Lecture notes on metric space and Gromov-Hausdorff distance

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Contents

1 Introduction

This notes is based on the lecture given by Ilaria Mondello for the 2017-2018 master day of Paris-Saclay. This 3×1.5 course covers a quick but detailed introduction of metric space and Gromov-Hausdorff distance and finishes by the precompact theorem of Gromov-Hausdorff space. This note tries to recap this excellent introduction course.

At the beginning of the notes, I have to add some words about the Gromov-Hausdorff space since this may be one of the most tremendous distance in maths - generally it measures the distance between compact object - it appears in many branches like differential geometry, geometric group theory and also nowadays probability theory, to treat positive curvature problem, the isometric embedding problem and random object. Although sometimes we need only one of the several equivalent definitions, maybe a comprehensive understanding helps better apply this good idea in our work and find other surprising properties.

The lecture note is organized as following : In the first part, we recall the basic definitions of metric space and the definition of compactness by ϵ-net. The second part compares the advantages and disadvantages of some classical distance, which leads the motivation of the Hausdorff distance and Gromov-Hausdorff distance. The third and forth part treat the definition of these two distances and their various equivalent definitions which work better in some specific situations. Generally speaking, the Hausdorff distance compares the distance of two compact sets in the same metric space, while Gromov-Hausdorff distance permits us to compare the distance of two compact metric spaces. (So Gromov-Hausdorff space is a space of compact space !) Finally, we finish by an important precompact theorem of Gromov-Hausdorff space, whose proof is a little technical but shares the same idea of Ascoli-Arzela theorem and the completion of metric space.

In the remaining part of the first section, we recall the definition of metric space, the compactness and the completion of metric space, which the reader may be already familiar with and can be found in may text books for an introduction in detail.

1.1 Metric space

The metric space is the main role of this course, here we recall its definition.

Definition 1.1 (Metric space)**.** Given set X, $d : X \times X \to \mathbb{R}$ is a distance on X if and only if it satisfies the following three properties

1. **Positivity** $d(x, y) \geq 0$ and $d(x, y) = 0 \Leftrightarrow x = y$.
2. **Symmetry** $d(x, y) = d(y, x)$.
3. **Triangle inequality** $d(x, y) \leq d(x, z) + d(z, y)$.

, and we call (X, d) a metric space.

There are a lot of examples of metric space, for example the Euclid space $(\mathbb{R}^n, d)$. Moreover, for a same set, we can define different distances. For example, the circle $\mathbb{S}^{n-1} \subset \mathbb{R}^n$ can be equipped by a distance induced by the distance of $(\mathbb{R}^n, d)$ or another distance as the arc length defined by $d(x, y) = \arccos\langle x, y\rangle$. This is also an example of the metric space of Riemann manifold with geodesic distance.

To verify the definition of metric space, maybe the triangle inequality is the most non-trivial property and sometimes requires technique. However, we cannot forget the positivity, otherwise it will be another concept a little weaker than distance.

Definition 1.2 (Semi-metric space)**.** A space (X, d) is called semi-metric space if it satisfies all the three properties except $d(x, y) = 0 \Leftrightarrow x = y$.

Remark. A semi-metric space isn't very far from the metric space and in fact, we define an equivalence class by $\overset{d}{\sim}$ if $d(x, y) = 0$. Then the equivalence class forms a metric space $(X/\overset{d}{\sim}, d)$.

One example of semi-metric space is $(\mathcal{L}^p(\mathbb{R}^n), m)$ and we take the equivalence class $(L^p(\mathbb{R}^d), m)$.

The topology on metric space isn't so abstract since the open ball is clear.

Definition 1.3 (Topology on metric space)**.** The topology $\mathcal{T}$ on metric space (X, d) is generated by the open ball

$$B_r(x) = \{y \in X | d(x, y) < r\}$$

and we denote the convergence by $x_n \to x$ which means $d(x_n, x) \to 0$.

When we take two metric spaces, one most special relation is isometry - it means two spaces "look same".

Definition 1.4 (Isometry)**.** Given two metric space (X, d^X), (Y, d^Y), a function $f : X \to Y$ preserves the distance if $\forall x, x' \in X, d^Y(f(x), f(x')) = d^X(x, x')$. Moreover, if f is a bijection, then we call it an isometry between (X, d^X) and (Y, d^Y).

Finally, we generalize a little the Lipschitz function as a function between two metric space.

Definition 1.5 (Lipschitz)**.** Given two metric spaces (X, d^X), (Y, d^Y), a function $f : X \to Y$ is said Lipschitz if and only if $\exists C > 0, \forall x, x' \in X, d^Y(f(x), f(x')) \leq d^X(x, x')$.

1.2 Completion and compactness

When we study the limit behavior, we need some good property : at least we need completion and we hope to have compactness if possible. The completion of metric space is described as following :

Theorem 1.1 (Completion of metric space)**.** *Every metric space (X, d) admits a completion : there is a complete metric space $(\hat{X}, \hat{d})$ such that (X, d) is its dense subspace and $\hat{d}|_{X\times X} = d$. The completion is unique up to an isometry.*

We skip the proof but just give a main idea. In fact, just analogue to pass the rational number to the real number, we can identify the Cauchy sequence as the element in $\hat{X}$ and the take the limit of distance as the limit on it.

This completion is good since it preserves Lipschitz function. More generally, the Lipschitz function defined on a dense subset has a natural extension.

Theorem 1.2 (Extension of Lipschitz function). *If $f : X' \to Y$ is Lipschitz where X', Y are metric spaces and $X' \subset X$ is dense while Y is complete. Then there exists unique extension $\hat{f} : X \to Y$ which is also Lipschitz.*

Finally, we would like use a new idea of ϵ-net (Sometimes it is also called finite Lebesgue number theorem) to describe the compactness since this description is more convenient in metric space and paves way for the latter part of this note.

We need at first the distance between a point and a set.

Definition 1.6 (Distance between point and set). Given a metric space (X, d) and $S \subset X, x \in X$, then we define $d(x, S) = \inf_{y \in S} d(x, y)$.

