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o-Minimal structures in Algebaic Geometry

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Contents

1 Amador Martin-Pizzaro: Introduction to o-minimality

1.1 Introduction: Definable Chow

Theorem 1.1 (Chow)**.** *Let $X \subset \mathbb{P}^n(\mathbb{C})$ be a closed analytic subset. Then X is algebraic.*

Theorem 1.2 (Peterzil, Starchenko)**.** *Let X be a closed analytic subset of $\mathbb{C}^n$ (or, more generally, a closed analytic subset of a quasi-projective variety Y over $\mathbb{C}$), such that X is definable in an o-minimal structure expanding the ordered field $\mathbb{R}$. Then X is algebraic.*

Remark 1.3. Since every closed analytic subset of $\mathbb{P}^n(\mathbb{C})$ is definable in the o-minimal structure $\mathbb{R}_{\text{an}}$, we have Definable Chow $\implies$ Chow's Theorem.

1.2 Structures expanding an ordered field

Definition 1.4. A structure (expanding the ordered field $\mathbb{R}$) is given by a collection $\mathcal{F}$ of distinguished functions on $\mathbb{R}^k$ (where k varies with respect to the function). Write

$\mathbb{R}_{\mathcal{F}} = (\mathbb{R}, +, \cdot, <, \{f\}_{f\in\mathcal{F}})$, where each function $f \in \mathcal{F}$ is a function $f : \mathbb{R}^k \to \mathbb{R}$ for some k.

Examples 1.5.

- $\mathcal{F} = \varnothing$, write $\mathbb{R}$ instead of $\mathbb{R}_{\varnothing}$.
- $\mathcal{F} = \{\exp : \mathbb{R} \to \mathbb{R}_{>0}\}$. This gives $\mathbb{R}_{\exp}$.
- $\mathcal{F} = \{f : [0,1]^k \to \mathbb{R}\}$, where the f are analytic on some open neighborhood of $[0,1]^k$ (and 0 elsewhere). This gives $\mathbb{R}_{\text{an}}$. These are called the restricted analytic functions.
- $\mathcal{F} = \{\text{restricted analytic functions}\} \bigcup \{\exp\}$, this gives $\mathbb{R}_{\text{an},\exp}$.

Definition 1.6. Given a structure $\mathbb{R}_{\mathcal{F}} = (\mathbb{R}, +, \cdot, <, \{f\}_{f\in\mathcal{F}})$, the collection of definable sets in $\mathbb{R}_{\mathcal{F}}$ is the smallest sequence $\mathcal{D} = (D_n)_{n\in\mathbb{N}}$ satisfying the following:

1. Each D_n is a boolean subalgebra of $\mathscr{P}(\mathbb{R}^n)$.
2. $A \in D_n \implies A \times R, R \times A \in D_{n+1}$.
3. If $A \in D_{n+1}$ and $\pi : \mathbb{R}^{n+1} \to \mathbb{R}^n$ is the projection onto the first n coordinates, then $\pi(A) \in D_n$.
4. $\forall P \in \mathbb{R}[X_1, \ldots, X_n]$, $Z(P) = \{x \in \mathbb{R}^n : P(x) = 0\} \in D_n$.
5. For every $f \in \mathcal{F}$ of arbitrary k, the graph $\Gamma_f = \text{Graph}(f) \in D_{k+1}$.

Definition 1.7. $X \subset \mathbb{R}^n$ is definable with respect to $\mathbb{R}_{\mathcal{F}}$ if $X \in D_n$. A function $X \to \mathbb{R}$ is definable if its graph is definable.

Examples 1.8.

- Suppose $A \in D_n$ and $B \in D_m$. Then $A \times B \in D_{n+m}$. Proof: exercise. (Take the intersection $A \times \mathbb{R}^m \cap \mathbb{R}^n \times B = A \times B \subset \mathbb{R}^n \times \mathbb{R}^m$.)

- If $A \in D_n$ and $\sigma \in S_n$ then $\{x \in \mathbb{R}^n : x_\sigma \in A\} \in D_n$. (This says that permuting the coordinates sends a definable set to a definable set).
- semialgebraic sets of $\mathbb{R}^n$ belong to D_n.
- Suppose $f \in \mathcal{F}$ is a unary function. Then $\{x \in \mathbb{R} : f(x^3 + 1 - 3x) < x^3\} \in D_1$.

Definition 1.9. The collection of terms in the structure $\mathbb{R}_\mathcal{F}$ is the smallest collection of functions containing $\mathcal{F}$ and all the polynomials (in finitely many variables) over $\mathbb{R}$ that is closed under composition and the field operations.

Examples 1.10.
- polynomials are terms (even if $\mathcal{F} = \varnothing$)
- $x^2 + \exp^{\exp^{\cdot\cdot}} +$..etc is a term in $\mathbb{R}_{\exp}$
- $\sin(\exp(x) + x\cos(x) + ..)$ is a term in $\mathbb{R}_{\mathrm{an},\exp}$.

Definition 1.11. A basic relation is obtained from two terms t_1 and t_2 by considering the sets

$$\{x \in \mathbb{R}^n : t_1(x) = t_2(x)\} \quad \text{or} \quad \{x \in \mathbb{R}^n : t_1(x) < t_2(x)\}.$$

Definition 1.12 ('Pseudo definition')**.** A definable set in $\mathbb{R}^n$ is the collection of tuples which verify a property expressible by a first-order formula, that is, a mathematical property obtained from finitely many basic relations using logical connectives $\{\neg, \wedge, \vee, \implies\}$ and qualifiers $\{\forall, \exists\}$, where the latter run over elements in the structure (that is, elements of $\mathbb{R}$).

Examples 1.13.
- If $f \in \mathcal{F}$ is a unary function $\mathbb{R} \to \mathbb{R}$, then $\mathrm{Im}(f)$ is definable. Indeed, $\mathrm{Im}(f) = \{x \in \mathbb{R} : \exists y \in \mathbb{R} : f(y) = x\}$.
- If $f \in \mathcal{F}$ unary, the points where f is discontinuous is definable: this set S can be described as

  $$S = \{x \in \mathbb{R} : \exists \epsilon : \forall \delta \exists y \in \mathbb{R} : \epsilon > 0 \text{ and } \delta > 0 \text{ and } (x-y)^2 < \delta^2 \text{ but } (f(x)-f(y))^2 > \epsilon^2\}.$$

- If $X \subset \mathbb{R}^n$ is definable, then so is its closure $\bar{X} \subset \mathbb{R}^n$. Indeed,

  $$\bar{X} = \{z \in \mathbb{R}^n : \forall \epsilon \in \mathbb{R}, (\epsilon > 0 \implies \exists x \in \mathbb{R} : x \in X \text{ and } |z - x|^2 < \epsilon^2)\}.$$

- However, the multiplicative subgroup generated by 2 (which is $\{x \in \mathbb{R} : \exists n \in \mathbb{N} : (x = 2^n \text{ or } x \cdot 2^n = 1)\}$) is not definable.

Definition 1.14. A structure $\mathbb{R}_\mathcal{F}$ is o-minimal if every definable subset X of $\mathbb{R}$ is a finite union of points and intervals.

Example 1.15 ('non-example')**.** If $\sin : \mathbb{R} \to \mathbb{R}$, $\sin \in \mathcal{F}$, then $\mathbb{R}_\mathcal{F}$ is not o-minimal, because $X = \{x \in \mathbb{R} : \sin(x) = 0\}$ would be definable, but it is infinite and discrete.

Theorem 1.16 (Tarski-Seidenberg)**.** *If $\mathcal{F} = \varnothing$, the structure of $\mathbb{R}_{\mathcal{F}}$ is o-minimal. Indeed, definable sets in n variables are exactly the semialgebraic sets. That is, a definable set is given by a formula with no quantifiers. That is, the projection of a semialgebraic set is again semialgebraic.*

Example 1.17.

$\{(a,b,c) \in \mathbb{R}^3 : a \neq 0 \text{ and } \exists x \in \mathbb{R} : ax^2+bx+c = 0\} = \{(a,b,c) \in \mathbb{R}^3 : a \neq 0 \text{ and } b^2-4ac > 0\}.$

Lemma 1.18. *If $X \subset \mathbb{R}^n$ is definable and discrete, then X is finite.*

Proof. $n = 1$: X cannot contain an interval, so X is a finite union of points.
$n = 2$: Let $a \in \mathbb{R}$, and $X_a = \{b \in \mathbb{R} : (a,b) \in X\} \subset \mathbb{R}$ (called the fiber over a). Then X_a is definable and discreet, therefore finite by induction. So X_a is finite for all a. Define $\pi : \mathbb{R}^2 \to \mathbb{R}$ by $\pi(a,b) = a$. Then $\pi(X) = \{x \in \mathbb{R} : \exists y : (x,y) \in X\} \subset \mathbb{R}$ is definable. If $\pi(X)$ were infinite, it would contain an open interval $I \subset \pi(X)$. Then, for all $a \in I$, $X_a \neq \varnothing$, because $a \in \pi(X)$ so $\exists b \in \mathbb{R} : (a,b) \in X$ - but X_a is finite. Choose the smallest element $b \in X_a$, and call it b_a. Consider now

$$g : I \to \mathbb{R}, \quad a \mapsto b_a.$$

Then $(x,y) \in \Gamma_g \iff x \in I$ and $(x,y) \in X$ and $\forall z, ((x,z) \in X \implies y \leqslant z)$. It follows that g is continuous! (Possibly after shrinking the interval). Choose $a \in I$. Then there exists a sequence $(a_n) \in I : a_n \neq a, a_n \longrightarrow a$. But then $(a_n, g(a_n)) \longrightarrow (a, g(a))$. As $(a, g(a)) \in X$ and $(a_n, g(a_n)) \in X$ for each $n \in \mathbb{N}$, this shows that X is not discrete. ☐

Definition 1.19. A cell $C \subset \mathbb{R}^n$ in n-dimensional space is defined inductively on n:
$n = 1$: a cell is a point or an open interval (possibly unbounded).
$n = m + 1$: there is a cell $C' \subset \mathbb{R}^m$, and for definable functions f, g with domain C', C is the graph of f or $C = \{(x,y) \in \mathbb{R}^m \times \mathbb{R} : x \in C' \text{ and } f(x) < y < g(x)\}$ (such a set is called a band).

What of this is relevant? Cells are definable, and for some choice of coordinates, $\pi(C) \subseteq \mathbb{R}^k$ is open and homeomorphic to C. The dimension of a singleton is 0 and the dimension of an interval is 1. If C is a cell of $\mathbb{R}^n$, then the dimension of C is $\dim p_{n,n-1}(C)$ if C is a graph and $\dim p_{n,n-1}(C) + 1$ if C is a band. The dimension of a nonempty definable set X is the supremum of all $d \in \mathbb{N}$ such that there exists an injective definable map $\mathbb{R}^d \to X$. The dimension of the empty set is $-\infty$. (The following is true by [M. Coste]: take a $CDCD$ of $\mathbb{R}^n$ adapted to X. Then $\dim(X)$ is the maximum of the dimension of the cells contained in X.)

Theorem 1.20. *Every definable set in $\mathbb{R}^n$ can be decomposed into finitely many cells. For every $r \in \mathbb{N}$, the domain of every definable function f can be decomposed into sets such that $f|_{\text{Cell}}$ is a $\mathcal{C}^r$ function.*

Meaning to $\dim(X)$: we have $\dim(X) = 0 \iff X$ is finite and $\dim(\overline{X} \backslash X) < \dim(X)$.

Remark 1.21. A structure is o-minimal if and only if every definable subset of $\mathbb{R}^n$ has only finitely many definable components.

Theorem 1.22 (Wilko)**.** $\mathbb{R}_{\text{exp}}$ *is o-minimal.*

Theorem 1.23 (Van den Dries, Macintyre, Marker)**.** $\mathbb{R}_{\text{an,exp}}$ *is o-minimal.*

2 Bruno Klingler: o-minimality and transcendence, I

2.1 Remark after Amador's talk

Superiority of model theory: the second definition as given by Amador is often more convenient. For example, try to show that for $A \subset \mathbb{R}^n$ definable, its closure $\bar{A} \subset \mathbb{R}^n$ is definable, using both definitions.

2.2 More general o-minimal structures

Definition 2.1. A structure expanding $(\mathbb{R}, +, \cdot)$ is a collection $\mathscr{S} = (S_n)_{n\in\mathbb{N}}$ where $S_n \subset \mathscr{P}(\mathbb{R}^n)$ such that

1. Algebraic subsets are in S_n.
2. $S_n \subset \mathscr{P}(\mathbb{R}^n)$ is a Boolean subalgebra.
3. $A \in S_p, B \in S_q \implies A \times B \in S_{p+q}$.
4. If $p : \mathbb{R}^{n+1} \to \mathbb{R}^n$ is the projection on the first n coordinates, then $p(A) \in S_n$ for $A \in S_{n+1}$.

Definition 2.2. A structure expanding $(\mathbb{R}, +, \cdot, <)$ is o-minimal if

5. any set in S_1 is a finite union of points and intervals.

We can generalise this to a totally ordered ring $(R, +, \cdot, <)$. R is naturally a topological ring, for which polynomials are continuous. R is characteristic 0 (because $0 < 1 < 1 + 1 < \ldots < p < \ldots$). It is a fact that a field R can be totally ordered if and only if R is formally real, meaning that 0 is not a sum of squares. For example, $\mathbb{Q}_p$ cannot be ordered (by Hensel's lemma).

Lemma 2.3. *Suppose S is an o-minimal structure extending $(\boldsymbol{R}, +, \cdot, <)$. Then $\boldsymbol{R}$ is a real closed field (that is, every positive element in $\boldsymbol{R}$ is a square of an element in $\boldsymbol{R}$ and any polynomial of odd degree over $\boldsymbol{R}$ has a root in $\boldsymbol{R}$).*

2.3 Basic properties

Proposition 2.4 (Definable choice). *Let $Y \subset \boldsymbol{R}^{m+n}$ be definable, and let Y be seen as a definable family of definable subsets of $\boldsymbol{R}^m$ parameterised by $\boldsymbol{R}^n$. (Indeed, for each $b \in \boldsymbol{R}^n$, the fiber over b is $Y_b = \{a \in \boldsymbol{R}^m : (a, b) \in Y\}$ and this is a definable subset of $\boldsymbol{R}^m$). Then there is a definable function $f : \boldsymbol{R}^n \to \boldsymbol{R}^m$ such that for all $b \in \boldsymbol{R}^n$, if $Y_b \neq \varnothing$, then $f(b) \in Y_b$.*

Proof. See [Coste, 3.1]. □

Definition 2.5. A cylindrical definable cell decomposition (CDCD) of $\mathbf{R}^n$ is a finite partition of $\mathbf{R}^n$ into definable sets $(C_i)_{i\in I}$, called cells, having the following definition:
$n = 1$: the CDCD is $a_1 < \ldots, a_p$ of $\mathbf{R}$, where the cells are the $\{a_i\}$ and the (a_i, a_{i+1}).
$n > 1$: a CDCD of $\mathbf{R}^n$ is a CDCD of $\mathbf{R}^{n-1}$ together with, for each cell D of $\mathbf{R}^{n-1}$, continuous definable functions

$$f_{D,0} := -\infty < f_{D,1} < \ldots < f_{D,m_D} < +\infty.$$

The cells of $\mathbf{R}^n$ are the graphs $\{(x, f_{D_i}(x)) : x \in D\}$ and the bands $(f_{D,k}, f_{D,k+1}) = \{(x, y) : x \in D \text{ and } f_{D,k}(x) < y < f_{D,k+1}(x)\}$.

Theorem 2.6 (Cell decomposition CDCD$_n$). *Let $A_1, \ldots, A_k \subset \boldsymbol{R}^n$ be definable. Then there is a CDCD of $\boldsymbol{R}^n$ such that each A_i is a union of cells.*

Theorem 2.7 (Piecewise continuity PC$_n$). *Let $A \subset \boldsymbol{R}^n$ be definable, $f : A \to \boldsymbol{R}$ a definable function, and $p \in \mathbb{Z}_{\geqslant 0}$. Then there exists a $\mathcal{C}^p$ CDCD of $\boldsymbol{R}^n$ adapted to A: for each cell C of A, the function $f|_C$ is $\mathcal{C}^p$.*

Theorem 2.8 (Uniform finiteness UF$_n$). *Let $A \subset \boldsymbol{R}^n$ be definable. Suppose for each $x \in \boldsymbol{R}^{n-1}$, the fiber $A_x = \{y \in \boldsymbol{R} : (x, y) \in A\}$ is finite. Then there is a $k \in \mathbb{N}$ such that $\#A_x < k$ for each $x \in \boldsymbol{R}^{n-1}$.*

Theorem 2.9 (Stratification). *Let $A \subset \boldsymbol{R}^n$ be definable. Fix p. Then A admits a finite stratification, where a stratum consists of a $\mathcal{C}^p$-manifold M together with a definable graph of a function $f : U \to \boldsymbol{R}^n$, where $U \subset \boldsymbol{R}^k$ is an open definable subset, and f is a $\mathcal{C}^p$ function with bounded derivatives.*

Corollary 2.10. *Let $A \subset \boldsymbol{R}^n$ be definable, of dimension k, bounded. Then $\mathrm{vol}_k(A)$ is finite.*

Theorem 2.11 (Trivialisation). *Let $f : X \to Y$ be a continuous and definable function. Then there exists a partition $Y = \bigsqcup Y_i$, with Y_i definable, and*

$$\begin{array}{ccc} f^{-1}(Y_i) & \xrightarrow{f} & Y_i \\ \Big\downarrow{\scriptstyle\sim} & & \Big\downarrow{\scriptstyle=} \\ Y_i \times Z & \xrightarrow{p} & Y_i \end{array}$$

Theorem 2.12 (Monotonicity Theorem)**.** *Let $f : (a, b) \to \boldsymbol{R}$ be definable. Then there exists a partition $a = a_0 < a_1 < \ldots < a_d = b$ such that $f|_{(a_i, a_{i+1})}$ is continuous and either constant or strictly monotonous.*

Proof. The proof of CDCD, UF, PC follow simultaneously by induction on n (see [Coste]). □

3 Bruno Klingler: o-minimality and transcendence, II

3.1 The Pila-Wilkie counting theorem

3.1.1 Statement

Define a height function

$$H : \mathbb{Q} \to \mathbb{N}, 0 \mapsto 0, (a/b) \mapsto \max |a|, |b| \quad \text{ for } a, b \in \mathbb{Z}\backslash\{0\} : (a, b) = 1.$$

Remark that $H(\pm x^{\pm 1}) = H(x)$. This extends naively to $\mathbb{Q}^n$: take $H : \mathbb{Q}^n \to \mathbb{N}$, defined by $H(x) = \max_i H(x_i)$.

For $X \subset \mathbb{R}^n$, and $t \in \mathbb{R}_{\geqslant 1}$, we define

$$X(\mathbb{Q}, t) = \{x \in \mathbb{Q}^n, x \in X, H(x) \leqslant t\}, \quad N(X, t) = \#X(\mathbb{Q}, t).$$

Remark that $N(\mathbb{R}^n, t) \sim C \cdot t^{2n}$, and if $X = \Gamma_{f_{\mathbb{R}}}$ for $f \in \mathbb{Z}[t_1, \ldots, t_{n-1}]$ of degree d, then $N(X, t) \sim t^{2(n-1)/d}$. This gives the slogan "semialgebraic subsets of $\mathbb{R}^n$ have polynomically many rational points."
Remark that for $X = \Gamma_{\sin(\pi x)} \subset \mathbb{R}^2$, we have $N(X, t) \geqslant t^2$.

