f}_1^{-1} \circ \mathrm{f}_2\right)}_{\text{differentiable}}.$$

Outline for section 2

Outline

Differential Form on Manifold

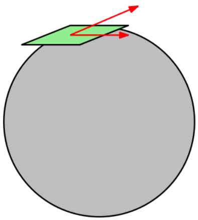
Definition (Sub-manifold)

M is a sub-manifold of $\mathbb{R}^m$ if $M \subset \mathbb{R}^m$ and M is a manifold.

In the case of sub-manifold, the tangent space T_pM at $p \in M$ is easy to define: let f be a local chart around p, and $f(0) = p$ then

$$T_pM := \mathrm{Vect}\{df_0(e_1), df_0(e_1) \cdots df_0(e_n)\}.$$

Then the k-form on M at p is defined as $\Lambda^k(T_pM)^*$.



Differential Form on Manifold

Theorem (Whitney Embedding)

Every n-dimensional smooth manifold can be smoothly embedded in $\mathbb{R}^{2n}$.

Thus you can treat the manifold as sub-manifold in some sense.

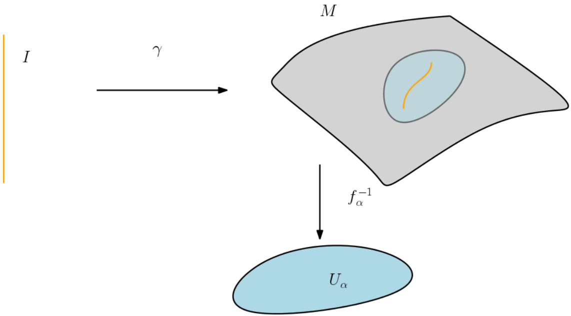
Differential Form on Manifold

A more abstract way to define the tangent plane.

Definition (Differentiable Curve on M)

Let $I = (a, b), \gamma : I \to M$ be a differential map, we call γ a differentiable curve on M.

Remark: A special example of differential map.



Differential Form on Manifold

Definition (Tangent Vector)

Let $\gamma : I \to M$ a differential curve on a manifold M, with $\gamma(0) = p \in M$, and let D be the set of functions differentiable at p. Then we define the tangent vector at p to be $\gamma'(0)$, which is a derivative operator $\gamma'(0) : D \to \mathbb{R}$ that

$$\gamma'(0)\phi := \frac{d}{dt}(\phi \circ \gamma)|_{t=0}.$$

Differential Form on Manifold

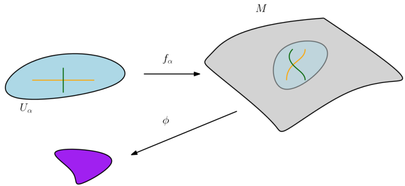
For local chart f_α of M, it gives a local canonical basis so that $\phi \in D$

$$\phi \circ f_\alpha = \phi(x_1, x_2, \cdots x_n),$$

then the differentiable curve $\{x_i\}_{1 \leqslant i \leqslant n}$ gives a canonical partial derivative

$$\frac{\partial}{\partial_{x_i}} \phi := \frac{\partial}{\partial_{x_i}} (\phi \circ f_\alpha).$$

Then $\left\{\frac{\partial}{\partial_{x_i}}\right\}_{1 \leqslant i \leqslant n}$ construct a canonical basis for $T_p M$ under the local chart f_α.



Differential Form on Manifold

Moreover, for $\gamma : \mathrm{I} \to \mathrm{M}$ passing p ($\gamma(0) = \mathrm{p}$), we have

$$f_\alpha^{-1} \circ \gamma = (x_1(t), x_2(t) \cdots x_n(t)).$$

and then

$$\begin{aligned}
\gamma'(0)\phi &= \frac{d}{dt}(\phi \circ \gamma)|_{t=0} \\
&= \frac{d}{dt}((\phi \circ f_\alpha) \circ (f_\alpha^{-1} \circ \gamma))|_{t=0} \\
&= \frac{d}{dt}(\phi(x_1(t), x_2(t) \cdots x_n(t)))|_{t=0} \\
&= \sum_{i=1}^{n} x_i'(0) \left(\frac{\partial}{\partial_{x_i}} \phi\right).
\end{aligned}$$

It does prove that T_pM is a tangent plane and we can construct the k-form on M at p is defined as $\Lambda^k(T_pM)^*$.