A ϵ-set is very intuitive : it says that we can cover the space by small balls of radius ϵ centering at a subset S. The definition totally bounded says that " the volume of space is finite ".

Definition 1.7 (ϵ-net and totally bounded). Given $\epsilon > 0$, we say that S is a ϵ-net of metric space (X, d) if $S \subset X$ and $\forall x \in X, d(x, S) \leq \epsilon$.

(X, d) is said totally bounded if and only if $\forall \epsilon > 0, \exists$ finite $\epsilon-$net.

We give the main theorem of compactness defined by ϵ-net

Theorem 1.3 (Compactness defined by ϵ-net). *A metric space (X, d) is compact if and only if (X, d) is complete and totally bounded.*

Proof. If (X, d) is compact, then every open covering contains a finite sub open covering. We know that obviously $\bigcup_{x \in X} B_\epsilon(x)$ covers X, then we extracts the centers of its sub open covering and this is a finite ϵ-net, so X is totally bounded. The completeness is directly the result of compactness.

Conversely, if (X, d) is totally bounded and complete, we would like prove the original definition of compactness. Given $\bigcup_{i \in I} O_i$ covers X and S^k is a finite ϵ_k-net where $\epsilon_k = \frac{1}{k}$, we define moreover that $\tilde{S}^n = \bigcup_{k=1}^n S^k$. Since S^k is finite, we can find finite open set O_i to cover them and therefore we know that to cover $\tilde{S}^n$ we need also only finite open subset O_i, so we denote it by U_n and without of generality we suppose that it's increasing.

Now we prove by absurd and suppose that the open covering doesn't have open sub covering so that $U_n \neq X, \forall n$. Then we know finite intersection of $X \backslash U_n$ isn't empty and this happens in at least one ϵ_k neighborhood of some point on S^K. We can continue this argument with n thanks to the totally bounded property. Finally, we extract a subset of decreasing neighborhood and use the fact that $X \backslash U_n$ is closed to get the limit which doesn't belong to infinite intersection of $X \backslash U_n$ and this is a contradiction. □

2 Idea of convergence

In this part, we discuss a classical definition of convergence called uniform convergence. It gives a lot of information of the metric space but it may be sometimes too strict and this gives us the motivation to find some other distance.

At first, we introduce the definition of distortion.

Definition 2.1 (Distortion)**.** Given two metric space $(X, d^X), (Y, d^Y)$ and a function between the two spaces $f : X \to Y$. The distortion of f is defined as

$$dis(f) = \sup_{x,x' \in X} |d^X(x, x') - d^Y(f(x), f(x'))|$$

.

As an example, we see if f is an isometry, the distortion is 0 so that X and Y are perfectly matched. The uniform convergence shares also the same spirit.

Definition 2.2 (Uniform convergence)**.** A family of metric space $\{(X_n, d_n)\}$ converges uniformly to a metric space (X, d) if $\exists$ homogeneous $f_n : X_n \to X$ such that $dis(f_n) \xrightarrow{n \to \infty} 0$.

Therefore, if X_n converges in this sense, we can apply homogeneous deformation to change it little by little and finally to X. However, we notice that all the space should be at least homogeneous, so some strange examples cannot apply this idea.

Example 2.1. First example is the following pictures : a series of torus whose diameter of handle becomes smaller and smaller and finally disappears. However, in the sense of uniform convergence, it doesn't converge since there will never be a homogeneous function between a torus and a sphere.

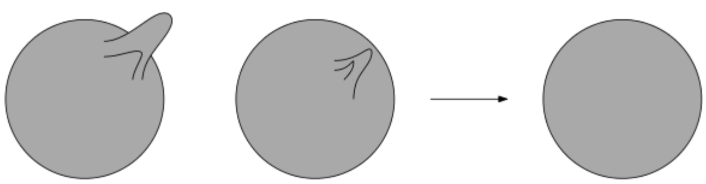

Figure 1: A series of torus "converge" to a sphere

Example 2.2. Another example is the distortion of two torus. Luckily, they are homogeneous at least, but the handled is attached to the small sphere in the left one and is attached to the large sphere in the right one. This makes the distortion so large than expecting no matter how small the handle is.

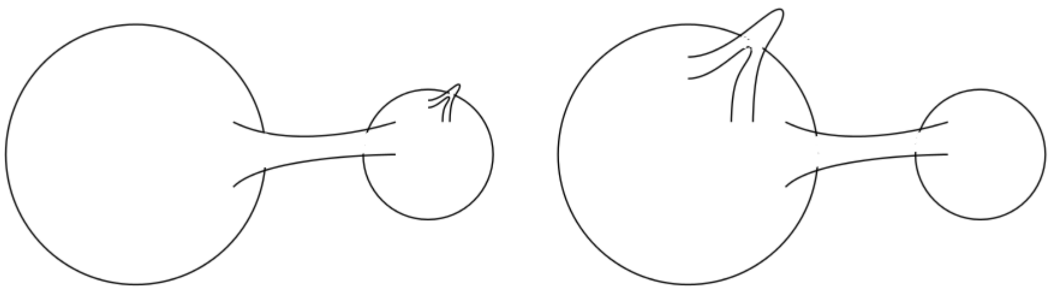

Figure 2: Two homogeneous torus

In conclusion, the uniform convergence just measures the distance between a family of homogeneous equivalent metric space. In some other situation, since we would study just the metric property and don't care other topological properties, we could neglect the restriction of f. We will discuss and develop it in the further part of the notes.

3 Hausdorff distance

In this section, we try to give the idea of Hausdorff distance, which allows us to measure the distance of two subsets from purely the viewpoint of metric. We will finally prove that it is well defined as a distance for all the compact set of a metric space.

3.1 Definition

Maybe someone would generalize the definition of distance between a point and a set, however it's easy to check that a naive generalization like $d(A, B) = \inf_{x\in A, y\in B} d(x, y)$ isn't a distance in cause of the triangular inequality. For example, we set $A = [0, 1], B = [1, 2], C = [2, 3]$, then $d(A, C) = 1 > d(A, B) + d(B, C) = 0 + 0 = 0$.