Theorem 3.1 (Weak Pila-Wilkie)**.** *Let $X \subset \mathbb{R}^n$ be definable. Suppose that X contains no positive dimensional semialgebraic sets. Then $\forall \epsilon > 0 \exists c = c(X, \epsilon) : N(X, t) \leqslant c \cdot t^{\epsilon}$.*

Example 3.2 (Bombieri-Pila)**.** Consider a curve C in $\mathbb{R}_{\text{an}}$. If C has "many" rational points, then C is algebraic.

Can we say something interesting if X contains some positive dimensional semialgebraic set?

Definition 3.3. Let $X \subset \mathbb{R}^n$. Then we define X^{alg} to be the union of all positive dimensional connected semialgebraic subsets contained in X, and $X^{\text{tr}} = X\backslash X^{\text{alg}}$.

Remark that X^{alg} is usually not semialgebraic.

Theorem 3.4 (Pila-Wilkie)**.** *Let $X \subset \mathbb{R}^n$ be definable. Then*

$$\forall \epsilon > 0 \quad \exists c = c(X, \epsilon) > 0 : \quad N(X^{\text{tr}}, t) \leqslant c \cdot t^{\epsilon}.$$

Remark that as X^{tr} is usually not definable, Theorem 3.4 is stronger than Theorem 3.1.

Theorem 3.5 (Pila-Wilkie in family)**.** *Let $X = \{X_b\}_{b\in B} \subset \mathbb{R}^n \times \mathbb{R}^m$ be a definable family of definable subsets of $\mathbb{R}^n$, parameterised by B. Then $\forall\epsilon > 0$ there exists a constant $c = c(X, \epsilon)$ and a definable family $Y = \{Y_b\}_{b\in B}$ such that for all $b \in B$, $Y_b \subset X_b^{\mathrm{alg}}$ and $N(X_b\backslash Y_b, t) \leqslant c \cdot t^{\epsilon}$.*

One can replace rational points in the above construction and definitions by algebraic points of small degree (and $\mathbb{Q}$ by $\bar{\mathbb{Q}}$).

Definition 3.6. A (semialgebraic) block $B \subset \mathbb{R}^n$ is a positive dimensional connected definable set for which tehre exists a connected semialgebraic set Y$\subset \mathbb{R}^n$ such that $B \subset Y^{\mathrm{smooth}}$, and B agrees with Y in a open neighbourhood of every point $b \in B$.

Theorem 3.7 (Pila-Wilkie for blocks)**.** *Let $X \subset \mathbb{R}^n$ be definable. Then for all $\epsilon > 0$ there exists a $c = c(X, \epsilon)$ such that $X(\mathbb{Q}, t)$ is contained in at most $c \cdot t^{\epsilon}$ semialgebraic blocks of X.*

3.2 Proof of Theorem 3.5

Theorem 3.8. *Let $X \subset (0, 1)^n \subset \boldsymbol{R}^n$ be definable in some o-minimal structure expanding a real closed field $\boldsymbol{R}$. Fix $k \in \mathbb{N}_{>0}$. Then X admits a strong k-parametrization: a finite set I of maps $\phi : (0, 1)^{\dim X} \to X$ definable and $\mathcal{C}^k$, with $|\phi^{(\alpha)}(x)| \leqslant 1$ for all $x \in (0, 1)^{\dim(X)}$ for all $\alpha : |\alpha| \leqslant 1$, such that $\bigcup_{i\in I} \mathrm{Im}\phi_i = X$.*

Note that for $\mathbf{R} = \mathbb{R}$, this is an easy consequence of CDCD: write X as a finite union of $\mathcal{C}^k$ cells; hence we can suppose X is a cell, and therefore a bounded cell. A bounded cell has image of a ball under a $\mathcal{C}^k$ map with bounded derivatives. Decomposing the domain into finitely many pieces, and making a change of variables, we may arrange that the bound is one.

Note that for $\mathbf{R} \neq \mathbb{R}$, the hypothesis that $X \subset [-N, N]^n$ for some positive integer N is stronger than the hypothesis that X is bounded by an element in $\mathbf{R}$. Indeed, consider $\mathbf{R} = \mathbb{R}(x)$, let $P = \{S(x)/T(x) :$ the leading coefficients have the same sign $\}$. If $f \in \mathbf{R}$, either $f \in P$, $f = 0$, or $-f \in P$. If $f, g \in P$ then $f + g \in P$ and $f \cdot g \in P$. We can therefore define $f > 0$ if and only if $f \in P$. Then $x > N$ for each $N \in \mathbb{N}_{>0}$, because $x - N = \frac{x-N}{1}$ where x and 1 have the same sign.

Corollary 3.9. *Theorem 3.8 implies that if $\{X_b\}_{b\in B}$ is a definable family of definable sets of $(0, 1)^k, k \in \mathbb{N}_{>0}$, then there are finitely many definable families of definable functions $\{\{\phi_{i,b} : (0, 1)^{\dim X_b} \to (0, 1)^n\}_{b\in B}\}_{i\in I}$ so that for each $b \in B$, $\{\phi_{i,b}\}_{i\in I}$ is a strong k-parametrization of X_b.*

Theorem 3.10 (Bombieri-Pila)**.** *Given $m < n$, $d \in \mathbb{N}_{>0}$, there is another $k \in \mathbb{N}_{>0}$, $\epsilon = \epsilon(m, n, d), c = c(m, n, d)$ such that $\phi : (0, 1)^m \to \mathbb{R}^n$ is a $\mathcal{C}^k$-function with bounded derivatives ($|\phi^{(\alpha)}(x)| \leqslant 1 \forall\alpha : |\alpha| \leqslant k$) with image X. Furthermore, $X(\mathbb{Q}, t)$ is contained in the union of at most (not nececessarily irreducible) hypersurfaces of degree d. Finally, $\epsilon(m, n, d) \to 0$ as $d \to \infty$.*

Corollary 3.11. *If $\{X_b\}_{b\in B}$ is a definable family of definable subsets of $(0,1)^n \subset \mathbb{R}^n$, each of dimension strictly smaller than n, and $\epsilon > 0$, there is a number $d \in \mathbb{N}_{>0}$ and a constant c, depending only on ϵ and X, such that $\forall b \in B$, $X_b(\mathbb{Q}, t)$ is contained in the union of $c \cdot t^\epsilon$ hypersurfaces of degree d.*

Proof. (of Theorem 3.5): omitted. □

4 Bruno Klingler: o-minimality and transcendence, III

4.1 Transcendence: Intro

Definition 4.1. Let $K \subset L$ be a field extension. Then $\alpha \in L$ is algebraic over K if there exists a $P \in K[x]$ such that $P(x) = 0$. Otherwise, α is transcendental. More generally, $\alpha_1, \ldots, \alpha_n \in L$ are algebraically dependent over K if there exists a nonzero $P \in K[x_1, \ldots, x_n]$ such that $P(\alpha_1, \ldots, \alpha_n) = 0$. Otherwise, these elements are algebraically independent over K. A subset $\Gamma \subset L$ is algebraically independent over K if every finite subset of Γ is. A transcendence basis for L over K is a subset L over K which is algebraically independent over K and maximal for this property.

Lemma 4.2. *The set* $\{x \in L : x \text{ is algebraic over } K\}$ *is a subfield of* L.

Lemma 4.3. *Any two transcendental bases for L over K have the same cardinality (which we denote by $\mathrm{tr}_K(L)$).*

Example 4.4. Consider

$$\mathbb{Q} \subset \bar{\mathbb{Q}} \subset \mathbb{C}\,.$$

Because $\bar{\mathbb{Q}}$ is countable, and $\mathbb{C}$ is not, most elements of $\mathbb{C}$ are transcendental.

Theorem 4.5 (Liouville (1844), Hermite (1873), Lindemann (1882))**.** *The following numbers are transcendental:*

$$\sum_{n\in\mathbb{N}} 10^{-n!}, \quad e, \quad \pi.$$

Classical transcendence of the $e^\bullet$ function:

$$e^\bullet : \mathbb{C} \to \mathbb{C}^*$$

is expected to be highly transcendental. Indeed, if $\alpha_1, \ldots, \alpha_n \in \mathbb{C}$ are algebraically dependent, we expect that $e^{\alpha_1}, \ldots, e^{\alpha_n}$ are algebraically independent, and conversely. But $e^\bullet$ is a group homomorphism, so if $\sum \lambda_i \alpha_i = 0$ for $\lambda_i \in \mathbb{Q}$, then $\prod_i (e^{\alpha_i})^{\lambda_i} = 1$, from which we can obtain an algebraic relation between the e^{α_i}.

Conjecture 4.6 (Schanuel)**.** *If $\alpha_1, \ldots, \alpha_n \in \mathbb{C}$ are $\mathbb{Q}$-linearly independent, then*

$$\mathrm{tr}_{\mathbb{Q}}\, \mathbb{Q}(\alpha_1, \ldots, \alpha_n, e^{\alpha_1}, \ldots, e^{\alpha_n}) \geqslant 1.$$

Remark 4.7.
1. Conjecture 4.6 says nothing about when $\alpha_1, \ldots, \alpha_n$ are algebraically independent.

2. For $n = 1$: this is the Hermite-Lindemann Theorem. For all nonzero $\alpha \in \mathbb{C}$, $\mathrm{tr}_{\mathbb{Q}}\, \mathbb{Q}(\alpha, e^\alpha) \geqslant 1$. This has the following consequences: for $\alpha = 1$ we reobtain that e is transcendental. For $\alpha = \pi i$, we reobtain that π is transcendental.

3. For $n \geqslant 2$ this is open. Even for $\alpha_1 = 1, \alpha_2 = \pi i$,

$$\mathrm{tr}_{\mathbb{Q}}\, \mathbb{Q}(1, \pi i, e, -1) = \mathrm{tr}_{\mathbb{Q}}\, \mathbb{Q}(\pi, e) = 2 \quad ?$$

   This is not known.

Theorem 4.8 (Gelfond-Schneider). *For $a, b \in \bar{\mathbb{Q}}^*$, then $\log b / \log a$ is either rational or transcendental (to be precise: if $a \in \bar{\mathbb{Q}}$, $a \neq 0, 1$ and $r \in \bar{\mathbb{Q}} \backslash \mathbb{Q}$, then a^r is transcendental).*

For instance, $2^{\sqrt{2}}$ is transcendental, $e^\pi = (-1)^{-i}$ is transcendental, and $\log 3/\log 2$ is transcendental.

Theorem 4.9 (Lindemann-Weierstrass). *Suppose that $\alpha_1, \ldots, \alpha_n \in \bar{Q}$ are $\mathbb{Q}$-linearly independent. Then*

$$\mathrm{tr}_{\mathbb{Q}}\, \mathbb{Q}(e^{\alpha_1}, \ldots, e^{\alpha_n}) = n.$$

4.2 Functional transcendence

Replace $\mathbb{Q} \subset \mathbb{C}$ by $\mathbb{C} \subset \mathbb{C}((t_1, \ldots, t_k))$. Does the analogue of Conjecture 4.6 hold? Is it true that for all $f_1, \ldots, f_n \in \mathbb{C}[[t_1, \ldots, t_k]]$,

$$\mathrm{tr}_{\mathbb{C}}\, \mathbb{C}(f_1, \ldots, f_n, e^{f_1}, \ldots, e^{f_n}) \geqslant n \quad ?$$

1. Any relation

$$\mathbb{C} \ni C = r_1 f_1 + \ldots r_n f_n, \quad r_i \in \mathbb{Q}$$

   leads to an algebraic relation between the e^{f_i}. Therefore, we need to assume the f_i are $\mathbb{Q}$-linearly independent modulo a constant in $\mathbb{C}$.

2. There might be formal relations $P(f_1, \ldots, f_n) = 0$ for $P \in \mathbb{C}[[z_1, \ldots, z_k]]$, while the f_i are still $\mathbb{C}$-algebraically independent. For example, $f_1(t) = t$ and $f_2(t) = e^t$, with $p(z_1, z_2) = e^{z_1} - z_2$.

3. The number of independent formal relations is equal to the dimension over $\mathbb{C}((t_1, \ldots, t_k))$ of the formal kernel of the Jacobian $J(f_1, \ldots, f_n) := (\partial f_i / \partial t_j)_{i,j}$. So the formal transcendence degree of $f_1, \ldots, f_n$ over $\mathbb{C}$ is defined to be the rank of $J(f_1, \ldots, f_n)$.

Theorem 4.10 (Ax-Schanuel). *Let $f_1, \ldots, f_n \in \mathbb{C}[[t_1, \ldots, t_k]]$ be $\mathbb{Q}$-linearly independent modulo $\mathbb{C}$. Then*

$$\mathrm{tr}_{\mathbb{C}}\, \mathbb{C}(f_1, \ldots, f_n, e^{f_1}, \ldots, e^{f_n}) \geqslant n + \mathrm{rank}\ J(f_1, \ldots f_n).$$

Corollary 4.11.

$$\mathrm{tr}_{\mathbb{C}}\, \mathbb{C}(f_1, \ldots, f_n) + \mathrm{tr}_{\mathbb{C}}\, \mathbb{C}(e^{f_1}, \ldots, e^{f_n}) \geqslant n + \mathrm{rank} J(f_1, \ldots, f_n).$$

Corollary 4.12. *If $\mathrm{tr}_{\mathbb{C}}\, \mathbb{C}(f_1, \ldots, f_n) = \mathrm{rank}\ J(f_1, \ldots, f_n)$, then $\mathrm{tr}_{\mathbb{C}}\, \mathbb{C}(e^{f_1}, \ldots, e^{f_n}) = n$.*

4.3 Geometric functional transcendence

Define $\pi : \mathbb{C}^n \to (\mathbb{C}^*)^n$ by $(z_1, \dots, z_n) \mapsto (e^{2\pi i z_1}, \dots, e^{2\pi i z_n})$.

Definition 4.13. An irreducible algebraic subvariety of $\mathbb{C}^n$ is said to be bi-algebraic if $\dim V = \dim \overline{\pi(V)}^{\mathrm{Zar}}$. An irreducible subvariety $Y \subset (\mathbb{C}^*)^n$ is said to be bi-algebraic if $\dim \pi^{-1}(Y)^{\text{irr.comp.}} = \dim Y$.

Theorem 4.14 (Ax)**.** *Let* Graph $= \Delta \subset \mathbb{C}^n \times (\mathbb{C}^*)^n$*. Let $W \subset \mathbb{C}^n \times (\mathbb{C}^*)^n$ be an irreducible algebraic subvariety. If W intersects Δ atypically, meaning that there exists an analytic irreducible component U of $W \cap \Delta$ such that* $\operatorname{codim}_{\mathbb{C}^n \times (\mathbb{C}^*)^n}(U) < \operatorname{codim}_{\mathbb{C}^n \times (\mathbb{C}^*)^n}(W) + \operatorname{codim}_{\mathbb{C}^n \times (\mathbb{C}^*)^n}(\Delta)$*, then U is contained in the graph of a proper bi-algebraic $V \subset \mathbb{C}^n$.*

Theorem 4.15 (Weak Ax-Schanuel ($W = V \times Y$))**.** *Let $V \subset \mathbb{C}^n$ an algebraic subvariety and $Y \subset (\mathbb{C}^*)^n$ an algebraic subvariety. If there is an atypical analytic irreducible component U of $V \cap \pi^{-1}(Y)$, then W is contained in a proper bi-algebraic $V' \subset \mathbb{C}^n$.*

Theorem 4.16 (Ax-Lindemann-Weierstrass)**.** *Let $V \subset \mathbb{C}^n$ be an irreducible algebraic subvariety. Then $\overline{\pi(V)}^{\mathrm{Zar}}$ is bi-algebraic in $(\mathbb{C}^*)^n$.*

4.4 Proof of the Ax-Lindemann-Weierstrass Theorem using o-minimality

Proposition 4.17. *$Y \subset (\mathbb{C}^n)^*$ is bi-algebraic if and only if Y is a translate of a subtorus.*

Proof. Omitted. □

Lemma 4.18. *If $i_* \pi_1(Y)$ is not of finite index, then Y is contained in a coset of a proper algebraic subtorus.*

Proof. Omitted. □

Proof. (of Theorem 4.16). Let $V \subset \mathbb{C}^n$ be algebraic irreducible, $\mathbb{C}^n \xrightarrow{\pi} (\mathbb{C}^*)^n$. By Proposition 4.17, we want to show that $Y = \overline{\pi(V)}^{\mathrm{Zar}} \subset (\mathbb{C}^*)^n$ is a translate of a torus. We can assume that Y is not contained in a translate of a strict subtorus. We can assume that $V \subset \pi^{-1}(Y)$ is maximal among the irreducible algebraic subvarieties of $\mathbb{C}^n$ contained in $\pi^{-1}(Y)$ (otherwise, replace V by the maximal one). Consider

$$\Sigma(v) = \{v \in \mathbb{R}^n : \dim_{\mathbb{R}}((V + \sigma) \cap \pi|_F^{-1}(Y) = \dim_{\mathbb{R}}(V)\}.$$

V is irreducible, and $v \in \Sigma(v)$ if and only if $V + v$ meets F, $V + v \subset \pi^{-1}(Y)$.

Step 1: $\Sigma(v)$ is $\mathbb{R}_{\mathrm{an,exp}}$ definable. *Proof:* Omitted.
Step 2: $\mathrm{Stab}_{\mathbb{Z}^n}(V)$ is infinite. *Proof:* Omitted.
Step 3: Induction. Note that $\mathrm{Stab}_{\mathbb{C}^n}(V)$ is an algebraic group. By Step 2 we get a complex line $\mathbb{C}$ defined over $\mathbb{Q}$. Decompose $\mathbb{C}^n = \mathbb{C}^{n-1} \oplus \mathbb{C}$ over $\mathbb{Q}$. Then $V = V' \times \mathbb{C}$, so we can apply the induction hypothesis on $Y' = Y \cap (\mathbb{C}^*)^{n-1}$. □

5 Bruno Klingler: o-minimality and transcendence, IV

5.1 Bi-algebraic geometry

Last lecture we discussed $\pi : \mathbb{C}^n \to (\mathbb{C}^*)^n$, and the transcendence of this map. We make the following observations.

1. Both $\mathbb{C}^n$ and $(\mathbb{C}^*)^n$ are algebraic varieties, and π is transcendental.
2. The map $\mathbb{C}^n \to (\mathbb{C}^*)^n$ can shown to be a universal cover by a monodromy argument.
3. There exists an $F \subset \mathbb{C}^n$ semialgebraic, such that $\pi|_F : F \to (\mathbb{C}^*)^n$ is definable in some o-minimal structure.
4. We can try to replace $(\mathbb{C}^*)^n$ by a some algebraic variety S.
5. If we then consider the universal cover $\widetilde{S^{\mathrm{an}}} \to S^{\mathrm{an}}$, the problem will be that in general, as soon as $\pi_1(S^{\mathrm{an}})$ is infinite, $\widetilde{S^{\mathrm{an}}}$ has no algebraic structure.

Theorem 5.1 (Claudon, Höring, Kollàr)**.** *Assume that the abundance conjecture holds. Then, for any normal, projective variety X over $\mathbb{C}$ the following are equivalent.*

1. *The universal cover of X is biholomorphic to a quasi-projective variety.*
2. *There is a finite, é, Galois cover $X' \to X$ that is a fiber bundle over an abelian variety with simply connected fiber.*
3. *The universal cover $\tilde{X}$ is biholomorphic to a product $\mathbb{C}^m \times F$ where $m \geqslant 0$ and F is a projective, simply connected variety.*

If we relax condition 1 to the condition that the universal cover of X is biholomorphic to a semialgebraic variety, then $S = \Gamma\backslash\mathcal{D}$, a connected Shimura variety, where $\mathcal{D} \subsetneq \mathbb{C}^n$ bounded symmetric, Γ an arithmetic lattice in $G = \{\text{biholomorphic automorphisms}\}$.

