ALGEBRAIC STRUCTURE OBJECTS (AND DIAGRAM FRENZY)

Nataniel Marquis

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In february 2023, I simultaneously stumbled upon several questions. Some were concerning rings objects in the category of topological spaces with a continuous monoid action. Others were concerning the quotient of a product of objects by the product action of a product of groups. Convincing myself that the statements I hoped for were true, I found myself drawing beautiful diagrams on the board, enriching them, dissecting them. Even if the results of this text are basic, they are established with great generality and give the occasion to commutative diagram addicts to satisfy their vice.

1 Algebraic structure objects

For every category $\mathcal{C}$ and every object X of $\mathcal{C}$, there exists a contravariant functor from $\mathcal{C}$ to Ens called the presheaf associated to X, which is defined by $h_X : Y \mapsto \mathrm{Hom}_{\mathcal{C}}(Y, X)$. An algebraic structure object X will roughly be an object together with a factorization of the presheaf h_X through the desired category of algebraic structures. This is reformulated as the datum of an algebraic structure on each $\mathrm{Hom}_{\mathcal{C}}(Y, X)$, fonctorial in Y. Translating everything categorically remak that an algebraic structure is the datum of maps making some diagrams commute. Explicitely, a group structure on a set G is the data of three maps

$$\mu : G \times G \to G, \ \ e : * \to G \text{ and } i : G \to G$$

respectively called multiplication, neutral et inverse, which makes the following diagrams commute :

$$\begin{tikzcd} G\times G\times G \arrow[r, "\mathrm{Id}_G\times\mu"] \arrow[d, "\mu\times\mathrm{Id}_G"'] & G\times G \arrow[d, "\mu"] \\ G\times G \arrow[r, "\mu"] & G \end{tikzcd} \qquad \begin{tikzcd} G \arrow[r, "\mathrm{Id}_G\times e"] \arrow[d, "e\times\mathrm{Id}_G"'] \arrow[dr, "\mathrm{Id}_G"] & G\times G \arrow[d, "\mu"] \\ G\times G \arrow[r, "\mu"] & G \end{tikzcd}$$

$$\begin{tikzcd} G \arrow[r, "\Delta_G"] \arrow[d, "\Delta_G"'] \arrow[dr] & G\times G \arrow[r, "\mathrm{Id}_G\times i"] & G\times G \arrow[dd, "\mu"] \\ G\times G \arrow[d, "i\times\mathrm{Id}_G"'] & * \arrow[dr, "e"] & \\ G\times G \arrow[rr, "\mu"] & & G \end{tikzcd}$$

These diagrams exhibit the associativity of the multiplication, the fact that e is a neutral element and that i is an inverse map. Thanks to the Yoneda lemma, giving ourselves such functorials maps on $\mathrm{Hom}_{\mathcal{C}}(-, X)$ translates into similar maps and diagrams on the object of $\mathcal{C}$. For a category $\mathcal{C}$ with final object and fiber products, a group object will be defined as an object G together with morphisms μ, e and i as above (the product being seen in $\mathcal{C}$ and $*$ standing for the final object) which makes the previous diagram commute.

This section aims to prove the following :

Proposition 1.1. *Let $\mathcal{C}$ and $\mathcal{D}$ be two categories with final object and every fiber product. Let $F : \mathcal{C} \to \mathcal{D}$ be a covariant functor such that the natural morphism $F(Y \times_X Y') \to F(Y) \times_{F(X)} F(Y')$ is an isomorphism for every fiber product. Let T be an endomorphism of the functor F.*

For every monoid object Z of $\mathcal{C}$, there exist a natural structure of monoid object on $F(Z)$, which thus improves $T(Z)$ as a morphism of monoid objects.

The same statement is correct if we replace every occurence of monoid by group, abelian group, ring, R-module or R-algebra for a ring R.

Let's lay some groundwork.