That is the reason to define the r-neighborhood and the Hausdorff distance. Generally speaking, a Hausdorff distance is the least radius that we shrink one figure and can touch the other.

Definition 3.1 (r-neighborhood)**.** Given (X, d) a metric space and $A \subset X, r > 0$. The r-neighborhood of A is defined as

$$U_r(A) = \{x \in X | d(x, A) < r\}$$

Definition 3.2 (Hausdorff distance)**.** Given (X, d) a metric space and its two subsets $A, B \subset X$, then we define the Hausdorff distance as

$$d_H(A, B) = \inf \{r > 0 | B \subset U_r(A), A \subset U_r(B)\}$$

.

We do calculate some easy examples to be familiar with the definition.

Example 3.1. Given $a < b < c < d$ and $A = [a, b], B = [c, d]$, we have $d_H(A, B) = (d - b) \vee (c - a)$.

Example 3.2. Given $I_\epsilon = [\epsilon, 1 - \epsilon], \epsilon > 0$. Then we have

$$d_H(I_\epsilon, [0, 1]) \xrightarrow{\epsilon \to 0} 0$$

. If we admits that it's a distance, informally, we can write $I_\epsilon \xrightarrow{d_H} [0, 1]$.

The following proposition is nearly direct from the definition, since the smallest radius to cover B from A means the largest distance from one point of A to the set B.

Proposition 3.1 (Another equivalent definition of d_H)**.** *In the same setting of the definition of Hausdorff distance, we have*

$$d_H(A, B) = \max \left\{ \sup_{b \in B} d(b, A), \sup_{a \in A} d(a, B) \right\}$$

3.2 A metric space for compact subset

We will prove that the Hausdorff distance is a distance for the compact subset by starting the proof of the following theorem which prepares for the main theorem.

Theorem 3.1 (d_H as semi-metric)**.** *Given a metric space (X, d) then*

1. *d_H is a semi-metric on $\delta(X) =$ all the subsets of X.*
2. *$\forall A \subset X, d_H(A, \bar{A}) = 0$.*
3. *$\forall A, B$ closed subset, $d_H(A = B) \Leftrightarrow A = B$.*

Proof. **Proof for (1)**It's obvious to verify that $d_H(A, B) \geq 0$ and $d_H(A, B) = d_H(B, A)$. To check the triangular inequality, we state a lemma as following.

Lemma 3.1 (Composition of r-neighborhood)**.** *Given S subset of a metric space (X, d), we have $U_{r_1}(U_{r_2}(S)) \subset U_{r_1+r_2}(S)$.*

The proof is direct. $\forall x \in U_{r_1}(U_{r_2}(S))$, we have $d(x, U_{r_2}(S)) < r_1$, which implies $\exists y \in U_{r_2}(S), d(x, y) < r_1$. By the same argument, $\exists z \in S$ such that $d(y, z) < r_2$, therefore $d(x, z) < d(x, y) + d(y, z) = r_1 + r_2$ and $x \in U_{r_1+r_2}(S)$ and we prove the lemma.

We remark that the converse $U_{r_1+r_2}(S) \subset U_{r_1}(U_{r_2}(S)$ isn't correct. A counter example could be $X = \mathbb{N}, S = \{0\}, r_1 = r_2 = 0.6$, then $U_{r_1+r_2}(S) = \{-1, 0, 1\}$ while $U_{r_1}(U_{r_2}(S)) = \{0\}$.

Thanks to this lemma, we prove the triangular inequality. Suppose that $d_H(A, C) = r_1$ and $d_H(C, B) = r_2$, then

$$\forall \epsilon > 0, B \subset U_{r_2+\frac{\epsilon}{2}}(C), C \subset U_{r_1+\frac{\epsilon}{2}}(A)$$

. We apply the lemma then

$$B \subset U_{r_2+\frac{\epsilon}{2}}(U_{r_1+\frac{\epsilon}{2}}(A)) \subset U_{r_1+r_2+\epsilon}(A)$$

. Another direction of inclusion follows the same proof. Because $\epsilon > 0$ is arbitrary, this means $d_H(A, B) \leq r_1 + r_2$ and we prove the triangular inequality.

Proof for (2) We use the equivalent definition of the Hausdorff distance

$$d_H(A, \bar{A}) = \max \left\{ \sup_{a \in A} d(a, \bar{A}), \sup_{a' \in \bar{A}} d(a', \bar{A}) \right\}$$

and the fact that $\sup_{a \in A} d(a, \bar{A}) = \sup_{a' \in \bar{A}} d(a', \bar{A}) = 0$, then we get the result desired.

Proof for (3) We prove by absurd and suppose that $A \neq B$. Then $\exists a \in A$ such that $d(a, B) > 0$ since B is closed. This means that $d_H(A, B) > 0$ and it's a contradiction. □

Theorem 3.2 (d_H is a distance for compact subset)**.** *Given a metric space (X, d), the Hausdorff distance on it makes the collection of compact subset of (X, d) a metric space.*

Proof. In the last theorem, we have prove it a semi-metric. Moreover, the third statement in last theorem implies that for two compact subset A, B, if $d_H(A, B) = 0$, then $A = B$. Therefore, the metric space is well defined. □

4 Gromov-Hausdorff distance

Although the Hausdorff distance permit us to compare the distance of compact, one may ask some further question. For example, for two compact sets different for a transition in the same space, we would like to say that they are "same" up to a transition, but their Hausdorff distance is not zero. How to solve this problem ? In this section, we will generalize the Hausdorff distance and define the Gromov-Hausdorff distance which compares the distance between two compact space, so this helps us treat our question since two compact sets can be treated as two compact metric space if we restrict the distance respectively on these two compact subsets. We will also give some other equivalent definitions to treat this distance in more practical may.

4.1 Definition by isometry

We give the original definition of Gromov-Hausdorff distance.