Conjecture 5.2 (Kollàr-Parden)**.** *Let S be a normal projective variety over $\mathbb{C}$. Then $\widetilde{S^{\mathrm{an}}}$ is biholomorphic to a definable analytic space if and only if*

$$\widetilde{S^{\mathrm{an}}} \cong \mathbb{C}^n \times D \times F^{\mathrm{an}}$$

where D is bounded and F^{an} is simply connected.

5.2 Bi-algebraic geometry

We emulate to put an algebraic structure on $\widetilde{S^{\mathrm{an}}}$ using an algebraic model.

Definition 5.3. Let $S/\mathbb{C}$ be an irreducible algebraic variety. A bi-algebraic structure on S is given by a holomorphic map $f : \widetilde{S^{\mathrm{an}}} \to X^{\mathrm{an}}$ and a group homomorphism $h : \pi_1(S) \to \mathrm{Aut}(X)$, where X is an algebraic variety over $\mathbb{C}$ and f is equivariant under h.

Examples 5.4. 1.
$$\begin{tikzcd} \mathbb{C}^n = (\mathbb{A}^n)^{\mathrm{an}} \arrow[r, "\mathrm{id}"] \arrow[d, "\pi"] & (\mathbb{A}^n)^{\mathrm{an}} \\ (\mathbb{C}^*)^n. \end{tikzcd}$$

2. For A be a complex abelian variety of dimension n:
$$\begin{tikzcd} \mathrm{Lie}(A) \cong \mathbb{C}^n = (\mathbb{A}^n)^{\mathrm{an}} \arrow[r, "\mathrm{id}"] \arrow[d] & (\mathbb{A}^n)^{\mathrm{an}} \\ A^{\mathrm{an}}. \end{tikzcd}$$

3.
$$\begin{tikzcd} \widetilde{S^{\mathrm{an}}} = \mathcal{D} \arrow[r, hook] \arrow[d] & (\mathcal{D}^\vee)^{\mathrm{an}} \\ S^{\mathrm{an}} = \Gamma\backslash\mathcal{D} \end{tikzcd}$$

4.
$$\begin{tikzcd} \widetilde{S^{\mathrm{an}}} \arrow[r, "\psi"] \arrow[d] & \mathcal{D} \arrow[r, hook] & (\mathcal{D}^\vee)^{\mathrm{an}} \\ S^{\mathrm{an}} \end{tikzcd}$$

where ψ is the period map.

Definition 5.5. Let S be endowed with a bi-algebraic structure.

1. An analytic irreducible $V \subsetneq \widetilde{S^{\mathrm{an}}}$ is called bi-algebraic if $V = [f^{-1}((V')^{\mathrm{an}})]^0 = f^{-1}(\overline{f(V)}^{\mathrm{Zar}})^0$, where $V' \subsetneq X$ algebraic.
2. An algebraic subset $Y \subsetneq S$ is bi-algebraic if one (equivalently, any) irreducible component of $\pi^{-1}(Y)$ is bi-algebraic.

Example 5.6. The Noether Lefschetz locus of a bi-algebraic subvariety is bi-algebraic.

Definition 5.7. Let $S/\mathbb{C}$ be an irreducible algebraic variety. A $\bar{\mathbb{Q}}$-bi-algebraic structure $(f : \widetilde{S^{\mathrm{an}}} \to X^{\mathrm{an}}, h : \pi_1(S) \to \mathrm{Aut}(X))$ is a bi-algebraic structure if S is defined over $\bar{\mathbb{Q}}$, X is defined over $\bar{\mathbb{Q}}$, and $h : \pi_1(S) \to \mathrm{Aut}(X)$ takes values in $\mathrm{Aut}_{\bar{\mathbb{Q}}} X_{\bar{\mathbb{Q}}}$. The $\mathbb{Q}$-bi-algebraic points are the arithmetic points.

Examples 5.8. 1. The map $\pi = (e^{2\pi i \bullet}, \ldots, e^{2\pi i \bullet}) : \mathbb{C}^n \to (\mathbb{C}^*)^n$ defines a $\bar{\mathbb{Q}}$-bi-algebraic structure for $\mathbb{A}^n/\mathbb{C}$. Here $\tilde{S} = X^{\mathrm{an}} = \mathbb{C}^n$

Lemma 5.9. *$s \in (\mathbb{C}^*)^n$ is arithmetic if and only if it is a torsion point.*

2. Consider $\mathrm{Lie}\, A_{\mathbb{C}} = \mathbb{C}^n \to A$, where A is a simple CM-type abelian variety over $\bar{\mathbb{Q}}$. Then there exists a CM-field F of degree $2g$ over $\mathbb{Q}$ such that $\mathrm{End}^0(A) = F$. We have $\mathrm{Lie}\, A_{\mathbb{C}} = V_F \otimes F_{\mathbb{C}}$ where V_F is a rank one F-module. Take $V_F \otimes_F \bar{\mathbb{Q}}$ as $\bar{\mathbb{Q}}$-algebraic structure on $\mathrm{Lie}\, A_{\mathbb{C}}$.

Theorem 5.10 (Masser)**.** *Let A be a complex Abelian variety of dimension g with CM. Let $V_{\bar{\mathbb{Q}}}$ be the $\bar{\mathbb{Q}}$-vector space generated by* $\mathrm{Ker}(\pi) \subset \mathrm{Lie}(A)$. *Arithmetic bi-algebraic points for this $\mathbb{Q}$-bi-algebraic structure on A are exactly the torsion points of A.*

Example 5.11 (Scheinder, '37)**.** Consider a connected Shimura variety $S = \Gamma\backslash\mathcal{D}$, with $\mathcal{D} \subsetneq \mathcal{D}^\vee$. Consider

$$\begin{tikzcd} \mathbb{H} \arrow[r, hook] \arrow[d] & \mathbb{C} \arrow[r, hook] & \mathbb{P}^1_{\mathbb{C}} \\ Y_0(1) \cong \mathbb{C} & & \end{tikzcd}$$

For $\tau \in \bar{\mathbb{Q}}$, $j(\tau) \in \bar{\mathbb{Q}}$ if and only if τ is imaginary quadratic, E_τ is CM.

Theorem 5.12 (Ullmo-Yafaev)**.** *$Y \subset S$ is bi-algebraic if and only if Y is weakly special, that is, we have*

$$Y = \Gamma_H\backslash\mathcal{D}_H \times \{\text{pt.}\} \subseteq \Gamma_H\backslash\mathcal{D}_H \times \Gamma_{G'}\backslash\mathcal{D}_{G'} \subset \Gamma\backslash\mathcal{D},$$

where

$$(H \times G', D_H \times D_{G'}) \subseteq (G, D)$$

is a Shimura subdatum, and $H \times G' = \mathrm{MT}_{\mathrm{gen}}(\varphi)$ for H an algebraic monodromy group.

[Cohen-Shiga-Wolfart]

Theorem 5.13. *If S is of abelian type, then $s \in S$ is arithmetic if and only if s is a CM point.*

Example 5.14. $S \to \Gamma\backslash\mathcal{D}$ a ZHS period map.

Theorem 5.15 (Klingler, ...)**.** *$Y \subset S$ is bi-algebraic if and only if $Y \supset \mathrm{NL}(v)$ is weakly special.*

Conjecture 5.16. *If S is defined over $\bar{\mathbb{Q}}$, and $\bar{\phi}$ is of generic origin. Then Y is $\bar{\mathbb{Q}}$-bi-algebraic if and only if Y is a component of the Noether-Lefschetz locus. This is essentially equivalent to* (Hodg $\implies$ absolute Hodg).

5.3 Tame bi-algebraic structures

Proper case: $f(\widetilde{S^{\mathrm{an}}}) \subset \mathcal{D} \subsetneq_{\mathbb{R}_{\mathrm{alg}}\text{-def.}} X$. $h(\pi_1(S^{\mathrm{an}}))$ acts properly disctoninuous on $\mathcal{D}$. In that case, we ask: does there exists a semialgebraic subset $F \subset \mathcal{D}$ such that

$$\pi|_F : F \to \Gamma\backslash\mathcal{D}$$

is definable?

5.4 Atypical intersection heuristic

Suppose S is endowed with a tame bi-algebraic structure. Then any irreducible subvariety of S containing a Zariski set of $\bar{\mathbb{Q}}$-bi-algebraic points is a $\bar{\mathbb{Q}}$-bi-algebraic subvariety.

1. Manin's Theorem
2. Raynard's Theorem
3. Shimura varieties

5.5 Functional transcendence heuristic

5.5.1 Ax-Schanuel Principle

Theorem 5.17 (..?). *Let S be endowed with a tame bi-algebraic structure. Let $\Delta \subset \widetilde{S^{\mathrm{an}}} \times S^{\mathrm{an}}$ be the graph of $W \subset_{\text{algebraic}} \widetilde{S^{\mathrm{an}}} \times S^{\mathrm{an}}$. Let W be an analytic irreducible component of $\Delta \cap W$. Then $\operatorname{codim}(X) \geqslant \dim \overline{W}^{\text{bi-alg}}$, where $\overline{W}^{\text{bi-alg}}$ is the smallest bi-algebraic subvariety containing W.*

5.5.2 Ax-Lindemann-Weierstrass principle

Theorem 5.18 (..? Klingler). *Let S as above. If $V \subsetneq \widetilde{S^{\mathrm{an}}}$ is an irreducible algebraic subvariety, then $\overline{\pi(V)}^{\mathrm{Zar}}$ is bi-algebraic.*

5.5.3 Some ingredients in the hyperbolic case

<u>Setting:</u> Let $S = \Gamma\backslash\mathcal{D}$ be a Shimura variety.

Theorem 5.19 (Wilmo-Yafaev). *Bi-algebraic = weakly special.*

Proof. Omitted. □

Theorem 5.20 (Deligne). *Let $\mathbb{V}$ be a $\mathbb{Q}$-variation of Hodge structures on Y, with $\mathrm{MT}_{\mathrm{gen}}(Y) = G$. Then $H \lhd G^{\mathrm{der}}$. So $G = H_Y \times G'$, and $\mathcal{D} = \mathcal{D}_Y \times \mathcal{D}'$.*

5.6 Hyperbolic geometry

Proof. (of Ax-Lindemann Theorem 5.18 for Shimura varieties). Suppose $V \subsetneq \mathcal{D}$ is irreducible and algebraic, with $\pi : \mathcal{D} \to \Gamma\backslash\mathcal{D}$. We need to show that $\overline{\pi(V)}^{\mathrm{Zar}}$ is b-algebraic, hence weakly special.

Proposition 5.21.

$$H_V := [\overline{G(\mathbb{Z}) \cap \mathrm{Stab}_{G(\mathbb{R})}(V)}^{\mathrm{Zar}/\mathbb{Q}}]^0$$

is positive dimensional and stabilises V.

Proposition 5.22. *Let $V \subsetneq \mathcal{D}$ be an irreducible algebraic subvariety. Then there exists a $C_1 > 0$ such that*

$$\gamma \in G(\mathbb{Z}) : V \cap \gamma F \neq \varnothing, H(\gamma) \leqslant t\} \geqslant t^4.$$

<u>Height:</u>

$$G/\mathbb{Q} \hookrightarrow \mathrm{GL}(E)/\mathbb{Q}\,.$$

Let v be a place of $\mathbb{Q}$, then we get a norm $|| \ ||_v$ on E_v. Choose $|| \ ||_\infty$ to be K-invariant. Then $|| \ ||_v$ is an operator norm on E_∞/E_v.

Definition 5.23. Let $\varphi \in \mathrm{End}(E)$. We define H by

$$H(\varphi) = \prod_{v \in P} \max(1, ||\varphi||_v).$$

For $g \in G(\mathbb{Z})$ we define

$$H(g) = \max(1, ||g||_\infty).$$

$\square$

6 Benjamin Bakker: Definable analytic geometry, I

Fix an o-minimal structure S expanding $\mathbb{R}$. When we say definable, we mean definable with respect to S.
Goal of today: Define a category of complex analytic varieties, locally modelled on definable sets.
Why:

- locally quite flexible (e.g. $\mathbb{R}_{\text{an}}$ = restricted analytic functions, and these are accessible)
- $\mathbb{R}_{\text{an,exp}}$ = place where a lot of Hodge theory lives
- retain algebraicity globally.

6.1 Definable Chow

Theorem 6.1. *Closed analytic subvarieties $\mathcal{Y} \subset (\mathbb{P}^n)^{\text{an}}$ are algebraic. In other words, $\mathcal{Y} = Y^{\text{an}}$ for some algebraic variety $Y \subset \mathbb{P}^n$.*

Example 6.2. Observe what can happen if we drop the properness condition of the space in which $\mathcal{Y}$ is embedded. Consider $\Gamma_{e^z} = \text{Graph}(\exp(z)) \subseteq \mathbb{C} \times \mathbb{C}^*$, this is a closed analytic subvariety, but not algebraic.

There is an $\mathbb{R}$-linear bijection $\mathbb{C} \cong \mathbb{R}^2$, and so $\mathbb{C}^n$ has a notion of a definable subset via such an isomorphism (note that this notion thus may depend on the chosen isomorphism).

Theorem 6.3. *[Peterzil, Starchenko] A closed analytic subvariety $\mathcal{Y} \subset (\mathbb{A}^n)^{\text{an}}$ whose underlying set is definable is algebraic.*

Remark 6.4.

1. $\mathbb{C}^n = (\mathbb{A}^n)^{\text{an}}$. The notion of definability depends on choice of coordinates, where we mean the coordinates are algebraic. (Exercise: show that the notion of a definable subset only depends on the algebraic structure).
2. Peterzil and Starchenko proved it for arbitrary real closed fields.

Loosely speaking, the result says that “holomorphicity + definability = algebraicity”.

Lemma 6.5. *An entire function $f : \mathbb{C} \to \mathbb{C}$ which is definable, is a polynomial.*

Proof. The fibers are discrete and definable subsets of $\mathbb{C} = \mathbb{R}^2$, and therefore finite. This implies that f cannot have an essential singularity. ☐

Two proofs of Theorem 6.3:

1. Ease o-minimality, hard complex analysis.
2. Cell decomposition, easy complex analysis.

Corollary 6.6. *Let $f : \mathbb{R}^m \supset A \to B \subset \mathbb{R}^n$ be definable. Then $A_n = \{b \in B : \dim(f^{-1}(b)) = n\}$ is definable.*

Proof. Observe from definition of cels that they are of constant (relative) dimension over their projection. Now just order the coordinates in reverse: Graph $\subset \mathbb{R}^m \times \mathbb{R}^n$. □

Corollary 6.7. *Let $f : A \to B$ be a definable function with finite fibers. Then $A_n = \{b \in B : \# f^{-1}(b) = n\}$ is definable, and the fiber size is uniformly bounded.*

6.2 First proof of Definable Chow

Proof.

Theorem 6.8 (Bishop)**.** *Let $\mathscr{X} \subset \mathcal{U} \subset \mathbb{C}^n$, where $\mathcal{U} \subset \mathbb{C}^n$ open and $\mathscr{X} \subset \mathcal{U}$ is a closed analytic subvariety. Suppose that $\mathcal{Y} \subset \mathcal{U} \backslash \mathscr{X}$ is a closed subvariety (in other words, let $\mathcal{Y}$ be a closed analytic subvariety in the complement of a closed analytic subvariety of an open subset of $\mathbb{C}^n$). Suppose that $\mathcal{Y}$ has locally finite volume in $\mathcal{U}$. Then $\overline{\mathcal{Y}}$ is analytic.*

Lemma 6.9. *A bounded k-dimensional definable subset $A \subset \mathbb{R}^n$ has finite k-dimensional volume.*

Proof. Obvious for $A \subset \mathbb{R}^k$ of dimension k. By CDCD, after passing to a cell decomposition, we can assume $A \subset \mathbb{R}^n$ is a cell. But then

$$\mathrm{vol}(A) \leqslant \sum_{\text{proj. onto } k \text{ coord.}} \deg(\pi)\mathrm{vol}(\pi(A)).$$

□

To finish the proof of Definable Chow 6.3, observe that $(\mathbb{A}^n)^{\mathrm{an}} = \mathbb{C}^n \subset \mathcal{P}^n = (\mathbb{P}^n)^{\mathrm{an}} \supset \mathcal{P}^{n-1}$. We have that $\mathcal{Y} \subset \mathbb{C}^n$ has locally bounded volume around $\mathcal{P}^{n-1}$, by definability and Lemma 6.9. By Theorem 6.8, $\overline{\mathcal{Y}} \subset \mathcal{P}^n$ is analytic. By Theorem 6.1, $\overline{\mathcal{Y}} \subset \mathcal{P}^n$ is algebraic. But then $\mathcal{Y} = \overline{\mathcal{Y}} \cap \mathbb{C}^n \subset \mathbb{C}^n$ must be algebraic, too. □

6.3 Second proof of Definable Chow

Proof.

Lemma 6.10. *A definable holomorphic function $f : \mathbb{C}^n \to \mathbb{C}^n$ is algebraic.*

Proof. This is true for $n = 1$ by Lemma 6.5. Write $\mathbb{C}^n = \mathbb{C} \times \mathbb{C}^{n-1}$. Then by induction, $f(z, w)$ is algebraic for each $w \in \mathbb{C}^{n-1}$, i.e. $f(z, w) \in \mathbb{C}(w)[z]$ for each $w \in \mathbb{C}^{n-1}$. Moreover, $\deg_z f(z, w)$ is uniformly bounded, by Corollary 6.7. Therefore, we can write

$$f(z, w) = \sum^{N} \frac{\partial^k f}{\partial z^k}(0, w)\frac{z^k}{k!}.$$

Then <u>Exercise:</u> each $\frac{\partial^k f}{\partial z^k}(0, w) : \mathbb{C}^{n-1} \to \mathbb{C}$ is definable. By induction, $\frac{\partial^k f}{\partial z^k}(0, w)$ is algebraic. Therefore, f is algebraic. □

We can now induct on $\dim \mathcal{Y}$. We may as well assume that $\mathcal{Y}$ is analytically irreducible of dimension d.
Observe: It is enough to algebrize $\varnothing \neq \mathcal{Y} \cap U^{\text{an}}$ for a Zariski open $U \subset \mathbb{A}^n$. Indeed, $\mathcal{Y} = \overline{\mathcal{U} \cap U^{\text{an}}}$.

Step 1: We have
$$\mathcal{Y} \subset \mathbb{C}^n \subset \mathcal{P}^n \supset \mathcal{P}^{n-1},$$
and $\partial\mathcal{Y} = \overline{\mathcal{Y}}\backslash\mathcal{Y}$ is definable of smaller dimension than $\mathcal{Y}$, thus with dimension $\leqslant 2d-1$. (The proof if this goes as follows: it is clear for cells, and we already saw this proof in Amador's talk).

Step 2: There exists a linear projection $\pi : \mathbb{C}^n \to \mathbb{C}^d$ for which the restriction $\pi|_{\mathcal{Y}}$ is proper. *Proof.* We have $d < n$ so $\dim \partial\mathcal{Y} < \dim_{\mathbb{R}}(\mathcal{P}^{n-1}) = 2n-2$. There is a point $p \in \mathcal{P}^{n-1}\backslash\overline{\mathcal{Y}}$. We thus obtain a projection $\pi : \mathbb{C}^n \to \mathbb{C}^{n-1}$. Any fiber L of π intersects $\mathcal{Y}$ boundedly, i.e. $\pi|_{\mathcal{Y}}$ is proper. By Bemmert + definability, this is closed analytic subvariety that is definable, so iterate.