Lemma 1.2. *Let $F : \mathcal{C} \to \mathcal{D}$ be a functor between two categories with fiber products such that the natural morphism $F(Y \times_X Z) \to F(Y) \times_{F(X)} F(Z)$ is always an isomorphism. Let f be a morpohism from the diagram defining $Y \times_X Z$ to the one defining $Y' \times_{X'} Z'$. It induces a morphism*

$$f_\times : Y \times_X Z \to Y' \times_{X'} Z'.$$

It also induces a morphism $F(f)$ from the diagram defining $F(Y) \times_{F(X)} F(Z)$ to the one defining $F(Y') \times_{F(X')} F(Z')$. The following square commutes :

$$\begin{tikzcd} F(Y\times_X Z) \arrow[r, "F(f_\times)"] \arrow[d, "\wr"'] & F(Y'\times_{X'} Z') \arrow[d, "\wr"] \\ F(Y)\times_{F(X)} F(Z) \arrow[r, "F(f)_\times"] & F(Y')\times_{F(X')} F(Z') \end{tikzcd}$$

In particular, for any endomorphism T *of the functor* F*, and for every fiber product* $Y \times_X Z$*, the following diagram commutes :*

$$\begin{tikzcd} F(Y \times_X Z) \arrow[r, "\mathrm{T}(Y\times_X Z)"] \arrow[d, "\wr"'] & F(Y \times_X Z) \arrow[d, "\wr"] \\ F(Y) \times_{F(X)} F(Z) \arrow[r, "\mathrm{T}(Y)\times_{\mathrm{T}(X)}\mathrm{T}(Z)"] & F(Y) \times_{F(X)} F(Z) \end{tikzcd}$$

Démonstration. By universal property of the fiber product, we can verify this after projection on each factor. Denote by p_1 every projection on the first factor. We enrich the diagram as follows :

$$\begin{tikzcd} F(Y \times_X Z) \arrow[rr, "F(f_\times)"] \arrow[d, "\wr"'] & & F(Y' \times_{X'} Z') \arrow[d, "\wr"] & \\ F(Y) \times_{F(X)} F(Z) \arrow[rr, "F(f)_\times"] \arrow[dr, "p_1"] & & F(Y') \times_{F(X')} F(Z') \arrow[dr, "p_1"] & \\ & F(Y) \arrow[rr, "F(f_Y)"] & & F(Y') \end{tikzcd}$$

By definition of $F(f)_\times$, the lower parallelepiped commutes. Verifying the commutativity after post-composition by the first projection thus restrict to proving the commutativity of

$$\begin{tikzcd} F(Y \times_X Z) \arrow[rr, "F(f_\times)"] \arrow[d, "\wr"'] & & F(Y' \times_{X'} Z') \arrow[d, "\wr"] & \\ F(Y) \times_{F(X)} F(Z) \arrow[dr, "p_1"] & & F(Y') \times_{F(X')} F(Z') \arrow[dr, "p_1"] & \\ & F(Y) \arrow[rr, "F(f_Y)"] & & F(Y') \end{tikzcd}$$

which we also enrich into :

$$\begin{tikzcd} F(Y \times_X Z) \arrow[rr, "F(f_\times)"] \arrow[d, "\wr"'] \arrow[ddr, bend left, "F(p_1)"] & & F(Y' \times_{X'} Z') \arrow[d, "\wr"'] \arrow[ddr, bend left, "F(p_1)"] & \\ F(Y) \times_{F(X)} F(Z) \arrow[dr, "p_1"] & & F(Y') \times_{F(X')} F(Z') \arrow[dr, "p_1"] & \\ & F(Y) \arrow[rr, "F(f_Y)"] & & F(Y') \end{tikzcd}$$

The triangles commute by definition of the natural vertical morphisms. We restricted ourselves to prove the commutativity of the following :

$$\begin{tikzcd} F(Y \times_X Z) \arrow[r, "F(f_\times)"] \arrow[dr, bend left, "F(p_1)"] & F(Y' \times_{X'} Z') \arrow[dr, bend left, "F(p_1)"] & \\ & F(Y) \arrow[r, "F(f_Y)"] & F(Y') \end{tikzcd}$$

which commutativity is true before application of F. We proved that the original diagram commutes after projection to $F(Y')$. By symmetry, it is true after projection to $F(Z')$ and the original diagram commutes.