Definition 4.1 (Gromov-Hausdorff distance)**.** Given two metric spaces $(X, d^X), (Y, d^Y)$, the Gromov-Hausdorff distance between them is defined as the lower bound when we do embedding of X, Y to a same metric space (Z, d^Z) i.e

$$d_{GH}(X,Y) = \inf \left\{ d_H^Z(\phi_X(X), \phi_Y(Y)) | \phi_X : X \to Z, \phi_Y : Y \to Z \text{ isometric } \right\}$$

.

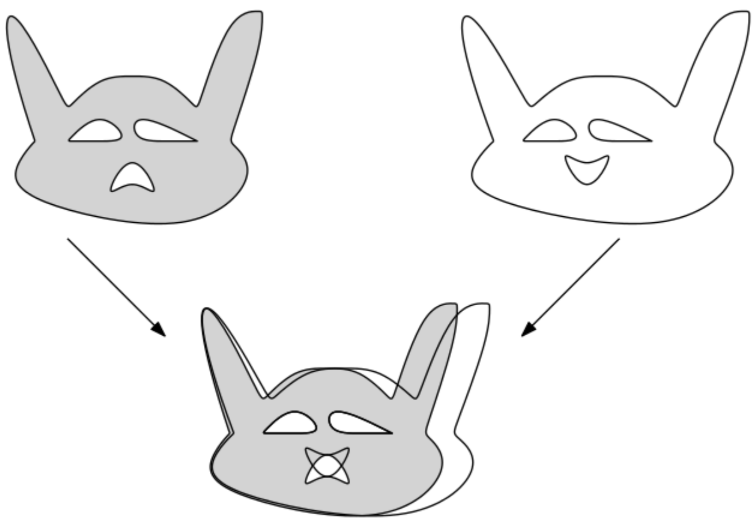

Figure 3: Embed two metric space to a third metric space and compare the distance

This is a nice definition, except finding the isometric embedding to a third metric space is very abstract. In the following parts and following section, we will develop other equivalent definition easier to implement in examples. A first equivalent definition allow us to design a semi-metric instead of searching for a metric space.

Proposition 4.1 (Equivalent definition of d_{GH}). *In the same setting of Gromov-Hausdorff distance, it can be defined as the lower bound of semi-metric on $d_H^{X \sqcup Y}$ i.e*

$$d_{GH}(X,Y) = \inf \left\{ d_H^{X \sqcup Y}(X,Y) | d^{X \sqcup Y}_{|X \times X} = d^X, d^{X \sqcup Y}_{|Y \times Y} = d^Y \right\}$$

Remark. Here $\sqcup$ means the disjoint union. In another word, if $Y = X$ or there is some point x in common, in the space $X \sqcup Y$ there will be two copies x_1, x_2.

Proof. Given a metric space and two isometric function (Z, ϕ_X, ϕ_Y), there is a natural semi-metric on the $X \sqcup Y$ defined by

$$f : (Z, \phi_X, \phi_Y) \to (\phi_X(X) \sqcup \phi_Y(Y), d^Z)$$

. In fact, d^Z restricted on the image is isometric and we add two copies make d^Z a semi-metric. Therefore, one embedding gives a correspondent semi-metric

$$d_{GH}(X,Y) \geq \inf \left\{ d_H^{X \sqcup Y}(X,Y) | d^{X \sqcup Y}_{|X \times X} = d^X, d^{X \sqcup Y}_{|Y \times Y} = d^Y \right\}$$

. On the other hand, once we defined a semi-metric on $X \sqcup Y$, we can also fabricate a metric space $Z = (X \sqcup Y / \sim d^{X \sqcup Y}, d^{X \sqcup Y})$and isometric function as projection on each space, so we get another inequality and we conclude. $\square$

4.2 Definition by correspondence

A second equivalent definition comes from a analogue of distortion of function. We will introduce a more general coupling called correspondence, which is more flexible than the homogeneous function.

Definition 4.2 (Correspondence). Given two sets X, Y, a correspondence is a subset $R \subset X \times Y$ which satisfies

- $\forall x \in X, \exists y \in Y, s.t(x,y) \in R.$
- $\forall y \in Y, \exists x \in X, s.t(x,y) \in R.$

The following proposition tells us how to construct a correspondence, although we don't really need this proposition very often. Since once we are familiar with this notation, we can design many correspondences in the specific situations as we wanted.

Proposition 4.2 (How to construct correspondence). *Given three sets X, Y, Z and f, g two surjective function $f : Z \to X, g : Z \to Y$, then $R = \{(f(z), g(z)) | z \in Z\}$ is a correspondence.*

Conversely, for every correspondence between X, Y, we can find a third set Z and two surjective function like above.

Proof. Given a set Z and two surjective functions f, g, we construct $\{(f(z), g(z))|z \in Z\}$ as a correspondence, since the surjective function assures to take all the values of X and Y.

On the other hand, given a correspondence R, we construct a set $Z = \{(x,y) \in X \times Y|(x,y) \in R\}$ and two projections $f = \Pi_X, g = \Pi_Y$ as surjective functions. □

We generalize the distortion of function to the distortion of correspondence.

Definition 4.3 (Distortion of correspondence)**.** Given two metric spaces $(X, d^X), (Y, d^Y)$ and R a correspondence between X and Y, then we can define the distortion of R by

$$dis(R) = \sup_{(x,y),(x',y') \in R} |d^X(x,x') - d^Y(y,y')|$$

Gromov-Hausdorff distance can also be given by find the lower bound of possible correspondence and it seems easier than the original definition or finding semi-metrics.