Step 3: The set $\mathcal{Y}_0 \subset \mathcal{Y}$ of points where $\pi|_{\mathcal{Y}}$ is not étale is definable. *Proof.* Closed analytic definable subvariety hence algebraic by induction, $\mathcal{Y}_0 = Y_0^{\text{an}}$. Closed analytic by complex analysis, definable by Corollary ... as it is the locus where fiber size is non-generic.

Step 4: $\mathcal{Y}$ is algebraic. *Proof.* We have $\mathbb{C}^n = \mathbb{C}^{n-d} \times \mathbb{C}^d$. So take π to be the projection onto the second factor. Let $\pi(\mathcal{Y}_0) = Z$. We know that $\pi|_{\mathcal{Y}\backslash\pi^{-1}(Z^{\text{an}})}$ has finite fibers of constant degree N (it is étale). Consider
$$F : \mathbb{C}^d\backslash Z^{\text{an}} \to \operatorname{Sym}^N \mathbb{C}^{n-d}, \quad z \mapsto \pi|_{\mathcal{Y}}^{-1}(z).$$
This map is definable, holomorphic and locally bounded over Z^{an}, therefore F extends to $\overline{F} : \mathbb{C}^d \to \operatorname{Sym}^N \mathbb{C}^{n-d}$. This map is also definable (because the graph of $\bar{F}$ is the closure of the graph of F). By Lemma 6.10, $\bar{F}$ is algebraic. Therefore, $\mathcal{Y}\backslash\pi^{-1}(Z^{\text{an}})$ is algebraic, being the pullback of the universal N-tuple. This implies that $\mathcal{Y}$ is algebraic. □

Remark 6.11. Peterzil-Starchenko's proof is basically the second proof. However, one needs to replace the Weierstrass-Casorati with a version of Liouville that works in general.

6.4 Definable topological spaces

Definition 6.12. A definable topological space is a topological space M with a definable atlas:

- a finite open covering $V_i \subset M$
- homomorphisms $\varphi_i : V_i \to U_i \subset \mathbb{R}^n$

such that

1. U_i and $U_{ij} = \varphi_i(V_i \cap V_j)$ are definable
2. $\varphi_{ij} = \varphi_j \circ \varphi_i^{-1} : U_{ij} \to U_{ji}$ is definable.

A morphism $f : M \to M'$ is a continuous map which is definable on charts, which means that $\varphi'_{i'} \circ f \circ \varphi_i^{-1} : \mathbb{R}^n \supset \varphi_i(V_i \cap f^{-1}(V'_{i'})) \to U_{i'} \subset \mathbb{R}^m$ is definable for each i, i'.

In this way, we get a category of definable topological spaces, which we denote by $(\mathsf{S\text{-}DefTopSp})$.

Examples 6.13. Consider the topological space $\mathbb{C}^* \subset \mathbb{C} \cong \mathbb{R}^{2n}$.

1. The set $\mathbb{C}^*$ is semialgebraic, therefore it is an $\mathbb{R}_{\text{alg}}$ definable topological space, which we denote by $\mathbb{G}_m^{\text{def}}$.
2. For $a \in \mathbb{R}$ we can define the 'slope a' fundamental domain: this is

$$F_a = \{z \in \mathbb{C} : a\Im(z) \leqslant \Re(z) \leqslant 1 + a\Im(z)\}.$$

Consider the image of F_a under $e : \mathbb{C} \to \mathbb{C}^*$, $e(z) = e^{2\pi i z}$. By taking slightly thinner strips, we get a covering of $\mathbb{C}^*$ by semialgebraic sets with semialgebraic transition functions. This gives a definable topological space which we denote by $\mathbb{C}_a^*$.

Remark 6.14. Let $S' \supset S$ be o-minimal structures expanding $\mathbb{R}$. Then there is a functor $(\mathsf{S\text{-}DefTopSp}) \to (\mathsf{S'\text{-}DefTopSp})$.

Examples 6.15.

- All $\mathbb{C}_a^*$ are isomorphic over $\mathbb{R}_{\text{alg}}$, by $x + iy \mapsto (x + ay) + iy, q \mapsto qe^{iq \log |q|}$.
- The map $\text{id} : \mathbb{C}_a^* \to \mathbb{C}_m^{\text{def}}$ is not definable in <u>any</u> o-minimal structure if $a \neq 0$. Indeed, $\text{id}^{-1}(\text{ray})$ has infinitely many components if $a \neq 0$.
- The map $\text{id} : \mathbb{C}_0^* \to \mathbb{G}_m^{\text{def}}$ is not $\mathbb{R}_{\text{alg}}$-definable if and only if

$$(e^{-2\pi i} \cos(2\pi x), e^{-2\pi i} \sin(2\pi x)) = e : F_0 \to \mathbb{G}_m^{\text{def}}$$

is not $\mathbb{R}_{\text{alg}}$-definable. This map is also not $\mathbb{R}_{\text{an}}$-definable, and not $\mathbb{R}_{\exp}$ definable.

7 Benjamin Bakker: Definable analytic geometry, II

7.1 Basic definable analytic spaces

Last time, we saw two examples of definable topological spaces, namely $\mathbb{C}_a^*$ for $a \in \mathbb{R}$ and $\mathbb{G}_m^{\text{def}}$. We shall now consider two other examples of definable topological spaces.

Example 7.1. 1. Consider the action on $\mathbb{H}$ of

$$\Gamma(2) = \{A \in \mathrm{PSL}_2(\mathbb{Z}) : A \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \bmod 2\}.$$

Define $Y(2)^{\mathrm{an}} = \Gamma(2)\backslash\mathbb{H}$. We can cover the fundamental domain by semialgebraic opens U, such that the glueing maps are semialgebraic (because the action of $\Gamma(2)$ on $\mathbb{H}$ is semialgebraic). In this way the $\{U, \gamma U\}$ form an $\mathbb{R}_{\mathrm{alg}}$-atlas for $Y(2)^{\mathrm{an}}$. Denote the definable topological space thus obtained by $\mathcal{Y}(2)$.

2. Note that $Y(2) \subset \mathbb{A}^2$ is affine algebraic, so $Y(2)(\mathbb{C})$ is an algebraic set in $\mathbb{A}^n(\mathbb{C}) = \mathbb{C}^n$, giving rise to a definable topological spaces denoted by $Y(2)^{\mathrm{def}}$.

<u>Exercise:</u> When is $Y(2)^{\mathrm{def}} \cong \mathcal{Y}(2)$ (that is, for which o-minimal structure S) ?

<u>Recall Definable Chow:</u> A definable closed analytic subspace (possibly non-reduced) $\mathcal{Y} \subset (\mathbb{A}^n)^{\mathrm{an}}$ is algebraic, $\mathcal{Y} = Y^{\mathrm{an}}$.

How does the notion of reducibility behave with respect to definability? Consider the zero locus of the ideal $I = (x, y)^2 + (x - e^z y)$ in $\mathbb{C}^3$. This gives an infinitesimal wiggle on the z-axis, and $Z(I)$ is not algebraic, so definability of the modelling set is not enough.

We would like to define $\mathcal{O}_{\mathbb{C}^n}$, a sheaf on $\mathbb{C}^n$, with $\mathcal{O}_{\mathbb{C}^n}(U) = \{$definable holomorphic$f : U \to \mathbb{C}$ for $U \subset \mathbb{C}^n$ open and definable. But this naive definition goes wrong.

Definition 7.2. Let X be a definable topological space. The definable site of X, denoted by $\underline{X}$, is defined as follows: $Ob(\underline{X})$ = open definable $U \subset X$, $Mor(\underline{X})$ = inclusions, and $Cov(\underline{X})$ is given by finite open coverings U_i of U in $\underline{X}$.

Example 7.3. $\mathcal{O}_{\mathbb{C}^n}$ is a sheaf on $\underline{\mathbb{C}^n}$.

Remark 7.4. For any morphism $f : X \to Y$ of definable topological spaces we get $f_* : \mathrm{Sh}(X) \to \mathrm{Sh}(Y)$, and $f^{-1} : \mathrm{Sh}(Y) \to \mathrm{Sh}(X)$.

Remark 7.5. A definable site $\underline{X}$ does not have enough points (we cannot check exactness on stalks). To be vague: there exists an "o-minimal spectrum" $X^{\#}$ obtained by adjoining points valued in real closed fields. Then $\mathrm{Sh}(\underline{X}) \cong \mathrm{Sh}(\underline{X^{\#}})$.

Definition 7.6. A locally ringed definable space over $\mathbb{C}$ is a definable topological space together with a sheaf of $\mathbb{C}$-algebras on its definable site for which the stalks are local $\mathbb{C}$-algebras.

Definition 7.7. A basic definable analytic space is given by the following data: an open subset $U \subset \mathbb{C}^n$ which is definable and a finitely generated ideal $I \subset \mathcal{O}_{\mathbb{C}^n}(U)$, which gives the underlying analytic space $X = |V(I)| \subset U$. Often we shall denote basic definable analytic spaces by $(X \subset U \subset \mathbb{C}^n)$, and the ideal that defines X by I_X. Such a definable topological space comes together with a sheaf $\mathcal{O}_X$ on $\underline{X}$, defined as

$$\mathcal{O}_X = j^{-1}(\mathcal{O}_U / I \cdot \mathcal{O}_U).$$

A morphism

$$(X \subset U \subset \mathbb{C}^n) \xrightarrow{\varphi} (Y \subset V \subset \mathbb{C}^n)$$

is a definable holomorphic map $\varphi : U \to V$ such that $\varphi^{-1} I_Y \subset I_X$.

Remark 7.8. From a morphism of definable topological spaces

$$\varphi : X \to Y$$

we get a morphism of definable sites

$$\varphi : \underline{X} \to \underline{Y}$$

and a morphism of sheaves

$$\varphi^{\#} : \varphi^{-1}(\mathcal{O}_Y) \to \mathcal{O}_X .$$

Remark 7.9. From a morphism of basic definable analytic spaces

$$\varphi : (X \subset U \subset \mathbb{C}^n) \to (Y \subset V \subset \mathbb{C}^n)$$

we get functors

$$\varphi_\star : \mathrm{Mod}(\mathcal{O}_X) \to \mathrm{Mod}(\mathcal{O}_Y)$$

and

$$\varphi^\star : \mathrm{Mod}(\mathcal{O}_Y) \to \mathrm{Mod}(\mathcal{O}_X), \quad F \mapsto \mathcal{O}_X \otimes_{\varphi^{-1}\mathcal{O}_Y} \varphi^{-1} F.$$

Definition 7.10. Let X be a basic definable analytic space. Then $F \in \mathrm{Mod}(\mathcal{O}_X)$ is called:

1. Locally finitely generated if there exists a cover X_i of X in $\underline{X}$ such that

$$\mathcal{O}_{X_i}^{n_i} \to F_{X_i} \to 0 \quad \text{ is exact.}$$

2. Coherent if it is locally finitely generated, and, for any definable open subset $U \subset X$, given an exact sequence

$$\mathcal{O}_U^n \xrightarrow{\varphi} F_U \to 0,$$

   $\mathrm{Ker}(\varphi)$ is locally finite generated.

3. Locally finitely presented if on some cover X_i of X, the sequence

$$\mathcal{O}_{X_i}^{M_i} \to \mathcal{O}_{X_i}^{N_i} \to F_{X_i} \to 0$$

   is exact (in other words, locally, F is the cokernel of some matrix with coefficients in $\mathcal{O}_X$).

<u>Exercise:</u> Let X be a basic definable analytic space. Show that $\mathsf{Coh}(X) \subset \mathsf{Mod}(X)$ is an abelian extension category. (That is, show that each hom-set in $\mathsf{Coh}(X)$ carries the structure of an abelian group such that the composition is bilinear, that $\mathsf{Coh}(X)$ admits finite coproducts, and that for any exact sequence $0 \to A \to B \to C \to 0$ in $\mathsf{Mod}(X)$ such that A and C are in $\mathsf{Coh}(X)$, B is in $\mathsf{Coh}(X)$.)

Theorem 7.11 (Bakker, Brunebarbe, Tsimerman)**.** *(OKA coherence) For a basic definable analytic space X, the sheaf $\mathcal{O}_X$ is coherent.*

Example 7.12. Locally finitely generated ideals are locally finitely presented.

Proof. Exercise. □

Example 7.13. Let X be a basic definable analytic space. A sheaf on $\underline{X}$ is coherent if and only if it is finitely presented.

Proof. Exercise. □

Lemma 7.14. *Given a basic definable analytic space $X \subset U \subset \mathbb{C}^n$ which is d-dimensional, there exists a covering U_i of U and linear projections $\pi_i : \mathbb{C}^n \to \mathbb{C}^d$ such that*

$$\pi_i : X \cap U_i = X_i \to \pi_i(U_i)$$

is finite (finite fibers and proper).

Lemma 7.15 (Weierstrass division)**.** *Let $V \subset \mathbb{C}^n$ be open definable, $P \in \mathcal{O}_{\mathbb{C}^n}(V)[t]$ monic (its coefficients are definable functions). Let $Y = V(P) \subset U \subset V \times \mathbb{C} \subset \mathbb{C}^{n+1}$, with U a definable open subset. Given $f \in \mathcal{O}_{\mathbb{C}^{n+1}}(U)$ there exist unique Q, R definable holomorphic functions such that $f = QP + R$, with $R \in \mathcal{O}_{\mathbb{C}^n}(V)[t]$.*

Proof. By the classical Weierstrass division, we only have to show that Q and R are definable. Let us assume P is irreducible, with $\deg(P) = d$. Let $V' \subset V$ be the dense open set (definable) where the roots of P are distinct (the étale locus). We have $R|_{V'} = \sum^{d-1} a_i t^i$ with $a_i \in \mathcal{O}_{\mathbb{C}^n}(V)$, and this agrees with f fiberwise, and is therefore definable. Now $P|_{V'}$ is also definable. Observe that the graph of R is the closure of the graph of $R|_{V'}$, and therefore definable (here the irreducibility is used). But then $Q = (f - R)/P$ is definable. □

Proof. (of Theorem 7.11). Case I: $X \subset \mathbb{C}^n$ is open and definable. We work by induction on n. Let $U \subset X$ be a definable open subset, and let f be the morphism

$$\mathcal{O}_U^n \to \mathcal{O}_U$$

$$e_i \mapsto f_i.$$

What do we need? We need to find a cover $\{U_i\}$ of U, and for each U_j we need relations R_l in $\mathrm{Ker}(\varphi)(U_j)$ that generate $\mathrm{Ker}(\varphi)$ on U_j. Clearly, there are some obvious relations

$$f_i e_j - f_j e_i.$$

If the f_i is invertible, then the $e_j - \frac{f_j}{f_i} e_i$ generate $\mathrm{Ker}(\varphi)$. Let $Y = V(f_i) \subset U$. Then on $U \backslash Y$ we have a generator.

Now by Lemma 7.14, we can assume that after taking a cover, we have $Y \subset U \subset \mathbb{C}^n$ and a projection $\mathbb{C}^n \to \mathbb{C}^{n-1}$ such that

$$\begin{tikzcd} Y \arrow[r] \arrow[dr] & U \arrow[r] \arrow[d] & \mathbb{C}^n \arrow[d] \\ & V \arrow[r] & \mathbb{C}^{n-1} \end{tikzcd}$$

commutes, and the map $Y \to V$ is finite. If P is minimal polynomial of Y (definable, same as above), $P \in \mathcal{O}_{\mathbb{C}^n}(V)[t]$. Weierstrass preparation gives f_1/P is a definable function. We may replace f_1 by P. By Weierstrass division 7.15, replace f_j with respect to $f_j \in \mathcal{O}(V)[t]$ with $\deg f_j < \deg P$. Any relation over U can be expressed in terms of $e_j - \frac{f_j}{f_1}e_1$, and $\sum g_i e_i$, with $g_i \in \mathcal{O}_(V)[t]$, $\deg g_i < \deg P$. Again, by Weierstrass division Lemma 7.15, these arise in the kernel of

$$\begin{tikzcd} \mathcal{O}_U^n \arrow[r] & \mathcal{O}_U \\ \mathcal{O}_V[t]^n_{<k} \arrow[r] \arrow[u, hook] & \mathcal{O}_V[t]_{<k^2} \arrow[u, hook] \end{tikzcd}$$

where $k = \deg P$. This gives a map $\mathcal{O}_V^{nk} \xrightarrow{\psi} \mathcal{O}_V^{k^2}$, and by induction, $\operatorname{Ker}(\psi)$ is locally finitely generated by R_l.
Claim: the $e_j - \frac{f_j}{f_1}e_1, R_l$ generate $\operatorname{Ker}\varphi$ is only true on sets of the form $\pi^{-1}(V)$.

Lemma 7.16. *Let $f : X \to Y$ be a finite continuous map between definable topological spaces, and given a cover $\{X_i\}_{i\in I}$ of X, there exists a refinement X_i' and a cover Y_j of Y such that $f^{-1}(Y_j) = \bigsqcup_{i\in I} X_i'$.*

By Lemma 7.16, we can extend the relations on U' to a saturated open.

Corollary 7.17. *For a morphism f of basic definable analytic spaces, the functor f_* is exact.*

Case II: Let $X \subset U \subset \mathbb{C}^n$ arbitrary. Corollary 7.17 finishes the proof. $\square$

7.2 Definable analytic spaces

We have a full faithful embedding

$$\begin{aligned} (\text{BasicDefAnSp}/\mathbb{C}) &\hookrightarrow (\mathbb{C}\,\text{locally ringed sites}). \\ (X \subset U \subset \mathbb{C}^n) &\mapsto (\underline{X}, \mathcal{O}_X). \end{aligned}$$

Consequences:

1. Exactness can be checked on stalks.
2. The analytification functor $(-)^{\text{an}} : \mathsf{Coh}(X) \to \mathsf{Coh}(X^{\text{an}})$ is faithful and exact.

Definition 7.18. If X is a definable topological space and $\mathcal{O}_X$ is a sheaf of rings on $\underline{X}$ making it a $\mathbb{C}$-locally ringed site, then we call $(X, \mathcal{O}_X)$ a definable analytic space if there exists a covering in $\underline{X}$ of X by basic definable analytic spaces.

Theorem 7.19. *For $(X, \mathcal{O}_X)$ a definable analytic space, the sheaf $\mathcal{O}_X$ is coherent.*

Example 7.20. We can analytify $Y(2)^{\mathrm{def}}$ and $\mathcal{Y}(2)$ to $\Gamma(2)\backslash\mathbb{H}$.

The thus constructed categories fit in the following diagram:

$$\begin{tikzcd}
(\text{ Finite type schemes over } \mathbb{C}\text{ }) \arrow[rr, "(-)^{\mathrm{def}}"] \arrow[dr] & & (\text{S-DefAnSp}/\,\mathbb{C}) \arrow[dl] \\
& (\text{AnSp}/\,\mathbb{C}). &
\end{tikzcd}$$

8 Benjamin Bakker: Definable analytic geometry, III

8.1 Analytification

We recall Definition 7.18.

Definition 8.1. If X is a definable topological space and $\mathcal{O}_X$ is a sheaf of $\mathbb{C}$-algebras on the definable site $\underline{X}$ of X whose stalks are local rings, we call $(X, \mathcal{O}_X)$ a definable analytic space if there exists a covering of X in $\underline{X}$ by basic definable analytic spaces X_i.

In other words, X is a definable topological space, and $\mathcal{O}_X$ is a sheaf of rings on the definable site $\underline{X}$ of X such that there exists a definable cover X_i of X such that $(X_i, \mathcal{O}_{X_i})$ are basic definable analytic spaces.

Example 8.2. $\mathcal{Y}(2)$ and $Y(2)^{\mathrm{def}}$ are definable analytic spaces.