For an endomorphism of functor, the maps $\mathrm{T}(X)$, $\mathrm{T}(Y)$ and $\mathrm{T}(Z)$ bring together a morphism of diagrams to which we apply the previous result. $\square$

Definition 1.3. Let $\mathcal{C}$ be a category with finite products. A monoid object is a triple (X, μ, η) where

$$\begin{aligned} &X \text{ est un objet de } \mathcal{C}, \\ &\mu \,:\, X \times X \to X \\ &\eta \,:\, * \to X \end{aligned}$$

such that the following two diagrams commute

$$\begin{tikzcd} X \arrow[r, "\mathrm{Id}_X \times \eta"] \arrow[d, "\eta \times \mathrm{Id}_X"'] \arrow[dr, "\mathrm{Id}_X"] & X \arrow[d, "\mu"] \\ X \arrow[r, "\mu"] & X \end{tikzcd} \qquad \begin{tikzcd} X \times X \times X \arrow[r, "\mathrm{Id}_X \times \mu"] \arrow[d, "\mu \times \mathrm{Id}_X"'] & X \times X \arrow[d, "\mu"] \\ X \times X \arrow[r, "\mu"] & X \end{tikzcd}$$

The morphisms of monoid objects are the morphisms $f \,:\, X \to Y$ which make the two following diagram commute

$$\begin{tikzcd} X \times X \arrow[r, "\mu_X"] \arrow[d, "f \times f"'] & X \arrow[d, "f"] \\ Y \times Y \arrow[r, "\mu_Y"] & Y \end{tikzcd} \qquad \begin{tikzcd} * \arrow[r, "\eta_X"] \arrow[dr, "\eta_Y"'] & X \arrow[d, "f"] \\ & Y \end{tikzcd}$$

The categories of group objects, abelian group objects, ring objects, R-module or R-algebra objects for a ring R are defined in an analoguous manner adding the suitable maps and diagrams. Beware that an R-module object contains the data of a family of endomorphisms of X indexed by R which represent the multiplication by the corresponding elements of R.

For a ring object A of $\mathcal{C}$, we can even create an intrisic version of modules : the category A-Mod of A-module has for objects the couples formed by an abelian group object $M$and a morphism $\nu \,:\, A \times M \to M$ which makes the suitable diagrams commute (see [§2, Notes sur les mathématiques condensées] for a more complete introduction, including new diagrams).

Lemma 1.4. *Let $\mathcal{C}$ and $\mathcal{D}$ be two categories with final object and fiber product (in particular the finite products). Let Z be a monoid object of $\mathcal{C}$. Let $F \,:\, \mathcal{C} \to \mathcal{D}$ be a covariant functor preserving the final object and verifying the hypothesis of lemma 1.2. The composition*

$$\mu_F \,:\, F(Z) \times F(Z) \xrightarrow{\sim} F(Z \times Z) \xrightarrow{F(\mu)} F(Z)$$

together with the morphism $F(\eta) \,:\, * \to F(Z)$ equpis $F(Z)$ with a monoid object structure. For every morphism $f \,:\, Z \to Z'$ of monoid objects, the morphism $F(f)$ is a monoid objects morphism.

The same statement is correct if we replace the occurences of monoid by group (resp abelian group) while adding $F(i)$ as inverse map on $F(Z)$, where i is the inverse map on Z.

The same statement is correct if we replace the occurences of abelian group by ring while adding

$$\pi_F \,:, F(Z) \times F(Z) \xrightarrow{\sim} F(Z \times Z) \xrightarrow{F(\pi)} F(Z)$$

as multiplication map on $F(Z)$.

The same statement is correct if we replace the occurences of abelian group by R-module (resp. R-algebra) while adding $F(f_\lambda)$ as multiplications.

Let A be a ring object of $\mathcal{C}$. The same statement is correct if we replace the occurences of abelian group object in $\mathcal{C}$ by A-module, the occurences of abelian group object in $\mathcal{D}$ by $F(A)$-module and by adding

$$\nu_F \,:\, F(A) \times F(M) \xrightarrow{\sim} F(A \times M) \xrightarrow{F(\nu)} F(M)$$

as external law.