Differential Form on Manifold: Pullback

Recap: Pull-Back is the idea of df_p.

- $f : \mathbb{R}^n \to \mathbb{R}^m$ a C^1 differential map.
- $df_p : \mathbb{R}^n \to \mathbb{R}^m$.
- $\varphi \in \Lambda^k(\mathbb{R}^m)^*$.
- "Pull-back" means pull the differential form $\mathbb{R}^m$ to that of $\mathbb{R}^n$.
- $f^*\varphi \in \Lambda^k(\mathbb{R}^n)^*$: for $v_1 \cdots v_k \in \mathbb{R}^n$

$$\boxed{(f^*\varphi)_p(v_1, v_2, \cdots v_k) := \varphi_{f(p)}(df_p(v_1), df_p(v_2) \cdots df_p(v_k))}. \tag{2.1}$$

Differential Form on Manifold: Pullback

Generalization: Pull-Back on manifold

- $f_\alpha : U_\alpha \to M$ a local chart.
- $df_0 : T_0 U_\alpha \to T_p M$, $\forall \phi \in D(M)$,

$$(df_\alpha)_0(v)(\phi) = d(\phi \circ f_\alpha)_0(v) = \sum_{i=1}^{n} [v]_i \left(\frac{\partial}{\partial_{x_i}} \right) \phi.$$

  In another word, $(df_\alpha)(v) = \sum_{i=1}^{n} [v]_i \frac{\partial}{\partial_{x_i}}$.

- $\varphi \in \Lambda^k (T_p M)^*$.
- "Pull-back" means pull the differential form M to that of U_α.
- $f_\alpha^* \varphi \in \Lambda^k (T_0 U_\alpha)^*$: for $v_1 \cdots v_k \in \mathbb{R}^n$

$$\boxed{(f_\alpha^* \varphi)_0(v_1, v_2, \cdots v_k) := \varphi(df_\alpha(v_1), df_\alpha(v_2) \cdots df_\alpha(v_k))}. \tag{2.2}$$

  We define that $\varphi_\alpha := f_\alpha^* \varphi$.

Outline

Integration

Definition (Integration of n-form on $\mathbb{R}^n$)

Let $\varphi_\alpha = \omega_\alpha \mathrm{dx}_1 \wedge \mathrm{dx}_2 \cdots \wedge \mathrm{dx}_n$, then we have

$$\int \varphi_\alpha = \int \omega_\alpha \, \mathrm{dx}_1 \cdots \mathrm{dx}_n.$$

Definition (Integration on Manifold)

Let φ a n-form on n-dimensional manifold M and $\{f_\alpha, U_\alpha\}_{\alpha \in A}$ local charts. Then we define that

$$\int_{M \cap f_\alpha(U_\alpha)} \varphi := \int_{U_\alpha} \varphi_\alpha. \tag{2.3}$$

Integration

Question: Is it well-defined?

Let f_α, f_β two charts such that $f_\alpha(U_\alpha) = f_\beta(U_\beta)$. Because we have

$$f := f_\beta^{-1} \circ f_\alpha : U_\alpha \to U_\beta, (x_1, x_2 \cdots x_n) \to (y_1, y_2, \cdots y_n)$$

$$\varphi_\alpha = (f)^*(\varphi_\beta) = \det\left(\frac{\partial(y_1, \cdots y_n)}{\partial(x_1, \cdots x_n)}\right)\omega_\beta(y, \cdots y) dx_1 \wedge dx_2 \cdots \wedge dx_n.$$