Proposition 4.3 (d_{GH} defined by correspondence)**.** *In the same setting of Gromov-Hausdorff distance, we have*

$$d_{GH}(X,Y) = \frac{1}{2} \inf\{dist(R)|R \textit{ correspondence between } X \textit{ and } Y\}$$

. The proof has two parties :

1. *If* $d_{GH}(X,Y) < r$*, then* $\exists R$ *correspondence s.t* $dis(R) < 2r$*.*
2. $d_{GH}(X,Y) \leq \frac{1}{2} dis(R)$*.*

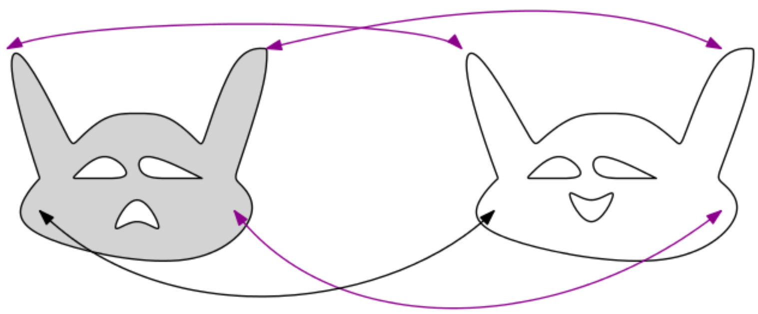

Figure 4:

Proof. **Proof of (1)** Once we have $d_{GH}(X,Y) < r$, we have a metric space (Z, d^Z) and two isometric function ϕ_X, ϕ_Y, we will construct a correspondence whose distortion is less than $2r$.

We define the correspondence R like this : $\forall x \in X, \exists y \in Y$, such that $d^Z(\phi_X(x), \phi_Y(y)) < r$. $\forall x' \in X, \exists y' \in Y$, such that $d^Z(\phi_X(x'), \phi_Y(y')) < r$. We put these couples $(x,y),(x',y')$ in the correspondence R.

By definition,

$$\begin{aligned} dis(R) &= \sup_{(x,y),(x',y') \in R} |d^X(x,x') - d^Y(y,y')| \\ &= \sup_{(x,y),(x',y') \in R} |d^Z(\phi_X(x), \phi_X(x')) - d^Z(\phi_Y(y), \phi_Y(y'))| \end{aligned}$$

. The construction of correspondence tells us

$$\begin{aligned} d^Z(\phi_X(x), \phi_X(x')) &\leq d^Z(\phi_X(x), \phi_Y(y)) + d^Z(\phi_Y(y), \phi_Y(y')) + d^Z(\phi_Y(y'), \phi_X(x')) \\ &\leq d^Z(\phi_Y(y), \phi_Y(y')) + 2r \\ d^Z(\phi_Y(y), \phi_Y(y')) &\leq d^Z(\phi_Y(y), \phi_X(x)) + d^Z(\phi_X(x), \phi_X(x')) + d^Z(\phi_X(x'), \phi_Y(y')) \\ &\leq d^Z(\phi_X(x), \phi_X(x')) + 2r \end{aligned}$$

Therefore, $dis(R) < 2r$.

Proof of (2) The second part is a little delicate. We use the equivalent definition and we will construct a semi-metric on $X \sqcup Y$ from the correspondence.

The main task is to construct $d^{X \sqcup Y}(x,y), x \in X, y \in Y$. Suppose that $dis(R) = r$ We define

- If $(x,y) \in R$, we define $d^{X \sqcup Y}(x,y) = r$.
- If $(x,y) \notin R$, we define $d^{X \sqcup Y}(x,y) = r + \inf\{d^X(x,x') | (x',y) \in R\}$

We will prove that it works. One useful inequality in the proof is that

$$\forall (x,y),(x',y) \in R, d^X(x,x') = |d^X(x,x') - d^Y(y,y)| \leq dis(R) = r$$

For any three points $x_1, x_2 \in X, y \in Y$, there are three situations.

1. $(x_1, y) \in R, (x_2, y) \in R$. In this case, $d^{X \sqcup Y}(x_1, x_2) \leq r = d^{X \sqcup Y}(x_1, y) + d^{X \sqcup Y}(x_2, y)$.
2. $(x_1, y) \notin R, (x_2, y) \notin R$. Then $\forall x_3$ such that $(x_3, y) \in R$,

$$\begin{aligned} d^{X \sqcup Y}(x_1, x_3) &\leq \inf_{(x_4,y) \in R} d^X(x_1, x_4) + d^X(x_4, x_3) \\ &\leq \inf_{(x_4,y) \in R} d^X(x_1, x_4) + r \\ &\leq d^{X \sqcup Y}(x_1, y) + \frac{r}{2} \end{aligned}$$

Therefore

$$\begin{aligned}
d^X(x_1,x_2) &\leq \inf_{(x_3,y)\in R} d^X(x_1,x_3)+d^X(x_3,x_2)\\
&\leq \inf_{(x_3,y)\in R} d^X(x_3,x_2)+d^{X\sqcup Y}(x_1,y)+\frac{r}{2}\\
&\leq d^{X\sqcup Y}(x_2,y)+d^{X\sqcup Y}(x_1,y)
\end{aligned}$$

The other two inequalities are easier to check by the same idea.

3. $(x_1,y)\notin R,(x_2,y)\in R$. The inequality $d^{X\sqcup Y}(x_2,y)<d^{X\sqcup Y}(x_1,y)+d^{X\sqcup Y}(x_1,x_2)$ is trivial. The other two inequalities follow by

$$\begin{aligned}
d^{X\sqcup Y}(x_1,y) &= \inf_{(x_3,y)\in R} d^X(x_1,x_3)+\frac{r}{2}\\
&\leq d^{X\sqcup Y}(x_1,x_2)+d^{X\sqcup Y}(x_2,y)\\
d^{X\sqcup Y}(x_1,x_2) &\leq \inf_{(x_3,y)\in R}[d^X(x_1,x_3)+d^{X\sqcup Y}(x_3,x_2)]\\
&\leq \inf_{(x_3,y)\in R} d^X(x_1,x_3)+r\\
&= d^{X\sqcup Y}(x_1,y)+d^{X\sqcup Y}(x_2,y)
\end{aligned}$$

The other case like $(x,y_1,y_2),y_1\in Y,y_2\in Y$ need similar analysis. We neglect the discussion and conclude the $d_{GH}(X,Y)\leq\frac{1}{2}dis(R)$. □

The definition of d_{GH} by correspondence allows us to deduce the triangular inequality.

Theorem 4.1 (Triangular inequality of d_{GH})**.** *d_{GH} satisfies triangular inequality.*

Proof. It is the direct result of the following result.