Démonstration. We won't do every verification but nonetheless give a complete proof for monoid objects. Let's begin by the left neutrality diagram, in other words by proving the commutativity of the following diagram :

$$
\begin{tikzcd}
F(Z) \arrow[r, "\mathrm{Id}_{F(Z)}\times F(\eta)"] \arrow[dr, "\mathrm{Id}_{F(Z)}"'] & F(Z)\times F(Z) \arrow[d, "\mu_F"] \\
& F(Z)
\end{tikzcd}
$$

Because $\mathrm{Id}_{F(Z)} = F(\mathrm{Id}_Z)$ we can enrich and precise the diagram as follows

$$
\begin{tikzcd}
F(Z) \arrow[r, "F(\mathrm{Id}_Z)\times F(\eta)"] \arrow[dr, "F(\mathrm{Id}_Z\times\eta)"] \arrow[ddr, "F(\mathrm{Id}_Z)"'] & F(Z)\times F(Z) \arrow[d, "\wr"] \\
& F(Z\times Z) \arrow[d, "F(\mu)"] \\
& F(Z)
\end{tikzcd}
$$

Le upper triangle commutes thanks to lemma 1.2 and the lower commutes by appliying F to the left neutrality diagram for the monoid object structure on Z.

Moove on to associativity, which is the commutativity of

$$
\begin{tikzcd}
F(Z)\times F(Z)\times F(Z) \arrow[r, "\mathrm{Id}_{F(Z)}\times\mu_F"] \arrow[d, "\mu_F\times\mathrm{Id}_{F(Z)}"'] & F(Z)\times F(Z) \arrow[d, "\mu_F"] \\
F(Z)\times F(Z) \arrow[r, "\mu_F"] & F(Z)
\end{tikzcd}
$$

We enrich the diagram as

$$
\begin{tikzcd}
F(Z)\times F(Z)\times F(Z) \arrow[r, "\sim"] \arrow[d, "\wr"'] & F(Z\times Z)\times F(Z) \arrow[r, "F(\mu)\times\mathrm{Id}_{F(Z)}"] \arrow[d, "\wr"'] & F(Z)\times F(Z) \arrow[d, "\wr"'] \\
F(Z)\times F(Z\times Z) \arrow[r, "\sim"] \arrow[d, "\mathrm{Id}_{F(Z)}\times F(\mu)"'] & F(Z\times Z\times Z) \arrow[r, "F(\mu\times\mathrm{Id}_Z)"] \arrow[d, "F(\mathrm{Id}_Z\times\mu)"'] & F(Z\times Z) \arrow[d, "F(\mu)"] \\
F(Z)\times F(Z) \arrow[r, "\sim"] & F(Z\times Z) \arrow[r, "F(\mu)"] & F(Z)
\end{tikzcd}
$$

The left upper square commutes and prove that the morphism $F(Z\times Z\times Z)\to F(Z)\times F(Z)\times F(Z)$ induces by the image through F of the three projections is an isomorphism (as in the previous lemma, the commutativity can be rigorously verified after projecting on each factor and with similar sleight of hands). Because $\mathrm{Id}_{F(Z)} = F(\mathrm{Id}_Z)$, the right upper and left lower squares are emphasized as commutative by the lemma 1.2. Proving the associativity for $F(Z)$ thus restricts to proving the commutativity of :

$$\begin{tikzcd}
F(Z)\times F(Z)\times F(Z) \arrow[dr, "\sim"] & & \\
& F(Z\times Z\times Z) \arrow[r, "F(\mu\times \mathrm{Id}_Z)"] \arrow[d, "F(\mathrm{Id}_Z\times\mu)"'] & F(Z\times Z) \arrow[d, "F(\mu)"] \\
& F(Z\times Z) \arrow[r, "F(\mu)"] & F(Z)
\end{tikzcd}$$

This is the associativity diagram for Z, carried through F.

Now, let $f : Z \to Z'$ be a morphism of monoid objects. Proving that $F(f)$ is a monid objects morphism means proving the neutral element preservation (which is just the neutral element preservation for f carried through F) and that the following diagram commutes

$$\begin{tikzcd}
F(Z)\times F(Z) \arrow[r, "\mu_F"] \arrow[d, "F(f)\times F(f)"'] & F(Z) \arrow[d, "F(f)"] \\
F(Z')\times F(Z') \arrow[r, "\mu'_F"] & F(Z')
\end{tikzcd}$$