But be aware: for this definition to make sense, we need

$$(\text{BasicDefAnSp}/\,\mathbb{C}) \xrightarrow{full} (\text{locally ringed definable topological spaces}).$$

This is “up to shrinking / covering ” (exercise).

Example 8.3.

$$\begin{tikzcd}
(\text{ Algebraic schemes over } \mathbb{C}\text{ }) \arrow[rr, "(-)^{\mathrm{def}}"] \arrow[dr] & & (\text{S-DefAnSp}/\,\mathbb{C}) \arrow[dl] \\
& (\text{AnSp}/\,\mathbb{C}). &
\end{tikzcd}$$

where (Algebraic schemes over $\mathbb{C}$) $\to$ (S-DefAnSp/ $\mathbb{C}$) is defined locally by

$$X = \operatorname{Spec}\mathbb{C}[X_i]/(f_j) \mapsto V(f_j) \subset \mathbb{C}^n,$$

and (S-DefAnSp/ $\mathbb{C}$) $\to$ (AnSp/ $\mathbb{C}$) by $(X \subset U \subset \mathbb{C}^n) \mapsto (X \subset U \subset \mathbb{C}^n)$, where the latter is now considered as an analytic space, forgetting the definable structure.

Let X be a definable analytic space. There is a functor

$$(-)^{\mathrm{an}} : \mathsf{Coh}(X) \to \mathsf{Coh}(X^{\mathrm{an}}), \quad F \mapsto \mathcal{O}_{X^{\mathrm{an}}} \otimes_{\mathcal{O}_X^{\mathrm{sh}}} F^{\mathrm{sh}}$$

where F^{sh} stands for the sheafification of the sheaf F (which was only a sheaf in the definable topology).

Example 8.4. If $\mathbb{R}_{\mathrm{an}} \subset S$, then $\mathcal{O}_{X^{\mathrm{an}}} = \mathcal{O}_X^{\mathrm{sh}}$ so this is just sheafification in the analytic topology.

Remark 8.5. Any $F \in \mathsf{Coh}(X)$ is locally finitely presented. Consider a local presentation

$$\mathcal{O}_X^n \xrightarrow{g} \mathcal{O}_X^m \to F_X \to 0.$$

This gives

$$\mathcal{O}_{X^{\mathrm{an}}}^n \xrightarrow{g^{\mathrm{an}}} \mathcal{O}_{X^{\mathrm{an}}}^m \to F^{\mathrm{an}} \to 0,$$

since sheafification is exact and the tensor product right exact.

Theorem 8.6. *The functor* $(-)^{\mathrm{an}} : \mathsf{Coh}(X) \to \mathsf{Coh}(X^{\mathrm{an}})$ *is exact and faithful.*

Proof. Step 1: Show that the local rings $\mathcal{O}_{X,x}$ are noetherian. We may assume X is a basic analytic space ($X \subset U \subset \mathbb{C}^n$). Consider the sequence $\mathcal{O}_{U,x} \to \mathcal{O}_{X,x} \to 0$ and reduce to $\mathcal{O}_{\mathbb{C}^n,x}$. It is enough to show that for all $f \in \mathcal{O}_{\mathbb{C}^n,x}$, $\mathcal{O}_{\mathbb{C}^n,x}/(f)$ is noetherian. By definable Noether normalisation, replace f by a definable Weierstrass polynomial. We get $V(f) \subset U \to V$ and by definable Weierstrass division, $\mathcal{O}_{\mathbb{C}^n,x}/(f)$ is finite over $\mathcal{O}_{\mathbb{C}^{n-1},y}$. By induction we are done.

Step 2: Use Artin-Rees, implying that if the stalks are noetherian, completion is exact.
Exercise: $\widehat{\mathcal{O}_{X,x}} \cong \widehat{\mathcal{O}_{X^{\mathrm{an}},x}}$ (hint: polynomials are definable so you get all power series).
Step 3: Exactness follows since completion in the definable category is exact, and this computes the completion of analytifications.
Step 4: Prove that $E \xrightarrow{f} F$ in $\mathsf{Coh}(X)$ such that $E^{\mathrm{an}} \xrightarrow{f^{\mathrm{aN}}} F^{\mathrm{an}}$ is the zero morphism, then $f = 0$. By exactness, it is enough to show $F = 0$ if $F^{\mathrm{an}} = 0$. Consider a cover of X with presentations

$$\mathcal{O}_X^n \to \mathcal{O}_X^m \to F \to 0.$$

If $F^{\mathrm{an}} = 0$, at every $x \in X^{\mathrm{an}}$, then $\mathcal{O}_{X^{\mathrm{an}}}^n \xrightarrow{g^{\mathrm{an}}} \mathcal{O}_{X^{\mathrm{an}}}^m$ has a section if and only if one of the minors of g^{an} is nonzero. Let X_i be the open cover where the $i'th$ minor is nonzero. This is a definable cover, and at this cover, $\mathcal{O}_{X_i}^n \to \mathcal{O}_{X_i}^m$ has a section. Because this is a rational function in the entries of g, this is a definable section. $\square$

Corollary 8.7.

$$(\text{BasicDefAnSp}/\mathbb{C}) \hookrightarrow (\text{locally ringed definable topological spaces}/\mathbb{C}).$$

Proof. Consider two basic definable analytic spaces $(X \subset U \subset \mathbb{C}^n)$ and $(Y \subset V \subset \mathbb{C}^m)$. Suppose we are given a morphism

$$f : (\underline{X}, \mathcal{O}_X) \to (\underline{Y}, \mathcal{O}_Y)$$

in the category of locally ringed definable topological spaces. This consists of a morphism $f : |X| \to |Y|$ of definable topological spaces, and a morphism $f^\# : f^{-1}\mathcal{O}_Y \to \mathcal{O}_X$ of sheaves rings on $\underline{X}$. By pulling back the coordinate functions of V, we get a definable holomorphic map $g : U \to V$, such that $g^{-1}I_Y \subset I_X$. This gives a map

$$g : (X \subset U \subset \mathbb{C}^n) \to (Y \subset V \subset \mathbb{C}^m).$$

What we now need to show, is that $f^\# = g^\# : g^{-1}\mathcal{O}_Y \to \mathcal{O}_X$. On completions, this is true, because by pulling back they send the coordinate functions to the same holomorphic equations, and the coordinate functions generate the completions. Therefore, $f^\#$ and $g^\#$ agree on completions, as they agree on coordinate functions. By faithfulness, because a section of $\mathcal{O}_X$ is determined by its value in $\widehat{\mathcal{O}_{X,x}}$ for all x, it follows that $f^\# = g^\#$. □

Corollary 8.8. *Let $E \subset F$ in $\mathsf{Coh}(X)$. If $s \in F(X)$ such that $s_x^{\mathrm{an}} \in E_{X^{\mathrm{an}},x}$ for all $x \in X$, then $s \in E(X)$.*

Exercise: Show that $\mathrm{Supp}(F) \subset X$, the support of $F \in \mathsf{Coh}(X)$, is a definable closed analytic subspace, cut out by the minors of a presentation. Furthermore, show that

1. $I_{\mathrm{Supp}(F)} = \mathrm{Ann}(F)$
2. $x \notin |\mathrm{Supp}(F)| \iff F_x = 0$
3. $\mathrm{Supp}(F^{\mathrm{an}}) = \mathrm{Supp}(F)^{\mathrm{an}}$.

8.2 Increasing sequences of coherent subsheaves

Theorem 8.9 (Definable noetherian induction)**.** *Let X be a definable analytic space. Let $F \in \mathsf{Coh}(X)$. Every increasing sequence*

$$\ldots F_1 \subset F_2 \subset F_3 \subset \ldots$$

of coherent subsheaves stabilises.

Corollary 8.10. *Any decreasing chain of closed definable analytic subspaces stabilises.*

Proof. Consider ideals. □

Example 8.11. This is false in the analytic category. Consider $Z_N = \{n \in \mathbb{N} : n \geqslant N\} \subset \mathbb{C}$. Then $Z_1 \supset Z_2 \supset \ldots$ does not stabilise.

Proof. (of Theorem 8.9). Since definable covers are finite, this is a local statement: assume ($X \subset U \subset \mathbb{C}^n$). Assume $\mathcal{O}_X^n \to \mathcal{O}_X^m \to F \to 0$, and $\mathcal{O}_U^m \to \mathcal{O}_X^m$. By pulling back the sequence, reduce to $\mathcal{O}_U^n$. By pushing forward and pulling back, we can reduce to the case $F = \mathcal{O}_U$ for an open definable subset $U \subset \mathbb{C}^n$, and a sequence of ideals

$$I_1 \subset I_2 \subset \ldots \subset \mathcal{O}_U \, .$$

We may assume there exists $0 \neq f \in I_1$, and we will quotient out by f to prove by induction. This gives us

$$I_1/(f) \subset I_2/(f) \subset \ldots \subset \mathcal{O}_Y$$

where $Y = V(f) \subset U \to V$, with $\pi : Y \to V$ the projection which we may assume to be finite by Lemma 7.14. Because π_* is exact, $\pi_* \, \mathcal{O}_Y$ is $\mathcal{O}_V^N$ by Weierstrass division [...], so reduce to $\mathcal{O}_V^N$, and use the induction hypothesis. □

8.3 Reduced spaces

In the analytic category, for $V(I) = \mathcal{X} \subset U \subset \mathbb{C}^n$, we have $\mathcal{X}^{\mathrm{red}} = V(I^{\mathrm{rad}})$. The stalks of $\mathcal{O}_{\mathbb{C}^n}$ are noetherian, and the radical is finitely generated. In the definable category, it is not clear if the radical of a coherent ideal is coherent.

Definition 8.12. Let X be a definable analytic space. Then $Y \subset X$ is a closed definable analytic subspace if one of the two following equivalent conditions holds:

1. $Y \subset X$ is a closed reduced subspace which is a definable analytic space,

2. $\mathcal{Y} \subset X^{\mathrm{an}}$ is a closed reduced analytic set, which is definable.

One implication is clear, we prove that 2 implies 1.

Proposition 8.13. *Let $U \subset \mathbb{C}^n$ be open and definable. Let $\mathcal{X} \subset U^{\mathrm{an}}$ be a closed, reduced, analytic subset which is definable. Let $I \subset \mathcal{O}_U$ be the ideal of definable holomorphic functions which vanish on $\mathcal{X}$. Then I is coherent, and analytifies to $I_{\mathcal{X}}$.*

Corollary 8.14. *Let X be a definable analytic space. Then there exists a subspace $X^{\mathrm{red}} \subset X$ which is a definable analytic subspace such that $(X^{\mathrm{red}})^{\mathrm{an}} = (X^{\mathrm{an}})^{\mathrm{red}}$.*

Proof. Consider $X^{\mathrm{red}} := V(I_X) \subset X \subset U \subset \mathbb{C}^n$. □

Definition 8.15. We say that a definable analytic space X is reduced if $X^{\mathrm{red}} = X$.

Corollary 8.16. *Let X be a definable analytic space. There is a bijection between the closed reduced definable analytic subsets $Y \subset X$ and the reduced and closed analytic subsets $\mathcal{Y} \subset X^{\mathrm{an}}$ such that $|\mathcal{Y}| \subset |X|$ is definable.*

Corollary 8.17. *Let X and Y be reduced definable analytic spaces. Then*

$$\mathrm{Hom}_{DefAnSp}(X, Y) \cong \mathrm{Hom}_{AnSp}(X^{\mathrm{an}}, Y^{\mathrm{an}}).$$

Proof. Consider the graph. $\square$

Proof. (of Proposition 8.13). The key of this proof lies in the fact that it is enough to produce a coherent idea $J \subset \mathcal{O}_U$ with $J \subset I$ which analytically cuts out $\mathcal{X}$.
If there is some open $U' \subset U$ with a function $f \in I(U') \backslash J(U')$ then $J \subset J' \subset I$, with $J' = J + (f)$, and $J^{\mathrm{an}} = (J')^{\mathrm{an}}$. By faithfulness, $J = J' = J + (f)$, hence $f \in J$, contradiction, so $I|_{U'} = J|_{U'}$.
By Lemma 7.14 we may assume we are in the following situation:

$$\begin{tikzcd} \mathcal{X} \arrow[r] \arrow[dr, "\text{finite}"'] & U \arrow[r] \arrow[d] & \mathbb{C}^n \arrow[d] \\ & V \arrow[r] & \mathbb{C}^d, \end{tikzcd}$$

where $d = \dim(\mathcal{X})$. Let $\mathcal{X}_0$ be the (non-étale locus of $\pi)^{\mathrm{red}}$. The non-étale locus is cut out by finitely many polynomials which makes it definable, hence this is a closed definable analytic subset. By induction $\mathcal{X}_0 = X_0^{\mathrm{an}}$ is a closed definable analytic subspace. Let $Y = \pi(X_0)$. Proceed as in Lecture 1:

$$V \backslash Y \to \operatorname{Sym}^N \mathbb{C}^{N-d}$$

where N is the generic degree of the map π. We can pull back and get a definable holomorphic function which cuts out $\mathcal{X} \backslash \pi^{-1}(Y)$. Therefore, we have a coherent ideal J' on U cutting out $\mathcal{X} \backslash \pi^{-1}(Y)$. By induction, $I_{\mathcal{X}_0}$ is coherent, and we can take $J = I_{\mathcal{X}_0} + J'$. $\square$

Corollary 8.18 (Definable Nullstellensatz)**.** *Let X be a definable analytic space. Then there exists an $n \in \mathbb{N}$ such that $I^n_{X^{\mathrm{red}}} = 0$.*

Example 8.19. This is not true in the analytic category. Consider Z_n, the n^{th} order neighbourhood of an element $x \in \mathbb{C}$, and $Z = \bigcup Z_n$.

Proof. We go local, where it holds in the analytic category. Let $E \subset E'$ a subsheaf. Then $\operatorname{Supp}(E) \subset \operatorname{Supp}(E')$. Therefore, $\operatorname{Supp}(I^k_{\mathrm{red}})$ is decreasing. Analytically, eventually it contains a fixed point. By Theorem 8.9, $\operatorname{Supp}(I^n_{X^{\mathrm{red}}}) = \varnothing$, which implies $I^n_{X^{\mathrm{red}}} = 0$. $\square$

Corollary 8.20. *If $X \supset Z$ then there exists a positive integer n such that $I^n_{Z^{\mathrm{red}}/X} \subset I_{Z/X}$.*

Remark 8.21. Consider the functors

$$(\mathsf{AlgSch}/\mathbb{C}) \quad \xrightarrow{\mathrm{def}} \quad (\mathsf{DefAnSp}/\mathbb{C}),$$

and for an algebraic scheme X over $\mathbb{C}$:

$$\mathsf{Coh}(X) \quad \to \quad \mathsf{Coh}(X^{\mathrm{def}}), \quad F \mapsto \mathcal{O}_{X^{\mathrm{def}}} \otimes_{\mathcal{O}_X^{\mathrm{sh}}} F^{\mathrm{sh}}.$$

Then $\mathcal{O}_X^n \xrightarrow{g} \mathcal{O}_X^m \to F$ in $\mathsf{Coh}(X)$ gives rise to $\mathcal{O}^n_{X^{\mathrm{def}}} \xrightarrow{g} \mathcal{O}^m_{X^{\mathrm{def}}} \to F^{\mathrm{def}}$ in $\mathsf{Coh}(X^{\mathrm{def}})$.

Theorem 8.22 (GAGA)**.** *For an algebraic scheme X over $\mathbb{C}$, the functor $(-)^{\mathrm{def}}$: $\mathsf{Coh}(X) \to \mathsf{Coh}(X^{\mathrm{def}})$ is fully faithful, exact, and closed under subobjects and quotients.*

Corollary 8.23 (Definable Chow)**.** *For an algebraic scheme X over $\mathbb{C}$, any closed definable analytic subspace $\mathcal{Y} \subset X^{\mathrm{def}}$ is algebraic: we have $\mathcal{Y} = Y^{\mathrm{def}}$ for Y in $(\mathsf{AlgSch}/\mathbb{C})$.*

9 Benjamin Bakker: Definable analytic geometry, IV

9.1 Definable GAGA

Theorem 9.1 (Serre, Grothendieck)**.** *Let X be a proper algebraic scheme over $\mathbb{C}$. Then*

$$(-1)^{\mathrm{an}} : \mathsf{Coh}(X) \xrightarrow{\sim} \mathsf{Coh}(X^{\mathrm{an}})$$

is an equivalence of categories.

Remark 9.2. If X is not proper there are counterexamples:

1. $(-1)^{\mathrm{an}}$ is not essentially surjective: let $\mathcal{Y} = \mathrm{Graph}(e^z) \subset \mathbb{C} \times \mathbb{C}^*$ and consider $\mathcal{O}_{\mathcal{Y}}$ in $\mathsf{Coh}(X^{\mathrm{an}})$.

2. $(-1)^{\mathrm{an}}$ is not full: consider the morphism $\mathcal{O}_{\mathbb{C}} \xrightarrow{\cdot e^z} \mathcal{O}_{\mathbb{C}}$ in $\mathsf{Coh}(X^{\mathrm{an}})$.

Example 9.3.

1. Let X be a non-proper genus $\geqslant 1$ curve. Then there are many non-isomorphic line bundles. However, the all the analytified line bundles on X^{an} are trivial.

2. Let $X = \mathbb{A}^2 \backslash \{0\}$. Now there are no non-trivial algebraic line bundles, but after analytifying, there are many non-isomorphic ones.

Hidden behind these statements there are essential singularities causing problems.

By Yohan's talk (see Section 12.2), quotients by closed étale equivalence relations exist in $\mathsf{DefAnSp}/\mathbb{C}$. As such quotients do not exist in the category of schemes, we have to turn to the category of algebraic spaces. We are thus in the following situation:

$$\begin{tikzcd}
(\mathsf{AlgSch}/\mathbb{C}) \arrow[rr] \arrow[dr, "(-)^{\mathrm{def}}"] & & (\mathsf{AlgSp}/\mathbb{C}) \arrow[dl, "(-)^{\mathrm{def}}"] \\
& (\mathsf{DefAnSp}/\mathbb{C}). &
\end{tikzcd}$$

Let us assume all objects are separated.

Theorem 9.4. *For an algebraic space X over $\mathbb{C}$ which is separated and of finite type over $\mathbb{C}$, the functor*

$$(-1)^{\mathrm{def}} : \mathsf{Coh}(X) \to \mathsf{Coh}(X^{\mathrm{def}})$$

is fully faithful, exact, and the essential image is closed under subobjects and quotients.

Example 9.5. Let $\mathcal{E} \subset F^{\mathrm{def}}$ be a coherent subsheaf of a coherent sheaf F^{def} that is the definabilization of a coherent sheaf F on the algebraic space X. Then $\mathcal{E} = E^{\mathrm{def}}$ for a coherent sheaf E on X.

Corollary 9.6 (Definable Chow$^+$)**.** *Let $\mathscr{Z} \subset X^{\mathrm{def}}$ be a definable analytic subspace. Then $\mathscr{Z} = Z^{\mathrm{def}}$ for an algebraic subspace $Z \subset X$.*

Proof. Consider the sequence $\mathcal{O}_X^{\mathrm{def}} \to \mathcal{O}_{\mathscr{Z}} \to 0$. □

Example 9.7. The functor $(-1)^{\mathrm{def}}$ in Theorem 9.4 is not essentially surjective. Let $X = \mathbb{G}_m$. Let V_λ be a rank 1 local system on X^{an} with monodromy $\alpha = e^{2\pi i\lambda}$. We glue these local systems on $\mathbb{G}_m^{\mathrm{def}}$ by identity maps and twists by α. This makes $\mathcal{L}_\lambda = V_\lambda \otimes \mathcal{O}_{X^{\mathrm{def}}}$ a definable line bundle. Then $\mathcal{L} = L^{\mathrm{def}}$ if and only if $\mathcal{L}$ has a nowhere zero definable section, that is, there exists a function

$$f : \mathbb{G}_m^{\mathrm{def}} \to \mathbb{C}^*$$

which is a multivalued definable holomorphic function with monodromy α. If we want to make such a function, we set

$$f = e^{\lambda \log q + g(q)} q^n$$

which we can define on Δ^* and extend to $\mathbb{C}^*$.