Lemma 4.1 (Composition of correspondence)**.** *Given three metric spaces $(X,d^X),(Y,d^Y),(Z,d^Z)$ and R_1 correspondence between X and Y, R_2 correspondence between Y and Z, then we define their composition*

$$R_1\circ R_2=\{(x,z)\in X\times Z|\exists y\in Y s.t.(x,y)\in R_1,(y,z)\in R_2\}$$

Moreover, we have the inequality

$$dis(R_1\circ R_2)\leq dis(R_1)+dis(R_2)$$

.

We prove the lemma at first.

$$\begin{aligned}
& \sup_{(x,z),(x',z')\in R_1\circ R_2} |d^X(x,x') - d^Z(z,z')| \\
= & \sup_{(x,z),(x',z')\in R_1\circ R_2} |d^X(x,x') - d^Y(y,y') + d^Y(y,y') - d^Z(z,z')| \\
\leq & \; dis(R_1) + dis(R_2)
\end{aligned}$$

We prove the main theorem. Note R_1, R_2, R_3 the correspondence between $(X,Y),(Y,Z),(X,Z)$.

$$\begin{aligned}
d_{GH}(X,Z) &= \frac{1}{2}\inf_{R_3} dis(R_3) \\
&\leq \frac{1}{2}\inf_{R_1\circ R_2} dis(R_1\circ R_2) \\
&\leq \inf_{R_1} dis(R_1) + \inf_{R_2} dis(R_2) = d_{GH}(X,Y) + d_{GH}(Y,Z)
\end{aligned}$$

□

4.3 Characterization by ϵ-isometry

Searching for a correspondence to minimize its distortion may be easier than searching for a good isometric embedding or a good semi-metric, but we would like to know what happens if we could not find the optimal correspondence ? Just like solving an ODE by its numerical schema, if the solution has some continuity by parameter, then we can use interpolation and finite difference method to approximate it. In this part, we will see that d_{GH} is also a very robust distance : we don't have to find the optimal approximation but just a good enough approximation.

The following concept defines another "isometry", which says that we treat a function as "isometry" if we are not so serious and allow some shrinking.

Definition 4.4 (ϵ-isometry)**.** (X, d^X) and (Y, d^Y) are two metric spaces, we define a ϵ-isometry $f : X \to Y$ if and only if it satisfies the following two conditions

1. $dis(f) < \epsilon$.
2. $f(X)$ is a ϵ-net of Y.

The following proposition gives a third definition of Gromov-Hausdorff distance.

Proposition 4.4 (ϵ-isometry and d_{GH})**.** *ϵ-isometry describes the Gromov-Hausdorff distance. In fact, for two metric spaces (X, d^X) and (Y, d^Y)*

1. $d_{GH}(X,Y) < \epsilon \Rightarrow \exists f : X \to Y$ *a* 2ϵ*-isometry.*

2. $\exists f : X \to Y$ *isometry* $\Rightarrow d_{GH}(X,Y) < 2\epsilon$.

Proof. **Proof of (1)** Using the definition of d_{GH} by the distortion of correspondence and given R a correspondence such that $\frac{1}{2}dis(R) < \epsilon$, then we design a function $f : X \to Y : \forall x \in X$, we find a y in its correspondence such that $(x, f(x)) \in R$. We will prove that this gives a 2ϵ-isometry.

We check the distortion of function

$$dis(f) = \sup_{x,x' \in X} |d^X(x,x') - d^Y(f(x), f(x'))| \le dis(R) < 2\epsilon$$

. We check the 2ϵ-net. That is to cover Y by $U_{2\epsilon}(f(X))$. $\forall y \in Y$, it has an associated element in X such that $(x,y) \in R$. Deducing from the distortion of correspondence

$$|d^X(x,x) - d^Y(y, f(x))| < dis(R) < 2\epsilon$$

Therefore $d^Y(y, f(x)) < 2\epsilon$ and we find a point in image that covers y by a 2ϵ ball.

Proof of (2) Conversely, we have to construct a good correspondence from a ϵ-isometry. Denoting f the ϵ-isometry, we define

$$R = \left\{(x,y) \subset X \times Y | d^Y(f(x), y) < \epsilon\right\}$$

. This is well defined since every point in Y can be covered by a ϵ-environment of the image. Then we check that it is a good correspondence. $\forall (x,y), (x',y') \in R$

$$\begin{aligned} |d^X(x,x') - d^Y(y,y')| &= |d^X(x,x') - d^Y(f(x), f(x')) + d^Y(f(x), f(x')) - d^Y(y,y')| \\ &\le |d^X(x,x') - d^Y(f(x), f(x'))| + |d^Y(f(x), f(x')) - d^Y(y,y')| \end{aligned}$$

By the definition of R

$$\begin{aligned} d^Y(f(x), f(x')) &\le d^Y(f(x), y) + d^Y(y,y') + d^Y(y', f(x')) \\ &\le d^Y(y,y') + 2\epsilon \\ d^Y(y,y') &\le d^Y(y, f(x)) + d^Y(f(x), f(x')) + d^Y(f(x'), y') \\ &\le d^Y(f(x), f(x')) + 2\epsilon \end{aligned}$$

The two estimation proves that $|d^Y(f(x), f(x')) - d^Y(y,y')| < 2\epsilon$ so

$$\sup_{(x,y),(x',y') \in R} |d^X(x,x') - d^Y(y,y')| \le \epsilon + 2\epsilon = 3\epsilon$$

So $d_{GH}(X,Y) < \frac{1}{2}dis(R) < \frac{3}{2}dis(R)$. $\square$

Remark. The proof of the proposition tells us a useful algorithm to transform between a good correspondence and a good ϵ-isometry.

After collecting all the tools, we come finally to prove that d_{GH} is a distance between compact space.

Theorem 4.2 (d_{GH} as a distance between compact metric spaces)**.** *The Gromov-Hausdorff distance is a finite distance for the compact metric space up to isometry.*

Proof. The symmetry is obvious and we have proved the triangular inequality by the correspondence. So d_{GH} is at least a semi-metric and the only problem is to prove that $d_{GH}(X,Y) = 0$ implies the isometry.