We enrich the diagram as follows :

$$\begin{tikzcd}
F(Z)\times F(Z) \arrow[r, "\sim"] \arrow[d, "F(f)\times F(f)"'] & F(Z\times Z) \arrow[r, "F(\mu)"] \arrow[d, "F(f\times f)"'] & F(Z) \arrow[d, "F(f)"] \\
F(Z')\times F(Z') \arrow[r, "\sim"] & F(Z'\times Z') \arrow[r, "F(\mu')"] & F(Z')
\end{tikzcd}$$

The left square commutes thanks to the naturality of the commutation of F to products. The right square is the law preserving diagram for F carried through F. This concludes. $\square$

Proposition 1.5. *Let $\mathcal{C}$ and $\mathcal{D}$ be two categories with final object and fiber products. Let $F : \mathcal{C} \to \mathcal{D}$ be a covariant functor verifying the hypothesis of lemma 1.2 and T an endomorphism of the functor F. The monoid object structure on $F(Z)$ given by proposition 1.4 upgrades $\mathrm{T}(Z)$ to a morphism of monoid objects.*

The same statement holds upon replacing monoid by group, abelian group, ring, R-module or R-algebra.

Démonstration. Let $\mu : Z\times Z \to Z$ be the multiplication map on Z. Thanks to the previous lemma, we can draw a big commutative diagram :

$$\begin{tikzcd}
F(Z)\times F(Z) \arrow[r, "\mathrm{T}(Z)\times\mathrm{T}(Z)"] \arrow[d, "\wr"'] & F(Z)\times F(Z) \arrow[d, "\wr"] \\
F(Z\times Z) \arrow[r, "\mathrm{T}(Z\times Z)"] \arrow[d, "F(\mu)"'] & F(Z\times Z) \arrow[d, "F(\mu)"] \\
F(Z) \arrow[r, "\mathrm{T}(Z)"] & F(Z)
\end{tikzcd}$$

where the composition of vertical arrows corresponds to the multiplication given by proposition 1.4. The envelop of this diagram therefore emphasizes that $\mathrm{T}(Z)$ is a monoid object morphism.

We can easily make it for R-module objects. Let $\lambda \in R$, and set $f_\lambda : Z \to Z$ to be the multiplication by λ on Z. The naturality of T gives a commutative diagram

$$\begin{tikzcd}
F(Z) \arrow[r, "\mathrm{T}(Z)"] \arrow[d, "F(f_\lambda)"'] & F(Z) \arrow[d, "F(f_\lambda)"] \\
F(Z) \arrow[r, "\mathrm{T}(Z)"] & F(Z)
\end{tikzcd}$$

where the vertical arrow is the multiplication by λ on $F(Z)$ give by proposition 1.4. We could have the sensation to completely forgot the compatibility of abelian group structure and external law. They don't matter for the property of being a morphism of R-module. $\square$

2 Quotients by an equivalence relation in a topos

Take two sets X_1 and X_2 on which two groups G_1 et G_2 and. The product $X_1 \times X_2$ comes with an action of $G_1 \times G_2$ and the identifies to the product ${}^{X_1}/_{G_1} \times {}^{X_2}/_{G_2}$. A reasonable framework to generalize this result is a topos. Even if the construction of the product action and the existence of a comparison morphism would be possible for group objects in any category, we will write it only inside of a topos for which the comparison morphism will indeed be an isomorphism. We fix a site $\mathcal{C}$.

Definition 2.1. Let X be an object of $\mathrm{Sh}(\mathcal{C})$. An equivalence relation on X is a subsheaf $\mathcal{R} \subset X \times X$ which contains the diagonal, is symetrical [1] and transitive [2]. Such an equivalence relation can also be seen as the data of an equivalence relation on $X(S)$ for every object of $\mathcal{C}$, functorial in S and with gluing condition.

A morphism of sheaf $f : X \to Y$ is said to be $\mathcal{R}$-invariant if the following two compositions coincide

$$\mathcal{R} \hookrightarrow X \times X \xrightarrow[f \times f]{} Y \times Y \rightrightarrows Y.$$

Proposition 2.2. *For every sheaf X on $\mathcal{C}$ and every equivalence relation $\mathcal{R}$ on X, the contravariant functor*

$$\mathrm{Sh}(\mathcal{C}) \to \mathrm{Ens}, \quad Y \mapsto \left\{ f \in \mathrm{Hom}_{\mathrm{Sh}(\mathcal{C})}(X, Y) \mid f \text{ est } \mathcal{R}\text{−équivariante} \right\}$$

is representable. We denote any representant by ${}^{X}/_{\mathcal{R}}$.