Then we have that

$$\begin{aligned}
\int_{f_\alpha(U_\alpha)} \varphi = \int_{U_\alpha} \varphi_\alpha &= \det\left(\frac{\partial(y_1, \cdots y_n)}{\partial(x_1, \cdots x_n)}\right)\omega_\beta(y, \cdots y) dx_1 \wedge dx_2 \cdots \wedge dx_n \\
&= \int_{U_\beta} \det\left(\frac{\partial(y_1, \cdots y_n)}{\partial(x_1, \cdots x_n)}\right)\omega_\beta(y, \cdots y) dx_1 dx_2 \cdots dx_n \\
&= \int_{U_\beta} \omega_\beta dy_1 \cdots dy_n \\
&= \int_{U_\beta} \varphi_\beta.
\end{aligned}$$

Integration-decomposition of unity

Definition (Decomposition of unity)

Given a covering $M = \bigcup_{i=1}^{N} V_i$, we say $\{\theta_i\}_{1 \leqslant i \leqslant N}$ is an associated decomposition of unity if

- $0 \leqslant \theta_i \leqslant 1$, differentiable and $\mathrm{supp}(\theta_i) \subset V_i$.
- $\sum_{i=1}^{N} \theta_i = 1$.

Integration

Definition (Integration on Manifold)

Let φ a n-form on n-dimensional manifold M and $\{f_\alpha, U_\alpha\}_{\alpha \in A}$ local charts. Then we define that

$$\int_M \varphi = \sum_\alpha \int_{U_\alpha} \varphi_\alpha, \tag{2.4}$$

using the decomposition of unity $\{\theta_i\}_{1 \leqslant i \leqslant N}$ associated to a covering of local charts $M = \bigcup_{i=1}^N V_i, V_i = f_i(U_i)$:

$$\int_M \varphi = \sum_{i=1}^N \int_M \varphi\theta_i. \tag{2.5}$$

Remark: the interest is that $\operatorname{supp}(\varphi\theta_i) \subset V_i$ and we can use a local chart to define its integration.

Integration

Question: Is it well-defined ? Does it depend on how we do the decomposition of unity?
Let $\{\theta_i\}_{1\leqslant i\leqslant N}$ be the decomposition of unity associated to the covering $\{V_i\}_{1\leqslant i\leqslant N}$, while $\{\eta_j\}_{1\leqslant j\leqslant L}$ the one associated to the covering $\{W_j\}_{1\leqslant j\leqslant L}$. Then, observe that

$$\{\theta_i\eta_j\}_{1\leqslant j\leqslant L, 1\leqslant i\leqslant N} \text{ decomposition of unity associated to } \{V_i \cap W_j\}_{1\leqslant j\leqslant L, 1\leqslant i\leqslant N}.$$

$$\begin{aligned}\int_M \varphi = \sum_{i=1}^{N}\int_M \varphi\theta_i = \sum_{i=1}^{N}\sum_{j=1}^{L}\int_M \varphi\theta_i\eta_j \\ = \sum_{j=1}^{L}\sum_{i=1}^{N}\int_M \varphi\theta_i\eta_j = \sum_{j=1}^{L}\int_M \varphi\eta_j.\end{aligned}$$

Integration-decomposition of unity

Theorem (Decomposition of unity)

Given a covering $M = \bigcup_{i=1}^{N} V_i$, the decomposition of unity exists.

Proof.

Sketch of the proof:
Step 1: Existence of the function C_c^∞.
Step 2: Change of coordinate.
Step 3: Normalization using the local compactness. □

Outline for section 3

Manifold with boundary

A n-dimensional smooth manifold with boundary is an object locally looks like $\mathbb{R}^n$ while the boundary ∂M looks like a hyper-plane $\mathbb{R}^{n-1}$.

Manifold (with or without boundary)

Examples of manifold (with boundary):

- $\mathbb{R}^n$. (without boundary)
- n-dimensional unit ball B_1. (with boundary)
- $(n-1)$-dimensional unit sphere $\mathbb{S}^{n-1}$. (without boundary)
- Torus. (without boundary)
- Projective space: $P^2 = (\mathbb{R}^3 \backslash 0)/\sim$. (without boundary)

Stokes' Theorem

Theorem (Stokes' Theorem)