<u>Fact:</u> Definable functions in $\mathbb{R}_{\mathrm{an}}$ are polynomially bounded.

If $S \subset \mathbb{R}_{\mathrm{an}}$, and $f = q^n e^{\lambda \log q}$, then this is not definable if $\lambda \notin \mathbb{R}$; and if we restrict to the ray $q = t \in \mathbb{R}_{>0}$, then $f = t^{n+\lambda}$ is not definable if $\lambda \notin \mathbb{R}$. It is only definable for $S \subset \mathbb{R}_{\mathrm{alg}}$ and $\lambda \in \mathbb{Q}$.

<u>Exercise:</u> Show that X is a definable topological space. Show that there is a definable cover $\{X_i\}$ such that each X_i is contractible. (Hint: use cell decomposition.)

Remark 9.8.

1. Proving the faithfulness and exactness in Theorem 9.4 is easy: this amounts to observing that the functors $\mathsf{Coh}(X) \to \mathsf{Coh}(X^{\mathrm{an}})$ and $\mathsf{Coh}(X^{\mathrm{def}}) \to \mathsf{Coh}(X^{\mathrm{an}})$ are faithful and exact, hence the functor $\mathsf{Coh}(X) \to \mathsf{Coh}(X^{\mathrm{def}})$ is so, too.

2. The fulness claim follows from the subobject claim, for a map $\varphi : E^{\mathrm{def}} \to F^{\mathrm{def}}$ between object in the essential image of $(-1)^{\mathrm{def}}$ is the same as giving a map

$$\mathrm{id} \oplus \varphi : E^{\mathrm{def}} \hookrightarrow (E \oplus F)^{\mathrm{def}}.$$

3. The subobject claim is true if and only if the quotient claim is true: consider a sequence

$$0 \to \mathcal{E} \to F^{\mathrm{def}} \to \mathcal{G} \to 0.$$

Proof. (of Theorem 9.4, Bakker, Brunebarbe, Tsimerman). Let X be a separated finite type algebraic space over $\mathbb{C}$. Let us prove the subobject claim: assume the existence of an exact sequence

$$0 \to \mathcal{E} \to F^{\mathrm{def}} \to \mathcal{G} \to 0 \tag{1}$$

in $\mathsf{Coh}(X^{\mathrm{def}})$.

Exercise / Step 1: Reduce to the case that X is reduced: show that, for $X' \subset X$ cut out by $I^2_{X'/X} = 0$, Theorem 9.4 for X' implies Theorem 9.4 for X.

Assume X is reduced. Step 2: If $\mathcal{E}, \mathcal{F}, \mathcal{G}$ in 1 are vector bundles, then $\mathcal{E} = E^{\mathrm{def}}$. Proof: we have $\mathbb{A}(\mathcal{E}) \subset \mathbb{A}(F)^{\mathrm{def}}$, where $\mathbb{A}(\mathcal{E})$ is the total space of $\mathcal{E}$, and the inclusion is as closed analytic definable subspaces. By Theorem 9.6, $\mathbb{A}(\mathcal{E})$ is algebraic. But if the total space of a coherent sheaf is algebraic, this implies that the sheaf is algebraic. Therefore, $\mathcal{E} = E^{\mathrm{def}}$.

Step 3: For an affine Zariski open $\varnothing \neq U \subset X$, the sheaf $\mathcal{E}|_{U^{\mathrm{def}}}$ is algebraic. That is, $\mathcal{E}|_{U^{\mathrm{def}}} \subset (F|_U)^{\mathrm{def}}$ is algebraic. Proof: We are in the situation of Step 2 on the complement of a definable analytic subspace $\mathscr{Z} \subset X^{\mathrm{def}}$ cut out by some minors of presentation matrices. Now $\mathscr{Z}^{\mathrm{red}} \subset X^{\mathrm{def}}$ has a definable structure, and therefore is algebraic by Theorem 9.6, i.e. $\mathscr{Z}^{\mathrm{def}} = Z'^{\mathrm{def}}$. Take $U = X \backslash Z'$.

Step 4: Use induction.
Exercise: We can assume $\mathcal{E}|_{U^{\mathrm{def}}} = (F|_U)^{\mathrm{def}}$, and by looking at sequence 1 we see that $\mathcal{G}$ is supported on a strictly smaller definable analytic subspace $\mathscr{Z} = \mathrm{Supp}(\mathcal{G}) \subset X^{\mathrm{def}}$. So sequence (1) extends to a diagram

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{E} \arrow[r] & F^{\mathrm{def}} \arrow[r] & \mathcal{G} \arrow[r] & 0 \\ & & & F^{\mathrm{def}}/I_{\mathscr{Z}}F^{\mathrm{def}}. \arrow[u] & \end{tikzcd} \tag{2}$$

Now $\mathscr{Z}$ is not algebraic, however, by Theorem 9.6, $\mathscr{Z}^{\mathrm{red}} = Z_0^{\mathrm{def}}$ is algebraic. By Corollary 8.18 we have $I^n_{\mathscr{Z}} \subseteq I_{\mathscr{Z}}$ (the nilpotents are bounded). Hence diagram (2) extends to a diagram

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{E} \arrow[r] & F^{\mathrm{def}} \arrow[rr] \arrow[d, two heads] & & \mathcal{G} \arrow[r] & 0 \\ & & F/I_{\mathscr{Z}^{\mathrm{red}}}F^{\mathrm{def}} \arrow[rr, two heads] \arrow[urr, two heads] \arrow[d, equal] & & F^{\mathrm{def}}/I_{\mathscr{Z}}F^{\mathrm{def}} \arrow[u] & \\ & & (F/I_{\mathscr{Z}^{\mathrm{red}}}F)^{\mathrm{def}}. & & & \end{tikzcd} \tag{3}$$

$\square$

9.2 Definable images

Recall Definable Chow$^+$ (Theorem 9.6), with slogan 'definable subspaces of algebraic spaces are algebraic'.

Theorem 9.9 (Definable images). *Suppose X is an algebraic space over $\mathbb{C}$, let $\mathcal{V}$ be a definable analytic space, and*

$$\varphi : X^{\mathrm{def}} \to \mathcal{V}$$

be a analytically proper morphism of definable analytic spaces. Then there exists a up to isomorphism unique morphism $g^{\mathrm{def}} : X^{\mathrm{def}} \to Y^{\mathrm{def}}$ fitting in a commutative diagram

$$\begin{tikzcd}
X^{\mathrm{def}} \arrow[rr, "\varphi"] \arrow[dr, "g^{\mathrm{def}}"'] & & \mathcal{V} \\
& Y^{\mathrm{def}} \arrow[ur, hook, "\iota"'] &
\end{tikzcd}$$

where ι is a closed immersion and where $g : X \to Y$ is a scheme theoretically surjective morphism of algebraic spaces, meaning that it is surjective on points and that the sequence $0 \to \mathcal{O}_Y \to g_* \mathcal{O}_X$ is exact.

Proof. Assume X is reduced and irreducible. The image of φ exists analytically (Remmert), and it exists in the definable analytic category. Assume φ is surjective, so that $\mathcal{Y} = \mathcal{V}$. We need to algebrise $\varphi : X^{\mathrm{def}} \to \mathcal{Y}$.

<u>Step 1:</u> Reduce to φ being a modification (note that φ is proper so the fibers are proper analytic subspaces of X). Let $\mathrm{Hilb}^0(X)$ be the compact subscheme of $\mathrm{Hilb}(X)$ containing a general fiber of φ. We obtain the diagram

$$\begin{tikzcd}
& & (Z_H)^{\mathrm{def}} \arrow[dl] \arrow[d] & \\
X_0^{\mathrm{def}} \arrow[r, hook] \arrow[ddr] & X^{\mathrm{def}} \arrow[ddr] & H^{\mathrm{def}} \arrow[d, equal] & (\mathrm{Hilb}^0(X))^{\mathrm{def}} \\
& & \overline{s(\mathcal{U})} \arrow[d] & s(\mathcal{U}) \arrow[l, hook'] \arrow[u, hook] \arrow[d, equal] \\
& \mathcal{Y}_0 \arrow[r, hook] & \mathcal{Y} & \mathcal{U}, \arrow[l, hook']
\end{tikzcd}$$

where $s(\mathcal{U})$ is closed in an open definable analytic subset of $(\mathrm{Hilb}^0(X))^{\mathrm{def}}$, for $\mathcal{U} \subset \mathcal{Y}$ a Zariski open in $(\mathsf{DefAnSp}/\mathbb{C})$ on which φ is flat.

Now ψ is definable analytic, and has the same image as φ. Therefore, it is enough to algebrise ψ. Then $\mathcal{Y}$ is algebraic, and so by Theorem 9.6, φ is algebraic.

Step 2: Induction. We have the reduced exceptional locus $W^{\mathrm{def}} \subset X^{\mathrm{def}} \xrightarrow{\varphi} \mathcal{Y}$, and the map $W^{\mathrm{def}} \to \mathcal{Y}$ factors through $W^{\mathrm{def}} \xrightarrow{g^{\mathrm{def}}} Z^{\mathrm{def}}$, of which the image is definable by induction.

Theorem 9.10 (Artin)**.** *φ is algebraic if and only if $\widehat{X_W} = \overline{W} \xrightarrow{\bar{\varphi}} \overline{Z} = \widehat{\mathcal{Y}_Z}$ is algebraic, i.e.. if and only if all the $\varphi_n : W_n \to Z_m$ for W_n and Z_m the n-th (resp. m-th) order neighborhoods are algebraic.*

One shows that the φ_n are algebraic by algebrising. Z_m is a non-reduced space, with reduced space Z. Use Theorem 9.6. $\square$

9.3 Applications

Theorem 9.11. *Let X be a reduced algebraic space over $\mathbb{C}$, and let*

$$\varphi : X^{\mathrm{an}} \to G(\mathbb{Z})\backslash\mathcal{D}$$

be a period map associated to a $\mathbb{Z}$-polarised variation of Hodge structures. Then there is uniquely (up to isomorphism) a factorisation

$$\begin{tikzcd} X^{\mathrm{an}} \arrow[rr, "\varphi"] \arrow[dr, "g^{\mathrm{an}}"'] & & G(\mathbb{Z})\backslash\mathcal{D} \\ & Y^{\mathrm{an}}, \arrow[ur, hook, "\iota"'] & \end{tikzcd}$$

where ι is a closed immersion and $0 \to \mathcal{O}_Y \to g_* \mathcal{O}_X$ is exact. Moreover, Y is quasi-projective.

Remark 9.12.

1. Theorem 9.11 was conjectured by Griffiths in 1969, and it was proved by him in the case X is proper. Also, some special cases were known, due to Sommese.

2. There is a natural line bundle L on $G(\mathbb{Z})\backslash\mathcal{D}$ defined as $L = \otimes_i \det F^i$. For curvature reasons, one expects this to be ample.

3. It is not strictly necessary to assume X is reduced, but in case one does not, one needs to assume φ is definable.

Proof. The first part of Theorem 9.11 will follow from Theorem 9.9 together with the definability of period maps (see...). As soon as we embed the period map into a proper period map, which can be done as Griffiths has shown. $\square$

Corollary 9.13 (Borel-Algebraicity-Theorem)**.** *Suppose X is an algebraic space with a quasi finite period map, i.e. $\varphi : X \to G(\mathbb{Z})\backslash\mathcal{D}$ having finite fibers (for example, consider the local Torelli map for an irreducible variety X). Suppose Z is a reduced algebraic space. Any period map ($Z^{\mathrm{an}} \xrightarrow{\psi} X^{\mathrm{an}}$) is algebraic.*

Proof. We obtain the diagram

$$\begin{tikzcd}
X^{\mathrm{an}} \arrow[r] & G(\mathbb{Z})\backslash\mathcal{D} \\
Z^{\mathrm{an}}. \arrow[u, "\psi"] \arrow[ur] &
\end{tikzcd}$$

Now ψ is definable as all period maps are, therefore algebraic by Theorem 9.6. □

Corollary 9.14. *Let Z be a reduced algebraic space. Any analytic family of polarised Calabi-Yau manifolds over Z^{an} is algebraic.*

Corollary 9.15. *Suppose $\mathcal{X}$ is a reduced separated Deligne-Mumford stack with a quasi-finite period map. Then its coarse moduli space is algebraic.*

10 Yohan Brunebarbe: Applications to Hodge theory, I

10.1 Intro: What is Hodge theory?

Let X be a smooth projective variety over $\mathbb{C}$. Let $k \geqslant 0$, and $H^k(X, \mathbb{Z})$ be the singular integral cohomology group of degree k. This has a polarizable $\mathbb{Z}$-Hodge structure.
There will be a functor: (geometric objects) $\rightarrow$ (linear algebra data).
Useful: - for 1. to study many moduli spaces of algebraic varieties (e.g. abelian varieties, curves, $K3$-surfaces, ...), and - for 2. the study of algebraic cycles on X. This is related to the Hodge conjecture. Loosely formulated: can we understand the existence of algebraic cycles on X in terms of linear algebra data?
A big drawback: of the theory is that the classifying spaces for Hodge structures are in general not algebraic. But they turn out to be definable in $\mathbb{R}_{\text{alg}}$, and the period maps are definable (in $\mathbb{R}_{\text{an,exp}}$).
Goals:

1. 'reprove' algebraicity of Hodge loci
2. 'algebraicity' of period maps (algebraic in some sense).

10.2 Recollections from Hodge theory

Definition 10.1. Let H be a finite $\mathbb{Z}$-module. A Hodge structure on H is a decomposition into complex vector spaces

$$H_{\mathbb{C}} := H \otimes_{\mathbb{Z}} \mathbb{C} = \bigoplus H^{p,q}$$

satisfying $\overline{H^{pq}} = H^{qp}$.

We write $h^{pq} = \dim H^{pq}$, these are called Hodge numbers. We say that the hodge structure is pure of weight n if $h^{pq} = 0$ for $p + q \neq n$. If H is a $\mathbb{Z}$-Hodge structure of weight n, then it is determined by its Hodge filtration $F^p = \oplus_{r \geqslant p} H^{r,s}$. Note that

$$H^{pq} = F^p \cap \overline{F^q}.$$

Conversely, a descending filtration $F^\bullet$ on $H_{\mathbb{C}}$ is a $\mathbb{Z}$-Hodge structure of weight n if and only if it satisfies $F^p \oplus \overline{F^{n-p+1}} = H_{\mathbb{C}}$ for all p. Let us proceed to the main example.

Theorem 10.2. *[Hodge, 40's] Let X be a smooth projective variety over $\mathbb{C}$. Then for any $k \geqslant 0$, the singular cohomology in degree k $H^k(X, \mathbb{Z})$ is equipped with a canonical pure Hodge structure of weight k, and this is functorial with respect to algebraic maps.*

Proof. We have $H^k(X, \mathbb{C}) = H^k_{dR}(X, \mathbb{C})$. By Hodge theory, these classes are in linear bijection with classes of harmonic forms, and so we can use the decomposition of Harmonic forms in terms of type. $\square$

There is an alternative proof using holomorphic Poincare

Lemma 10.3. *For a complex manifold X, and the constant sheaf $\mathbb{C}$ on X, the following complex defines a resolution of $\mathbb{C}$:*

$$0 \to \mathbb{C}_{X^{\mathrm{an}}} \to \mathcal{O}_{X^{\mathrm{an}}} \to \Omega_{X^{\mathrm{an}}} \to \ldots \to \Omega^n_{X^{\mathrm{an}}} \to 0$$

Proof. (of Theorem 10.2). By Lemma 10.3, we have $H^k(X^{\mathrm{an}}, \mathbb{C}) = \mathbb{H}(X^{\mathrm{an}}, \Omega^\bullet_{dR,X^{an}})$. We also have $F^p\Omega^\bullet = (0 \to 0 \to \ldots \to \Omega^p \to \ldots)$. This induces a filtration F^p on $\Omega^\bullet$, and hence a filtration on $\mathbb{H}^k(X^{\mathrm{an}}, \Omega^\bullet)$:

$$F^p\mathbb{H}^k(X^{an}, \Omega^\bullet) = \mathrm{Im}(\mathbb{H}^k(F^p\Omega^\bullet) \to \mathbb{H}^k(X, \Omega^\bullet)).$$

By Hodge theory, we get the same as via the Hodge filtration on $H^k(X, \mathbb{C})$. By Hodge theory, the corresponding spectral sequence

$$E_1^{p,q} = H^q(X^{an}, \Omega^p_{X^{an}}) \implies H^k(X, \mathbb{C})$$

degenerates at E_1, and this degeneracy implies the Hodge structure.

$$\mathrm{Gr}^p_F H^k(X, \mathbb{C}) = H^q(X, \Omega^p) \cong H^{p,q}.$$

In fact, $\mathbb{H}^k(X^{an}, \Omega^\bullet_{X^{an}}) = \mathbb{H}^k(X, \Omega^\bullet)$ (GAGA) . □

Definition 10.4. Suppose $H_\mathbb{Z}$ carries weight n Hodge structure, and let $q_\mathbb{Z}$ be a $(-1)^n$-symmetric bilinear form (symmetric if n is even, skewsymmetric otherwise) on $H_\mathbb{Z}$.

1. The Weil operator $C \in \mathrm{End}(H_\mathbb{R})$ is the real endomorphism acting by multiplication by $i^{p,q}$ on H^{pq}. (This commutes with complex conjugation, so is defined over $\mathbb{R}$ indeed).

2. The Hodge form is a hermitian form h on $H_\mathbb{C}$ defined by $h(u, v) = q_\mathbb{C}(C \cdot v, \bar{v})$¿

3. We say: the Hodge structure on $H_\mathbb{Z}$ is polarised by q if h is positive definite and the Hodge decomposition is h-orthogonal.

If $(H_\mathbb{Z}, \oplus H^{pq})$ is poarised by $q_\mathbb{Z}$, then the Hodge filtration $F^\bullet$ is $q_\mathbb{C}$-isotropic.: $(F^p)^\perp = F^{n+1-p}$.

Example 10.5. Let X be a smooth be a smooth projective variety. Consider $X \subset \mathbb{P}^N(\mathbb{C})$. There is a hyperplane class $\mu = [H|_X] \in H^2(X, \mathbb{Z})$ (which is $c_1(\mathcal{O}(1))|_X$). Then define

$$L : H^i(X, \mathbb{Z}) \to H^{i+2}(X, \mathbb{Z}), \quad \alpha \mapsto \mu \wedge \alpha.$$

For $k \leqslant \dim(X)$, we get a map

$$H^k(X, \mathbb{Z}) \xrightarrow{L^{n-k}} H^{2n-k}(X, \mathbb{Z}).$$

Theorem 10.6 (Lefschetz)**.** *If one replaces $\mathbb{Z}$ with $\mathbb{Q}$, this map becomes an isomorphism.*

Denote $H^k_{\mathrm{prim}}(X,\mathbb{Z}) := \mathrm{Ker}(L^{n-k+1}) \subseteq H^k(X,\mathbb{Z})$. We have $\mu \in H^2(X,\mathbb{Z}) \cap H^{1,1}$. Therefore, the wedge by μ is a morphism of Hodge structures. This implies that L^{n-k+1} is a morphism of Hodge structures, and so the kernel $H^k_{\mathrm{prim}}(X,\mathbb{Z})$ has a Hodge structure. The Lefschetz decomposition then says that

$$H^k(X,\mathbb{Q}) = \otimes_{i \geqslant 0} L^i H^{k-2i}_{\mathrm{prim}}(X,\mathbb{Q}).$$

There is a bilinear form $H^k_{\mathrm{prim}}(X,\mathbb{Z}) \times H^k_{\mathrm{prim}}(X,\mathbb{Z}) \to \mathbb{Z} = H^{2\dim(X)}(X,\mathbb{Z})$ sending $(\alpha,\beta) \mapsto (-1)^{k(k-1)/2}\mu^{n-k} \wedge \alpha \wedge \beta$. Claim: this is a well-defined bilinear form, and defines a polarisation on $H^k_{\mathrm{prim}}(X,\mathbb{Z})$.