Démonstration. We limit ourselves to stating that

$$X/\mathcal{R} = \left(S \mapsto X(S)/\mathcal{R}(S) \right)^{\sharp},$$

with sharp standing for sheafification, is a representant. $\square$

When G is a group object of $\mathrm{Sh}(\mathcal{C})$, an action of G on X is a morphism $\bullet : G \times X \to X$ which makes the following diagrams commute :

$$\begin{tikzcd} X \arrow[r, "e_G \times \mathrm{Id}_X"] \arrow[dr, "\mathrm{Id}_X"'] & G \times X \arrow[d, "\bullet"] \\ & X \end{tikzcd} \qquad \begin{tikzcd} G \times G \times X \arrow[r, "\mu \times \mathrm{Id}_X"] \arrow[d, "\mathrm{Id}_G \times \bullet"'] & G \times X \arrow[d, "\bullet"] \\ G \times X \arrow[r, "\bullet"] & X \end{tikzcd}$$

Such an action induces an equivalence relation on X, defined as the image of the morphism $G \times X \to X \times X$ obtained from $\bullet$ and projection X. We denote by ${}^{X}/_{G}$ the associated quotient without keeping track of the action.

Proposition 2.3. *Let X be an object of $\mathrm{Sh}(\mathcal{C})$. Let X_1 (resp. X_2) be an object of $\mathrm{Sh}(\mathcal{C})$, G_1 (resp. G_2) a group object acting on X_1 (resp. G_2). Let $f_1 : X_1 \to X$ (resp. $f_2 : X_2 \to X$) be a G_1-invariant morphism (resp. G_2-invariant).*

The following diagram commutes

$$\begin{tikzcd} (G_1 \times G_2) \times (X_1 \times_X X_2) \arrow[r, "p_1 \times p_1"] \arrow[d, "p_2 \times p_2"'] & G_1 \times X_1 \arrow[r, "\bullet_1"] & X_1 \arrow[dd, "f_1"] \\ G_2 \times X_2 \arrow[d, "\bullet_2"'] & & \\ X_2 \arrow[rr, "f_2"] & & X \end{tikzcd}$$

where p_i always stands for the i-th projection and the $\bullet_j$ for the action on X_j. The universal property gives a morphism $(G_1 \times G_2) \times (X_1 \times_X X_2) \to (X_1 \times_X X_2)$ which is a group action. Moreover, there exists a natural morphism

$$(X_1 \times_X X_2)/(G_1 \times G_2) \to \left(X_1/G_1 \right) \times_X \left(X_2/G_2 \right)$$

which is an isomorphism.

1. Symetricla means tha the image of $\mathcal{R}$ by inversion of coordinates on $X \times X$ is still $\mathcal{R}$ itself.
2. Transitive means that the image of $\mathcal{R} \times_X \mathcal{R}$ in $(X \times X) \times_{p_2, X, p_1} (X \times X) \cong X \times X$ identifies to a subsheaf of $\mathcal{R}$.

Démonstration. To prove that the first diagram commute, we enrich it as follows :

$$\begin{tikzcd}
(G_1 \times G_2) \times (X_1 \times_X X_2) \arrow[rr, "p_1 \times p_1"] \arrow[dr, "p_2"] \arrow[dd, "p_2 \times p_2"'] & & G_1 \times X_1 \arrow[r, "\bullet_1"] \arrow[d, "p_2"'] & X_1 \arrow[ddd, "f_1"] \\
& X_1 \times_X X_2 \arrow[r, "p_1"] \arrow[d, "p_2"] & X_1 \arrow[ddr, "f_1"] & \\
G_2 \times X_2 \arrow[r, "p_2"] \arrow[d, "\bullet_2"'] & X_2 \arrow[drr, "f_2"] & & \\
X_2 \arrow[rrr, "f_2"] & & & X
\end{tikzcd}$$

where the parallelepipeds with X_i as cornes commute thanks to the G_i-invariance of f_i, and the others commute by identities on fiber products and projections..