Let M be an n-dimensional manifold with boundary, compact and oriented. Let φ be a $(n-1)$-form. Then we have

$$\int_{\partial M} \omega = \int_{M} d\omega. \tag{3.1}$$

Stokes' Theorem - Examples

- Green's theorem.
- Kelvin–Stokes theorem: in $\mathbb{R}^3$, let $F = (P, Q, R)$ and Σ a closed surface, then

$$\int_{\partial \Sigma} F \, d\gamma = \int_{\Sigma} \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z} \right) dydz + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x} \right) dzdx + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dxdy$$

- Gauss's theorem: in $\mathbb{R}^3$, let $F = (P, Q, R)$ and S a closed surface and n the normal direction. Then

$$\int_{\partial S} F \cdot n = \int_{S} \nabla \cdot F.$$

Stokes' Theorem - Examples

Proof.

In Kelvin–Stokes theorem, we have $\omega = \mathrm{Pdx} + \mathrm{Qdy} + \mathrm{Rdz}$, which is a 1-form. Then we calculate its exterior differential

$$\begin{aligned} \mathrm{d}\omega = &\left(\frac{\partial \mathrm{P}}{\partial \mathrm{x}}\mathrm{dx} + \frac{\partial \mathrm{P}}{\partial \mathrm{y}}\mathrm{dy} + \frac{\partial \mathrm{P}}{\partial \mathrm{z}}\mathrm{dz}\right) \wedge \mathrm{dx} \\ &+ \left(\frac{\partial \mathrm{Q}}{\partial \mathrm{x}}\mathrm{dx} + \frac{\partial \mathrm{Q}}{\partial \mathrm{y}}\mathrm{dy} + \frac{\partial \mathrm{Q}}{\partial \mathrm{z}}\mathrm{dz}\right) \wedge \mathrm{dy} \\ &+ \left(\frac{\partial \mathrm{R}}{\partial \mathrm{x}}\mathrm{dx} + \frac{\partial \mathrm{R}}{\partial \mathrm{y}}\mathrm{dy} + \frac{\partial \mathrm{R}}{\partial \mathrm{z}}\mathrm{dz}\right) \wedge \mathrm{dz} \\ &= \left(\frac{\partial \mathrm{R}}{\partial \mathrm{y}} - \frac{\partial \mathrm{Q}}{\partial \mathrm{z}}\right) \mathrm{dy} \wedge \mathrm{dz} + \left(\frac{\partial \mathrm{P}}{\partial \mathrm{z}} - \frac{\partial \mathrm{R}}{\partial \mathrm{x}}\right) \mathrm{dz} \wedge \mathrm{dx} + \left(\frac{\partial \mathrm{Q}}{\partial \mathrm{x}} - \frac{\partial \mathrm{P}}{\partial \mathrm{y}}\right) \mathrm{dx} \wedge \mathrm{dy}. \end{aligned}$$

□

Stokes' Theorem - Examples

Proof.

In Gauss's theorem, we have $\omega = \mathrm{P}\mathrm{d}y \wedge \mathrm{d}z + \mathrm{Q}\mathrm{d}z \wedge \mathrm{d}x + \mathrm{R}\mathrm{d}x \wedge \mathrm{d}y$, which is a 2-form. Then we calculate its exterior differential

$$\begin{aligned}
\mathrm{d}\omega =& \left(\frac{\partial P}{\partial x}\mathrm{d}x + \frac{\partial P}{\partial y}\mathrm{d}y + \frac{\partial P}{\partial z}\mathrm{d}z\right) \wedge \mathrm{d}y \wedge \mathrm{d}z \\
&+ \left(\frac{\partial Q}{\partial x}\mathrm{d}x + \frac{\partial Q}{\partial y}\mathrm{d}y + \frac{\partial Q}{\partial z}\mathrm{d}z\right) \wedge \mathrm{d}z \wedge \mathrm{d}x \\
&+ \left(\frac{\partial R}{\partial x}\mathrm{d}x + \frac{\partial R}{\partial y}\mathrm{d}y + \frac{\partial R}{\partial z}\mathrm{d}z\right) \wedge \mathrm{d}x \wedge \mathrm{d}y \\
=& \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}\right) \mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z.
\end{aligned}$$

□