Remark 10.7. If $H_{\mathbb{Z}}$ carries a pure Hodge structure, then $H_{\mathbb{Z}}^{\otimes l}$, $\mathrm{Sym}^l(H_{\mathbb{Z}})$, $\bigwedge^l H_{\mathbb{Z}}$ inherit a pure Hodge structure. The same is true for polarised Hodge structures. This implies that the category of $\mathbb{Z}$- or $\mathbb{Q}$-Hodge structures is an abelian category with tensor products. The category of $\mathbb{Q}$-Hodge structures whose homogeneous component admit a polarisation, is moreover semi-simple.

10.3 Hodge structures in weight one

The functor

(Complex tori $\mathbb{C}^g / \mathbb{Z}^{2g}$) $\to$ (weight -1 Hodge structure with decomposition types $(-1,0)$ and $(0,-1)$)

defined by $X \mapsto H_1(X,\mathbb{Z})$ is an equivalence of categories. If you restrict this two abelian varieties, you get (abelian varieties) $\cong$ (polarisable weight -1 Hodge structures with type $(-1,0)$ and $(0,-1)$). Proof: exercise. Every weight -1 polarisable Hodge structure comes from geometry! This is not true in higher dimensions.

10.4 Classifying spaces

(= quotients of period domains)
Fix an integer n. Let H be a $\mathbb{Z}$ module of finite type, and q a $(-1)^n$-symmetric bilinear form on H, let $h^{p,q}$ be a collection of nonnegative integers such that $h^{pq} = h^{qp}$ and $(h^{pq} \neq 0 \implies p+q=n)$ and $\dim H_{\mathbb{C}} = \sum h^{pq}$.

The associated period domain:

$\hat{\mathcal{D}} \supset \mathcal{D} = \{$ Hodge structures of weight n on H polarised by q and with $\dim H^{pq} = h^{pq}\}$

where

$$\hat{\mathcal{D}} = \{ \text{ filtrations on } H_{\mathbb{C}},\ q_{\mathbb{C}}\text{-isotropic, and such that } \dim(F^p) = \sum_{r \geqslant p} h^{r,s} .\}$$

The condition of being $q_{\mathbb{C}}$-isotropic is a closed condition, therefore $\hat{\mathcal{D}} \subset \prod \mathrm{Grass}$ is a closed analytic subvariety. If $G = \mathrm{Aut}(H,q)$ (this is an algebraic subgroup of $\mathrm{GL}(H)$

defined over $\mathbb{Q}$), then $G(\mathbb{C})$ acts transitively on $\hat{\mathcal{D}}$, via $g \cdot F^p = g(F^p)$. So this gives an action of the Lie group $G(\mathbb{C})$, hence $\hat{\mathcal{D}}$ is in fact a complex manifold.
Now $\mathcal{D}$ is an open subset of $\hat{\mathcal{D}}$ in the Hausdorff topology, which is a smooth projective manifold, hence $\mathcal{D}$ is a complex manifold.

Note that $G(\mathbb{R})$ acts transitively on $\mathcal{D}$, and $G(\mathbb{C})$ on $\hat{\mathcal{D}}$. We get that

$$\{F^\bullet\} \in \mathcal{D} = G(\mathbb{R})\backslash K_{F^\bullet} \subset G(\mathbb{C})\backslash B_{F^\bullet} = \hat{\mathcal{D}}$$

where $K_{F^\bullet} = \mathrm{Stab}_{F^\bullet}$ =compact subgroup of $G(\mathbb{R})$.

Now $G(\mathbb{Z}) \subset G(\mathbb{C})$ and $G(\mathbb{Z})$ acts properly discontinuously on $\mathcal{D}$. The stabilizer $G(\mathbb{Z}) \cap K$ is discreet and compact, thus finite. We get a map

$$G(\mathbb{Z}) \cap \mathcal{D} \to G(\mathbb{Z})\backslash\mathcal{D}$$

where the latter is a normal complex analytic space (no fixed points $\implies$ manifold).

10.5 Period maps

Consier a smooth projective morphism $\mathcal{X} \to S$, for complex manifolds $\mathcal{X}$ and S, and let $s \in S$. We get a diagram

$$\begin{tikzcd}
 & \mathbb{P}^N \times S \arrow[dr] & \\
\mathcal{X} \arrow[ur] \arrow[rr] & & S \\
\mathcal{X}_S \arrow[u] \arrow[rr] & & s. \arrow[u]
\end{tikzcd}$$

Let $n \geqslant 0$. Locally on S, for every $s \in S$, there is an isomorphism $(H^n_{\mathrm{prim}}(\mathcal{X}_s, \mathbb{Z}), q) \cong (H, q)$ where H is a $\mathbb{Z}$-module of finite type, and q a $(-1)^n$-symmetric bilinear form on H. This gives, for each $s \in S$, a q-polarised Hodge structure of weight n on H. We obtain a map

$$S \xrightarrow{\pi} G(\mathbb{Z})\backslash\mathcal{D}$$
$$s \mapsto [H^n_{\mathrm{prim}}(\mathcal{X}_s, \mathbb{Z})]/_{\mathrm{iso}}.$$

(Alternatively, go to the universal cover of S first, and define the morphism from there).

Theorem 10.8 (Griffiths)**.** *The map π is locally liftable and holomorphic.*

$$\begin{tikzcd}
 & \mathcal{D} \arrow[d] \\
S \arrow[ur, dotted] \arrow[r, "\pi"] & G(\mathbb{Z})\backslash\mathcal{D}
\end{tikzcd}$$

(Remark that this is true if and only if the Hodge filtration on (H, q) varies holomorphically.)

Theorem 10.9 (Griffiths)**.** *The map π is a horizontal map, i.e. its differential takes values in the horizontal tangent bundle of $G(\mathbb{Z})\backslash\mathcal{D}$.*

Define $\mathrm{G}_{\mathbb{C}} = \mathrm{Lie}(G(\mathbb{C})) = \{x \in \mathrm{End}(H_{\mathbb{C}}) : q(xv,\mu) + q(\mu, xv) = 0 \forall v, \mu \in H_{\mathbb{C}}\} \subset \mathrm{End}(H_{\mathbb{C}})$. Then for all $\{F^\bullet\} \in \hat{\mathcal{D}}$, we get an induced filtration on $\mathrm{G}_{\mathbb{C}}$:

$$F^p\mathrm{G}_{\mathbb{C}} = \{x \in \mathrm{G}_{\mathbb{C}} : x(F^k H_{\mathbb{C}}) \subset F^{k+p}H_{\mathbb{C}} \quad \forall k\}.$$

At $\{F^\bullet\} \in \hat{\mathcal{D}}$, we have

$$T_{\hat{\mathcal{D}}}|_{\{F^\bullet\}} = \mathrm{G}_{\mathbb{C}}/b_{\mathbb{C}} \qquad (\text{ quotient by stabiliser, } b_{\mathbb{C}} = F^\bullet\mathrm{G}_{\mathbb{C}} \text{ }).$$

This implies that

$$F^{-1}/F^0 = \mathcal{H} \subset T_{\hat{\mathcal{D}}} = \mathrm{G}_{\mathbb{C}}/F^0.$$

Exercise: Consider a period domain $\mathcal{D} \subset \hat{\mathcal{D}}$ for which $\mathcal{H} \subsetneq T_{\hat{\mathcal{D}}}$. Prove that 'most' elements in $\mathcal{D}$ do not come from geometry (i.e., cannot be obtained as the primitive cohomology of a smooth projective variety over $\mathbb{C}$).
Exercise: Consider the following example: $\mathbb{Z}^2$, with weight 1, Hodge structures on $\mathbb{Z}^2$. Show that (open unit disc) $=: \Delta = \mathcal{D} \subset \hat{\mathcal{D}} = \mathbb{P}^1$.

11 Yohan Brunebarbe: Applications to Hodge theory, II

Problem: Let $\mathcal{X} \to S$ be a smooth projective morphism of varieties. Fix $s \in S$, and $Z_s \subset \mathcal{X}_s$ a closed algebraic subvariety. Question: Can we deform Z_s for $t \in S$ close to s?

11.1 Hodge Conjecture

Definition 11.1. A Hodge class of degree $2k$ in X is an element $\alpha \in H^{2k}(X,\mathbb{Q}) \cap H^{k,k}$. Denote $\mathrm{Hdg}^{2k}(X) = H^{2k}(X,\mathbb{Q}) \cap H^{k,k}$.

Lemma 11.2. *Let $\alpha \in H^{2k}(X,\mathbb{Q})$. Then $\alpha \in \mathrm{Hdg}^{2k}(X)$ if and only if $\alpha \in F^kH^{2k}(X,\mathbb{C})$.*

Proof. $H^{k,k} = F^k \cap \overline{F^k}$, as α has rational coefficients, it is stabilised by complex conjugation. $\square$

Remark 11.3. Let $Z \subsetneq X$ a closed irreducible subvariety of complex dimension k. To Z, one can associate a Hodge class as follows: take $\tilde{Z} \to Z$ a desingularization. The composite of algebraic maps ($\tilde{Z} \to Z \to X$) defines a morphism of $\mathbb{Q}$ or $\mathbb{Z}$ Hodge structures: $H^{2n-2k}(X,\mathbb{Q}) \to H^{2n-2k}(\tilde{Z},\mathbb{Q}) = \mathbb{Q}$. In this way we get an element of $\mathrm{Hom}(H^{2n-2k}(X,\mathbb{Q}),\mathbb{Q}) \cong H^{2k}(X,\mathbb{Q})$ (Poincaré duality), which we denote by $[Z] \in \mathrm{Hdg}^{2k}(X)$ (indeed, this element lands in $H^{k,k}$ because the Hodge decomposition is functorial).

Conjecture 11.4 (Hodge, 1950)**.** *Hodge classes on X are linear combinations with rational coefficients of classes $[Z]$ of algebraic cycles $Z \subset X$.*

Remark 11.5. Hodge asked this with $\mathbb{Z}$-coefficients, but this is false. Atiyah and Hirzebruch proved there exists torsion classes that are of even degree, but not algebraic.

11.2 Kollár counterexamples

Let $X \subset \mathbb{P}^4$ be a hyperplane of degree d. Then $H^4(X,\mathbb{Z}) = \mathbb{Z} \cdot \alpha$. For $h = [\text{Hyperplane}|_X] \in H^2(X,\mathbb{Z})$, we have $< h, \alpha >= 1$, but by Poincaré duality, this is false. Observe that $d \cdot \alpha = h^2$ (since both have intersection number d with h), hence $d \cdot \alpha$ is algebraic.

Theorem 11.6 (Kollár)**.** *Let p be a prime different from 2 and 3. Let $d \in \mathbb{Z}_{\geqslant 1}$ such that $p^3|d$. Then, for very general hypersurface of degree d in $\mathbb{P}^4$, the class α is not algebraic.*

Theorem 11.7 (Soulé-Voisin)**.** *The locus where α is algebraic is a dense countable union of proper algebraic subsets of the parameter space.*

Note that 11.7 indeed implies that for a general hypersurface, the class of α is not algebraic, because the complement cannot be the entire space by Baire's category theorem (a topological argument).

11.3 Hodge classes in families

Let $\mathcal{X} \to S$ be a smooth projective morphism.

Lemma 11.8 (Ehresmann)**.** *Any proper submersive surjective map between smooth manifolds is locally on the base a fibration. That is, let $0 \in S$ and let $0 \in U \subset S$ be a small open ball around $0 \in S$. Write $\mathcal{X}_0 = X_0$. Then there exists a $\mathcal{C}^\infty$ diffeomorphism $\mathcal{X}|_U \xrightarrow{\sim} X_0 \times U$ over U.*

Because U is contractible, we have, for any $s \in U$, canonical isomorphisms

$$H^k(\mathcal{X}_s,\mathbb{Z}) \cong H^k(\mathcal{X}_U,\mathbb{Z}) \cong H^k(X_0,\mathbb{Z}).$$

Let $\alpha \in \mathrm{Hdg}^{2k}(X_0,\mathbb{Z})$. Question: what is the locus $U_\alpha \subset U$ for which α remains Hodge?

Lemma 11.9. *U_α is a closed analytic subset of U.*

Proof. For all $t \in U$, we have $\alpha_t \in \mathrm{Hdg}^{2k}(\mathcal{X}_t)$ if and only if $\alpha_t \in F^k$ if and only if $[\alpha_t] = 0 \in H^{2k}(\mathcal{X}_t,\mathbb{C})$. So U_α is the vanishing locus of a section of a holomorphic vector bundle (corresponding to the local system $R^{2k}f_*\mathbb{C}$ with $f : \mathcal{X} \to S$), and therefore an analytic subset. ☐

In particular, the germ of U_α at s has a finite number of (analytic) irreducible components. The Hodge conjecture predicts that these are irreducible components of the germ of an algebraic subvariety of S. To sketch why, note that every point of $s \in S$ parameterises a closed subvariety of $\mathcal{X}$ via the Hilbert scheme: with $\mathcal{X} \to S$ we get $\mathrm{Hilb}_{\mathcal{X}/S} \xrightarrow{\varphi} S$, and $\mathrm{Hilb}_{\mathcal{X}/S}$ is a countable union of subvarieties of finite type over S. As $U_\alpha = \overline{\mathrm{Im}(\varphi)}$, this proves what we want.

Theorem 11.10 (Cattani-Deligne-Kaplan, 1995)**.** *The germ U_α at s is 'algebraic' (that is, algebraic is some sense).*

Definition 11.11. The Locus of Hodge Classes (LHC) is the subset of the total space $\mathcal{H}$ of the sheaf $R^{2k}f_*\mathbb{Q}\otimes\mathcal{O}_S$, where

$$\mathcal{H} = \{(s,\alpha) \in S \times \bigcup_{s\in S} H^{2k}(\mathcal{X}_s,\mathbb{C}) : s \in S, \alpha \in H^{2k}(\mathcal{X}_s,\mathbb{C})\},$$

consisting of pairs $(s,\alpha_s) \in \mathcal{H}$: such that $\alpha_s \in \mathrm{Hdg}^{2k}(\mathcal{X}_s)$.

It can be shown that LHC is a countable union of subvarieties of $\mathcal{H}$.

Theorem 11.12 (Cattani-Deligne-Kaplan)**.** *The connected components of the locus of Hodge classes are closed algebraic subsets.*

11.4 Mumford-Tate groups

Let H be a $\mathbb{Q}$-Hodge structure, pure of weight n. Then the group $\mathrm{GL}(H) \to \mathrm{Spec}\,\mathbb{Q}$ acts on

$$T^{a,b} := H^{\otimes a} \otimes (H^\vee)^{\otimes b},$$

where $T^{a,b}$ has an induced Hodge structure. Denote $T := \oplus_{a,b}T^{a,b}$, which is a bigraded $\mathbb{Q}$-algebra. We have $\mathrm{Hdg}^{\bullet,\bullet} \subset T^{\bullet,\bullet}$, a bigraded subalgebra.

Definition 11.13. The Mumford-Tate group $MT(H)$ of H is the subgroup of $\mathrm{GL}(H)$ of elements that fixes the Hodge classes.

By definition, $MT(H)$ is an algebraic subgroup of $\mathrm{GL}(H)$ over $\mathbb{Q}$.

Theorem 11.14.
1. *The elements of $T^{a,b}$ (with coefficients in $\mathbb{Q}$) that are fixed by $MT(H)$ are exactly the Hodge classes - that is, there are no other elements fixed by $MT(H)$ than the Hodge classes.*
2. *The $\mathbb{Q}$-subvector spaces of $T^{a,b}$ stabilised by $MT(H)$ are exactly the sub $\mathbb{Q}$-Hodge structures.* *(Exercise!)*

Theorem 11.15.
- *The algebraic group $MT(H)$ is connected.*
- *If H admits a polarisation q, then $MT(H) \subset \mathrm{Aut}(H,q)$, and $MT(H)$ is a reductive algebraic group.*

Theorem 11.16. *$MT(H)$ can be realised as the fixator of a well-chosen Hodge class (which is not in H).*

Consider $H_{\mathbb{C}} = \oplus_{p+q=n}H^{p,q}$. Define an action of $\mathbb{C}^*$ on $H_{\mathbb{C}}$ as following: $z\cdot v = z^{p-q}v$, for $z \in \mathbb{C}^*$ and $v \in H^{p,q}$. This gives a map

$$h : \mathbb{C}^* \to \mathrm{GL}(H_{\mathbb{C}}).$$

Let $M =$ smallest subgroup of $GL(H)$ defined over $\mathbb{Q}$ such that $h(\mathbb{C}^*) \subset M(\mathbb{C})$.
Claim 1: M verifies the conclusion of Theorem 11.14 and Theorem 11.15. *Proof.* An element $t \in T^{a,b} = \oplus L^{r,s}_{r+s=n(a-b)}$ is fixed by $\mathbb{C}^*$ if and only if $t \in L^{r,r}$. □
Take $t \in T^{a,b}$ fixed by M. In particular, it is fixed by $\mathbb{C}^*$, hence it is a Hodge class.

Proof. (of Theorem 11.15). True because $\mathbb{C}^*$ is connected. $\square$

The category of polarised Hodge structures is semi-simple, hence abelian and every object is a finite direct sum of simple objects (a simple object is an object for which there are precisely two quotient objects, namely the initial object and the object itself). We have $M \subset \mathrm{GL}(H)$ faithfully.

Claim 2: $M = MT(H)$.

Theorem 11.17 (Chevalley)**.** *Let V be a finite dimensional $\mathbb{C}$-vector space. Any subgroup of $\mathrm{GL}(V)$ can be realised as the stabiliser of a line in $T^{\bullet,\bullet}$ (tensor algebra).*

Apply Theorem 11.17 to M, then $M = \mathrm{Stab}(L)$ for a line L defined over $\mathbb{Q}$. Then M must be a sub-Hodge structure. Necessarily, $L = \mathbb{Q}(u)$. In fact, $M = \mathrm{Fix}(t)$ for all $t \in L\backslash\{0\}$. Therefore, t is a Hodge class.

Conclusion: Since M fixes Hodge classes, $\varnothing \neq M \subset MT(H)$. Conversely, $M = \mathrm{Fix}(t) \supset MT(H)$ for a Hodge class t. We conclude that $M = MT(H)$. $\square$

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12.1 Noether-Lefschetz loci

Let $\mathcal{D}$ be a period domain associated to (H, q).