We limit ourselves to proving that the first diagram of group action commutes. Thanks to universal property of fiber product, we can verify the commutativity after projection to X_1 (for instance) :

$$\begin{tikzcd}
X_1 \times_X X_2 \arrow[r, "e_{(G_1 \times G_2)} \times \mathrm{Id}_{(X_1 \times_X X_2)}"] \arrow[dr, "\mathrm{Id}_{(X_1 \times_X X_2)}"'] & (G_1 \times G_2) \times (X_1 \times_X X_2) \arrow[d, "\bullet_{(1 \times_X 2)}"] & \\
& X_1 \times_X X_2 \arrow[dr, "p_1"] & \\
& & X_1
\end{tikzcd}$$

After some enrichment :

$$\begin{tikzcd}
X_1 \times_X X_2 \arrow[r, "e_{(G_1 \times G_2)} \times \mathrm{Id}_{(X_1 \times_X X_2)}"] \arrow[ddr, "\mathrm{Id}_{(X_1 \times_X X_2)}"'] & (G_1 \times G_2) \times (X_1 \times_X X_2) \arrow[dr, "p_1 \times p_1"] \arrow[dd, "\bullet_{(1 \times_X 2)}"] & \\
& & G_1 \times X_1 \arrow[dd, "\bullet_1"] \\
& X_1 \times_X X_2 \arrow[dr, "p_1"] & \\
& & X_1
\end{tikzcd}$$

The right parellelepiped commutes by definition of the action. Thus, we only need the envelop of the diagram to commute. We only keep the envelop, then enrich again to obtain :

$$\begin{tikzcd}
X_1 \times_X X_2 \arrow[r, "e_{(G_1 \times G_2)} \times \mathrm{Id}_{(X_1 \times_X X_2)}"] \arrow[dr, "p_1"] \arrow[ddrr, "p_1"'] & (G_1 \times G_2) \times (X_1 \times_X X_2) \arrow[dr, "p_1 \times p_1"] & \\
& X_1 \arrow[r, "e_{G_1} \times \mathrm{Id}_{X_1}"] \arrow[dr, "\mathrm{Id}_{X_1}"] & G_1 \times X_1 \arrow[d, "\bullet_1"] \\
& & X_1
\end{tikzcd}$$

The upper part commutes thanks to identities on projections and the lower part thanks to the group action diagram for $\bullet_1$.

Pour construire le morphisme, démontrons que la composée des deux projections $\pi : X_1 \times_X X_2 \to X_1 \twoheadrightarrow {}^{X_1}/_{G_1}$ passe au quotient par l'action de $G_1 \times G_2$. Ceci équivaut à la commutation du diagramme suivant :

$$\begin{tikzcd}
(G_1 \times G_2) \times (X_1 \times_X X_2) \arrow[r, "\bullet"] \arrow[d, "p"'] & X_1 \times_X X_2 \arrow[d, "\pi"] \\
X_1 \times_X X_2 \arrow[r, "\pi"] & X_1/G_1
\end{tikzcd}$$

Nous l'étoffons comme suit :

$$\begin{tikzcd}
(G_1 \times G_2) \times (X_1 \times_X X_2) \arrow[rr, "\bullet"] \arrow[dd, "p"'] \arrow[dr, "p_1 \times p_1"] & & X_1 \times_X X_2 \arrow[d, "p_1"] \\
& G_1 \times X_1 \arrow[r, "\bullet_1"] \arrow[d, "p"] & X_1 \arrow[d, two heads] \\
X_1 \times_X X_2 \arrow[r, "p_1"] & X_1 \arrow[r, two heads] & X_1/G_1
\end{tikzcd}$$

où p sont les projections évidentes sur le deuxième facteur. Le quadrilatère du haut commute par définition de l'action $\bullet$; le quadrilatère de gauche commute de manière automatique et le carré du bas commute parce que $X_1 \twoheadrightarrow X_1/G_1$ est G_1-invariant. Nous laissons au lecteur ou à la lectrice le soin de vérifier que les deux morphismes $(X_1 \times X_2)/(G_1 \times G_2) \to X_i/G_i$ obtenus se rassemblent en un morphisme vers le produit fibré sur X. Le morphisme obtenu est un isomorphisme sur les sections des préfaisceaux, mais les préfaisceaux en jeu sont des faisceaux par commutation de la faisceautisation aux limites finies. $\square$