By the proposition of ϵ-isometry, we know $\forall \epsilon_n = \frac{1}{n} > 0, \exists f_n : X \to Y$ ϵ_n isometric. The compactness of X means that it's separable, for example, we can take the countable union of the finite ϵ-net. We denote S the countable dense subset of X and using the diagonal argument to find the limit for all the elements in S $f(x) = \lim_{n\to\infty} f_n(x)$.(The diagonal argument works only for countable set. That's where we use the compactness in the proof.)

Then we claim that f preserve the distance on S since

$$|d^Y(f_n(x_1), f_n(x_2)) - d^X(x_1, x_2)| \leq dis(f_n) < \epsilon_n \xrightarrow{\epsilon_n \to 0} 0$$

. We extend it to the whole X and we get a function which preserves the distance. The same argument works for the construction of a function from Y to X. So we prove the bijection and X, Y are isometric. □

Once we prove that d_{GH} is a metric space, we can define the Gromov-Hausdorff space (M, d_{GH}) where is M is the set of equivalence class of compact metric space. This is a metric space where the topology is given by d_{GH}, so we denote $M_n \xrightarrow{d_{GH}} M$ by $d_{GH}(M_n, M) \to 0$.

5 Precompact theorem of Gromov-Hausdorff space

In the last part, we talk about the precompact property in the Gromov-Hausdorff space. We will give an sufficient condition, by which we can always extract a sequence of sub sequence that converges to a limit.

We recall the classical Ascoli-Arzela theorem.

Theorem 5.1 (Ascoli-Arzela)**.** *Given $\mathcal{A} \subset C(K, F)$ where K is a compact metric space and F is a complete metric space. If $\mathcal{A}$ satisfies the following conditions*

1. *$\mathcal{A}$ is equi-continuous.*

2. *$\mathcal{A}$ is uniformly bounded.*

, then $\mathcal{A}$ is pre-compact.

We propose a condition called uniformly totally bounded, which plays the same role as equi-continuous condition in Ascoli-Arzela.

Definition 5.1 (Uniformly totally bounded)**.** Suppose that $\mathcal{A}$ is a family of compact metric space, it is uniformly totally bounded if and only if

1. There exists $D > 0$ such that $\forall X \in \mathcal{A}, diam(X) < D$.

2. $\forall \epsilon > 0, \exists N = N(\epsilon) \in \mathbb{N}$, such that every $X \in \mathcal{A}$ admits a ϵ-net whose cardinal is less than N.

Theorem 5.2 (Pre-compact)**.** *Every family of uniformly totally bounded compact metric space is pre-compact in the topology of Gromov-Hausdorff distance.*

Before proving the theorem, we do some preparation. As we have seen that d_{GH} is a very robust distance, in the compact metric space, it suffices to analyze the convergence in each ϵ-net.

Proposition 5.1 (ϵ-approximation)**.** *Given $\{(X_n, d_n)\}_{n\in\mathbb{N}}$ a family of compact metric space, then $(X_n, d_n) \xrightarrow{d_{GH}} (X, d)$ if and only if $\forall \epsilon > 0, \exists S_n$ a ϵ-net of X_n and S a ϵ-net of such that $S_n \xrightarrow{d_{GH}} S$.*

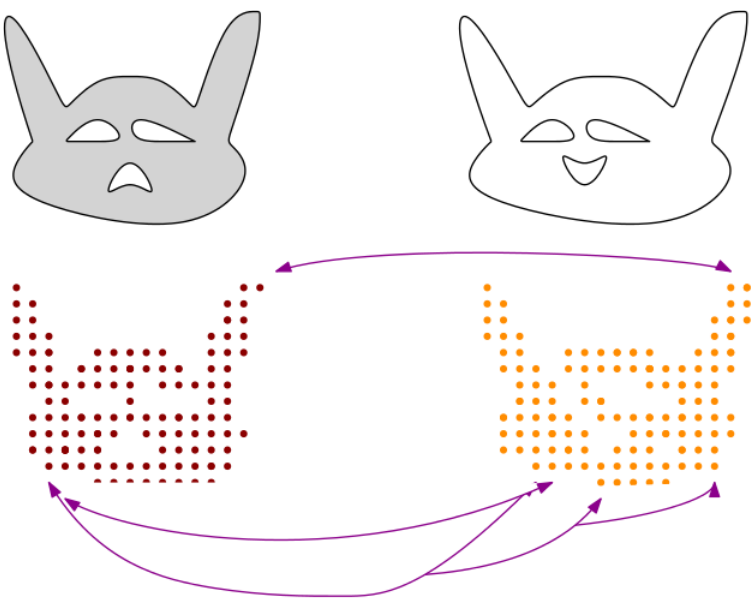

Figure 5:

Proof. If $X_n \xrightarrow{d_{GH}} X$, we can just put $S_n = X_n$ and $S = X$, then the convergence works for each n.

Conversely, if we have convergence for each ϵ-net. Given S_n a ϵ_1-net for each X_n and S a ϵ-net for X, we denote $R_n^{\epsilon_1}$ a correspondence between S_n and X, which realize that $dis(R_n^{\epsilon_1}) \le d_{GH}(S_n, S)$, then we construct a good correspondence $\bar{R}_n^{\epsilon_1}$ between X_n and X.

$$\begin{aligned}
c_n(x') &= x \in S_n \text{ s.t } d_n(x, x') < \epsilon_1 \text{ (We take just one if there are many choices)} \\
c(y') &= y \in S \text{ s.t } d(y, y') < \epsilon_1 \\
R_{n,0}^{\epsilon_1} &= R_n^{\epsilon_1} \\
R_{n,1}^{\epsilon_1} &= \{(x', y) \in X_n \times X | (c_n(x'), y) \in R_n^{\epsilon_1}\} \\
R_{n,2}^{\epsilon_1} &= \{(x, y') \in X_n \times X | (x, c(y')) \in R_n^{\epsilon_1}\} \\
\bar{R}_n^{\epsilon_1} &= R_{n,0}^{\epsilon_1} \cup R_{n,1}^{\epsilon_1} \cup R_{n,2}^{\epsilon_1}
\end{aligned}$$

We verify that this is a good correspondence.

$$\begin{aligned}
& \sup_{(x_1', y_1), (x_2', y_2) \in R_{n,1}^{\epsilon_1}} |d_n(x_1', x_2') - d(y_1, y_2)| \\
\le & \sup_{(x_1', y_1), (x_2', y_2) \in R_{n,1}^{\epsilon_1}} |d_n(x_1', x_2') - d_n(c_n(x_1'), c_n(x_2'))| + |d_n(c_n(x_1'), c_n(x_2')) - d(y_1, y_2)| \\
\le & \ 2\epsilon_1 + dis(R_n^{\epsilon_1})
\end{aligned}$$

$$\begin{aligned}
& \sup_{(x_1', y_1) \in R_{n,1}^{\epsilon_1}, (x_2, y_2') \in R_{n,2}^{\epsilon_1}} |d_n(x_1', x_2) - d(y_1, y_2')| \\
& \le \sup_{(x_1', y_1) \in R_{n,1}^{\epsilon_1}, (x_2, y_2') \in R_{n,2}^{\epsilon_1}} |d_n(x_1', x_2) - d_n(c_n(x_1'), x_2)| + |d_n(c_n(x_1'), x_2) - d(y_1, c(y_2'))| \\
& + |d(y_1, c(y_2')) - d(y_1, y_2')| \\
& \le 2\epsilon_1 + dis(R_n^{\epsilon_1})
\end{aligned}$$

The other inequalities are similar so we prove that

$$d_{G,H}(X_n, X) \le 2\epsilon_1 + 2d_{GH}(S_n, S)$$

. Since $d_{GH}(S_n, S) \xrightarrow{n \to \infty} 0$, we prove that

$$\forall \epsilon_1 > 0, \lim_{n \to} d_{GH}(X_n, X) < 2\epsilon_1$$

, which implies the convergence. $\square$

Finally, we come to prove this sufficient pre-compact property of Gromov-Hausdorff space.

Proof. Since we have a sequence of metric spaces, we have to construct its limit metric space. So we prove in three steps :

1. Construct a underlying set and a metric on it.

2. Prove that it's compact

3. Prove that it is the limit in the sense of d_{GH}.

Without of generality, we note (X_n, d_n) this family of metric space. Using the ϵ-approximation idea, for each $\epsilon_k = \frac{1}{k}$, we have a uniformly bounded number $N(\epsilon_k)$ such that in each there is a ϵ_k-net with cardinal $N(\epsilon_k)$. We also denote $N_0 = 0$ and $N_{k+1} = N_k + N(\epsilon_{k+1})$.

We extract the ϵ_k-net level by level in (X_n, d_n), i.e $\{x_{i,n}\}_{N_{k-1}<i\leq N_k}$ is the ϵ_k-net in the (X_n, d_n) and $\{x_{i,n}\}_{i\in\mathbb{N}}$ is a countable dense subset in (X_n, d_n). By the canonical diagonal extraction, we can get a sub-sequence indexed by p_n such that for i, j fixed, $d_{p_n}(x_{i,p_n}, x_{j,p_n})$ is a Cauchy sequence. To avoid too many notation, we still keep noting the index by n for the sub-sequence.

Now we start the step 1 to construct a limit space. Like the proof of completion, we design the sequence $x_i = \{x_{i,n}\}_{n\in\mathbb{N}}$ the element in space (X, d) and the "distance"

$$d(x_i, x_j) = \lim_{n\to\infty} d_n(x_{i,n}, x_{j,n})$$

between elements. However, we know this space is just a semi-metric space. To make it a candidate of the limit space, we make it at least a complete metric space $(\hat{X}, \hat{d}) = \text{Completion}((X/\overset{d}{\sim}, d))$ and we treat (X, d) as a dense subset in it.

The second step is to prove its compactness. As (X, d) is dense, it suffices to find a ϵ_k-net for element in (X, d). We know $\forall i \in \mathbb{N}, \forall k \in \mathbb{N}, \forall n \in \mathbb{N}, \exists N_k and j_n \leq N_k$ such that $d_n(x_{i,n}, x_{j_n,j}) < \epsilon_k$. Since N_k is finite, there exists a j repeating infinite times along a sub-sequence $\tilde{n}$. We obtain

$$d(x_i, x_j) = \lim_{\tilde{n}\to\infty} d_{\tilde{n}}(x_{i,\tilde{n}}, x_{j,\tilde{n}}) < \epsilon_k$$

. To treat the completion, we take the same strategy. Given $\lim_{i\to\infty} \hat{d}(x_i, \hat{x}) = 0$, for each x_i it associates a x_j in ϵ_k-net and there must be one repeating infinite times in the sequence. This one could be a good element associated with $\hat{x}$ in ϵ_k-net. In conclusion, for every ϵ_k, the cardinal of ϵ_k-net is at most N_k, so $(\hat{X}, \hat{d})$ is totally bounded and it is compact.

Finally, we prove that $(\hat{X}, \hat{d})$ is the limit in sense of Gromov-Hausdorff distance. This part is comparatively easy since we have a natural correspondence $R_n : (x_{i,n}, \{x_{i,n}\}_{n\in\mathbb{N}}$. We know that $\{x_{i,n}\}_{1\leq i\leq N_k}$ and $\{x_i\}_{1\leq i\leq N_k}$ are respectively two ϵ_k-net in (X_n, d_n) and $(\hat{X}, \hat{d})$. The pointwise convergence means

$$\max_{i,j\leq N_k} |d_n(x_{i,n}, x_{j,n}) - d(x_i, x_j)| \xrightarrow{n\to\infty} 0$$

, so the ϵ_k-net has d_{GH} convergence, so we have $(X_n, d_n) \xrightarrow[n\to\infty]{d_{GH}} (\hat{X}, \hat{d})$ by the ϵ-approximation. $\square$

6 What's more ?