Definition 12.1. For any $\varphi \in \mathcal{D}$, its Noether-Lefschetz locus is

$$\mathrm{NL}(\varphi) = \{\varphi' \in \mathcal{D} : \mathrm{Hdg}_{\varphi'}^{\bullet,\bullet} \supseteq \mathrm{Hdg}_{\varphi}^{\bullet,\bullet}\} = \{\varphi' \in \mathcal{D} : \mathrm{MT}(\varphi') \subseteq \mathrm{MT}(\varphi)\}.$$

Remark 12.2. For $\varphi \in \mathcal{D}$, $\mathrm{NL}(\varphi)$ is the intersection with $\mathcal{D}$ of an algebraic subvariety $\hat{\mathcal{O}}$. Indeed, there exists an element $v \in T^{a,b}$ such that $\mathrm{MT}(\varphi) = \mathrm{Fix}(v)$, therefore

$$\mathrm{NL}(\varphi) = \{\varphi' \in \mathcal{D} : v \text{ is Hodge at } \varphi'\} = \mathcal{D} \cap \{\varphi \in \hat{\mathcal{D}} : v \in F^m T^{a,b}\}.$$

Here $F^m T^{a,b}$ is the middle piece of the Hodge filtration.

Remark 12.3. Let $\{v_i\}$ be a collection of elements in $T^{\bullet,\bullet}$ (with coefficients in $\mathbb{Q}$), and $L \subset \mathrm{GL}(H)$ the fixator if these classes. Then the locus where the classes $\{v_i\}$ are Hodge is equal to

$$\mathrm{NL}(L) := \{\varphi' \in \mathcal{D} : \mathrm{MT}(\varphi') \subset L\}.$$

Then $\mathrm{NL}(\varphi) = \mathrm{NL}(\mathrm{MT}(\varphi))$.

Lemma 12.4. *Let $L \subset \mathrm{GL}(H)$ be a $\mathbb{Q}$-algebraic subgroup. Then any irreducible component $\mathrm{NL}(L)^0$ of $\mathrm{NL}(L)$ is an irreducible component of $\mathrm{NL}(\varphi)$ for every very general φ in $\mathrm{NL}(L)^0$.*

Proposition 12.5. *For every $\varphi \in \mathcal{D}$ with $\mathrm{MT}(\varphi) = M$, $\mathrm{NL}(\varphi) = M(\mathbb{R})\cdot\varphi$. In particular,*

$$\mathrm{NL}(\varphi) \cong M(\mathbb{R})/M(\mathbb{R}) \cap V \subset G(\mathbb{R})/V = \mathcal{D}$$

is a homogeneous complex manifold.

Recall that $\mathcal{D} = H/V$ where $H = G(\mathbb{R})^+$ for some $\mathbb{Q}$-group G.

Proof. We have $M(\mathbb{R}) \cdot \varphi \subset \mathrm{NL}(\varphi)$. Indeed, since $M(\mathbb{R})$ preserves the Hodge classes at φ, we have

$$\mathrm{Hdg}^{\bullet,\bullet}(m \cdot \varphi) \supseteq \mathrm{Hdg}^{\bullet,\bullet}(\varphi) \quad \forall m \in M(\mathbb{R}).$$

□

Theorem 12.6. *Let S be a smooth algebraic complex variety, and $S \to G(\mathbb{Z})\backslash\mathcal{D}$ a period map (i.e. locally liftable, holomorphic, horizontal, but not necessarily coming from geometry). Then the preimage of any Mumford-Tate subvariety of $G(\mathbb{Z})\backslash\mathcal{D}$ is algebraic.*

Definition 12.7. Given a period domain $\mathcal{D}$, a Mumford-Tate subdomain $\mathcal{D}' \subset \mathcal{D}$ is any subvariety of the form

$$\mathrm{NL}(\varphi) = M(\mathbb{R})/(V \cap M(\mathbb{R}))$$

for some $\varphi \in \mathcal{D}$. A Mumford-Tate subvariety of $G(\mathbb{Z})\backslash\mathcal{D}$ is the image by $\mathcal{D} \to G(\mathbb{Z})\backslash\mathcal{D}$ of a Mumford-Tate subdomain.

A Mumford-Tate subvariety of $G(\mathbb{Z})\backslash\mathcal{D}$ is then any analytic subvariety of $G(\mathbb{Z})\backslash\mathcal{D}$ of the form

$$M(\mathbb{Z})\backslash\mathcal{D}' \cong M(\mathbb{Z})\backslash M(\mathbb{R})/(V \cap M(\mathbb{R})).$$

Recall that $M \subset G = \mathrm{Aut}(H, q) \subset \mathrm{GL}(H)$.

Proof. (of Theorem 12.6, by Bakker-Klingler-Tsimerman). Assuming S is compact, the proof is easy: it is a consequence of Theorem 6.1 using that the period map is holomorphic, and that Mumford-Tate subvarieties are closed subvarieties of $G(\mathbb{Z})\backslash\mathcal{D}$. In general, the strategy of Bakker-Klingler-Tsimerman consists of endowing $G(\mathbb{Z})\backslash\mathcal{D}$ with a functorial $\mathbb{R}_{\mathrm{alg}}$ structure: Mumford-Tate subvarieties are definable. Then one proves that period maps are definable in $\mathbb{R}_{\mathrm{an,exp}}$. One concludes by using Theorem 6.3. □

Theorem 12.8. *For any (connected) linear semisimple $\mathbb{Q}$-group, any (torsion-free) arithmetic lattice $\Gamma \subset G(\mathbb{Q})$, and any connected compact subgroup $M \subset G(\mathbb{R})$, the arithmetic quotient*

$$\Gamma\backslash G(\mathbb{R})/M$$

admits a functorial structure of an $\mathbb{R}_{\mathrm{alg}}$ definable manifold.

Definition 12.9. A subgroup $\Gamma \subset G(\mathbb{Q})$ is arithmetic if it is commensurable to $G(\mathbb{Q}) \cap \mathrm{GL}(n, \mathbb{Z})$ for some embedding $G \subset \mathrm{GL}(n, \ )/\mathbb{Q}$.

In the case of quotients of period domains by arithmetic groups ($G(\mathbb{Z})\backslash\mathcal{D}$) or, more generally, a Mumford-Tate subvariety, this structure underlies a definable analytic structure.

Theorem 12.10 (Bakker-Klingler-Tsimerman)**.** *Let S be an algebraic variety. Then period maps*

$$S^{\mathrm{def}} \to \Gamma\backslash\mathcal{D}$$

are definable in $\mathbb{R}_{\mathrm{an,exp}}$.

12.2 Definable quotient spaces

Consider the category of definable topological spaces, with morphims the continuous and definable maps. We shall restrict ourselves to the definable topological spaces X where the diagonal $\Delta_X \subset X \times X$ is closed, i.e. which are Hausdorff as a topological space. If $X \subseteq \mathbb{R}^n$ we call it an affine definable topological space.

Let X be a definable topological space, and let $R \subset X \times X$ be a closed equivalence relation that is definable.

Does there exist a quotient X/R, and in which sense?

Definition 12.11. Let X be a definable topological space, and let $R \subset X \times X$ be a closed equivalence relation which is definable. A categorical quotient of X by R is a morphism $X \to Y$ which is constant on equivalence classes and such that any morphism $X \to Z$ that is constant on equivalence classes factorises uniquely through $X \to Y$.

Definition 12.12. Let X be a definable topological space, and let $R \subset X \times X$ be a closed equivalence relation which is definable. A geometric quotient of X by R is a surjective morphism $p : X \to Y$ where Y has the quotient topology such that the fibers of p are exactly the equivalence classes.

Remark 12.13. A geometric quotient of X by R is also a categorical quotient of X by R (exercise). In particular, it is unique if it exists (up to unique isomorphism).

Theorem 12.14 (Brumfiel, Van den Dries)**.** *Let X be a definable topological space, and let $R \subset X \times X$ be a closed equivalence relation which is definable. If R is definably proper over X, then the geometric quotient exists. Moreover, if X is affine, then X/R is affine.*

Definition 12.15. A morphism $f : X \to Y$ is called definably proper if for any definably compact subset $K \subset Y$, $f^{-1}(K)$ is compact.

Definition 12.16. A closed definable equivalence relation $R \subset X \times X$ is definably proper if the morphism $\mathrm{pr}_1 : R \to X$ is, which is equivalent to pr_2 being definably proper.

12.3 Application

Corollary 12.17. *Let M be a compact real algebraic group equipped with a continuous and semialgebraic action on a semialgebraic space X. Then the orbit space X/M admits a canonical structure of a semialgebraic space. Moreover, the canonical map $X \to X/M$ is semialgebraic.*

Proof. We have

$$X/G = X/R$$

where

$$R = \{(x, x_m) : x \in X, m \in M\} \subset X \times X, \quad G(/RR)/M, \quad M \subset G(\mathbb{R}).$$

□

Lemma 12.18. *Any compact subgroup of* $\mathrm{GL}(n, \mathbb{R})$ *is equal to* $L(\mathbb{R})$ *for* $L \subset \mathrm{GL}(n, \ )$ *an* $\mathbb{R}$*-algebraic subgroup.*

Example 12.19. Consider $\mathrm{SL}(2, \mathbb{Q})$. We have $U(1) \subset \mathrm{SL}(2, \mathbb{R})$, an action of $\mathrm{SL}_2(\mathbb{Z})$ on $\mathbb{H}$, and a diagram

$$\begin{tikzcd}
\mathrm{SL}(2,\mathbb{R})/U(1) \arrow[r, "\cong"] & \mathbb{H} \arrow[d] & F \arrow[l, hook'] \arrow[d] \\
& \mathrm{SL}_2(\mathbb{Z})\backslash \mathbb{H} \arrow[r, equal] & \text{“}\mathrm{SL}_2(\mathbb{Z})'' \backslash F
\end{tikzcd}$$

(where “$\mathrm{SL}_2(\mathbb{Z})$” stands for the equivalence relation obtained from the action by $\mathrm{SL}_2(\mathbb{Z})$ on F).

12.4 Closed étale equivalence relations

Theorem 12.20. *For $R \subset X \times X$ a definable and closed equivalence relation, such that* pr_1 *(or* pr_2*) is étale, the geometric quotient of X by R exists.*

Proof. There exists finitely many open definable subsets U_i of X: restrict to

$$R \cap (U_i \times U_i) = \Delta_i.$$

Remark that the U_i meet every equivalence class. □

13 Yohan Brunebarbe: Applications to Hodge theory, IV

13.1 Definable structures on quotients of period domains

Recall: $G \subset \mathrm{GL}_n$ is a semi-simple $\mathbb{Q}$-algebraic group, $M \subset G(\mathbb{R})$ a compact subgroup, and

$$\Gamma \subset G(\mathbb{Z}) = G(\mathbb{Q}) \cap \mathrm{GL}(n, \mathbb{Z})$$

a torsion-free, finite index subgroup. In Section 12 we have seen that $X = G(\mathbb{R})/M$ has a canonical structure of an $\mathbb{R}_{\mathrm{alg}}$ definable space, for the action of $G(\mathbb{R})$ is definable.

Remark 13.1. When $X = \mathcal{D}$ is a period domain, you can recover this semi-algebraic structure as follows: we have an action of $G(\mathbb{R})$ on $\mathcal{D} \subset \hat{\mathcal{D}}$ and an action of $G(\mathbb{C})$ on $\hat{\mathcal{D}}$, which is a smooth projective variety. You can then recover $\mathcal{D}$ as the orbit of a semialgebraic action, therefore semialgebraic.

Definition 13.2. A fundamental set for the action of Γ on X is an open definable subset $F \subset X$ such that

1. $\Gamma \cdot F = X$
2. $\mathrm{Stab}(F) = \{\gamma \in \Gamma : \gamma F \cap F \neq \varnothing\}$ is finite.

Proposition 13.3. *There exists a unique definable structure on* $\Gamma \backslash X$ *such that the map* $F \to \Gamma \backslash X$ *is definable.*

Proof. Let $R \subset X \times X$ be the equivalence relation associated to the action of Γ on X, that is,

$$R = \{(x, \gamma x) : x \in X, \gamma \in \Gamma\}.$$

Then, as a topological space,

$$\Gamma \backslash X = \Gamma \backslash \Gamma \cdot F = R_F \backslash F,$$

where $R_F = R \cap F \times F$. The equivalence relation R_F on the product $F \times F$ is closed, étale and analytically definable. We have

$$R_F = \bigsqcup_{\gamma \in \mathrm{Stab}(F)} \{(x, \gamma x) : x \in F, \gamma x \in F\},$$

which shows R_F is definable, because each $\{(x, \gamma x) : x \in F, \gamma x \in F\}$ is definable, and it is a finite disjoint union. Therefore $\Gamma \backslash X$ obtains a definable structure from the equality $\Gamma \backslash X = R_F \backslash F$. □

Remark 13.4.
1. The definable structure on $\Gamma \backslash X$ depends on the choice of F.
2. Two fundamental sets F and F' define the same structure on $\Gamma \backslash F$ if and only if F is contained in a finite union of translates of F'.
3. The map $F \to \Gamma \backslash X$ admits locally on the base some continuous definable section. Therefore, giving a morphism from a definable space Y to $\Gamma \backslash X$ is equivalent to giving a finite definable cover $Y = \cup Y_i$ and morphisms $Y_i \to F$ such that the induced maps $Y_i \to R_F \backslash F = \Gamma \backslash X$ coincide on overlaps.
4. This construction is compatible with the complex structure.

13.2 Specify the fundamental sets

1. Consider $\mathrm{GL}(n,\mathbb{Z})\backslash\mathrm{GL}(n,\mathbb{R})/O(n)$. Fact:

$$\mathrm{GL}(n,\mathbb{R}) = \{\begin{pmatrix}1 & * & \cdots & * \\ 0 & 1 & \cdots & * \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1\end{pmatrix}\begin{pmatrix}a_1 & 0 & \cdots & 0 \\ 0 & a_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_n\end{pmatrix}\}\cdot O(n)$$

Let $F_{A,B} \subset \mathrm{GL}(n,\mathbb{R})$ be the subset of matrices such that $|\star| < B$ and $a_i/a_{i+1} > A$.

Theorem 13.5 (Minkowski)**.** *The set $F_{A,B}$ is a fundamental set for the action of $\mathrm{GL}(n,\mathbb{Z})$ or $\mathrm{GL}(n,\mathbb{R})$, when $A < 1$ and $B > 1$. A forteriori, its image in $\mathrm{GL}(n,\mathbb{R})/O(n)$ is a fundamental set for the action of $\mathrm{GL}(n,\mathbb{Z})$.*

2. We proceed to the general case of a subgroup $G \subset \mathrm{GL}_n$. We have $M \subset G(\mathbb{R}) \subset \mathrm{GL}(n,\mathbb{R})$ and $M \subset g^{-1}O(n)g \subset \mathrm{GL}(n,\mathbb{R})$ for some $g \in \mathrm{GL}(n,\mathbb{R})$, and these inclusions commute. The map

$$\Gamma \circlearrowright G(\mathbb{R})/M \xrightarrow{p} \mathrm{GL}(n,\mathbb{R})/O(n) \circlearrowright \mathrm{GL}(n,\mathbb{Z})$$

is an equivariant continuous definable map.

Proposition 13.6. *One obtains a fundamental set for the action of Γ on $G(\mathbb{R})/M$ by taking the union of $\gamma p^{-1}(F_{A,B})$ for finitely many $\gamma \in G(\mathbb{Z})$.*

Remark 13.7. a) One can prove that fundamental sets obtained from different embeddings $G \subset \mathrm{GL}_n$ give the same quotient structure.

b) Consider the case of a period domain $\mathcal{D} = G(\mathbb{R})/V$,

$$G(\mathbb{R})/V \cong \mathcal{D} \to \{\text{ positive definite quadratic forms on } H_{\mathbb{R}}\} \cong \mathrm{GL}(n,\mathbb{R})/O(n),$$

$$\varphi \quad \mapsto \quad q(C(\varphi),\ ) = h(\varphi,\ )|_{H_{\mathbb{R}}}.$$

Therefore, lifting a period map $S \to \Gamma\backslash\mathcal{D}$ is equivalent to finding a flat basis of the corresponding $\mathbb{Z}$-local system in which the Hodge is Minkowski-reduced.

13.3 Structures on quotients of period domains

Now we have seen how to put a definable structure on quotients of period domains, we proceed to the following theorem.

Theorem 13.8 (Bakker-Klingler-Tsimerman)**.** *Let S be a smooth algebraic variety, and*

$$\pi : S^{\mathrm{an}} \to \Gamma\backslash\mathcal{D}$$

a period map, with $\Gamma \subset G(\mathbb{Z})$ torsion free of finite index. If one equips S^{an} and $\Gamma\backslash\mathcal{D}$ with the $\mathbb{R}_{\mathrm{an,exp}}$-structure coming from the canonical $\mathbb{R}_{\mathrm{alg}}$-structure, then π is $\mathbb{R}_{\mathrm{an,exp}}$-definable.

Proof. (in $\mathbb{R}_{\text{an,exp}}$). Let $\bar{S} \supset S$ be a log smooth compactification: this is a smooth compactifiaction such that $\bar{S} \setminus S$ is a normal crossing divisor. Let $\bar{S} = \cup V_i$ be a finite open cover such that $(V_i, V_i \cap S)$ is biholomorphic to $(\Delta^n, (\Delta^*)^r \times (\Delta^*)^{n-r})$. We need to prove that π is definable in restriction to $V_i \cap S$.

Assume for simplicity that $r = n$. Let

$$\mathscr{S} = \{x + iy \in \mathbb{H} : -\frac{1}{2} < x < \frac{1}{2}\}.$$

Then $\mathscr{S}^n$ is a fundamental set of $\mathbb{H}^n$ for $p : \mathbb{H}^n \to (\Delta^*)^n$ defined by $z \mapsto \exp(2\pi i z)$. We obtain the diagram

$$\begin{tikzcd} \mathscr{S}^n \arrow[r, "\tilde{\pi}"] \arrow[d, "p"] & \mathcal{D} \arrow[d] \\ (\Delta^*)^n \arrow[r, "\pi"] & \Gamma \setminus \mathcal{D}. \end{tikzcd}$$

It is sufficient to prove that the composition $\mathscr{S}^n \to \Gamma \setminus \mathcal{D}$ is definable.

By Borel's Monodromy Theorem, the monodromy is quasi-unipotent. By going to a finite étale cover of $(\Delta^*)^n$, one can assume that it is unipotent, so that $T_i = \exp(N_i)$ with N_i nilpotent.

Proposition 13.9. *The map $\tilde{\pi} : \mathscr{S}^n \to \mathcal{D}$ is definable.*

Proof. Recall the following theorem.

Theorem 13.10 (Schmid Nilpotent Orbit Theorem)**.** *If $\mathbb{V} = (\mathcal{H}, \nabla, F^\bullet, q)$ is a polarised variation of Hodge structures on $(\Delta^*)^n$ with unipotent monodromies, then the Hodge filtration $F^\bullet$ extends to Δ^n as a filtration by sub vector bundles in the Deligne extension of $(\mathcal{H}, \nabla)$.*

In coordinates, we have

$$\tilde{\pi}(z) = \exp(\sum_{j=1}^{n} z_j N_j) \cdot \psi(p(z)),$$

with $p : \Delta^n \to \hat{\mathcal{D}}$ holomorphic and definable. Therefore, $\tilde{\pi}$ is definable. We need to prove that the induced map $\mathscr{S}^n \to \Gamma \setminus \mathcal{D}$ is definable.

Theorem 13.11 (Bakker-Klingler-Tsimerman)**.** *There exist finitely many $\gamma \in G(\mathbb{Z})$ such that $\tilde{\pi}(\mathscr{S}^n)$ is contained in $\gamma \cdot F$, where F is a fundamental set.*

Equivalently, up to refining $\mathscr{S}^n$ by a finite cover, there exists a flat basis of the underlying $\mathbb{Z}$-local system which is Minkowski-reduced. In dimension 1, this is due to Schmid (1973). In dimension n, we omit the proof because of a lack of time. □

□

Remark 13.12. Not all quotients of period domains are algebraic. In fact, most quotients of period domains are not algebraic.

Let $\mathcal{D} = G(\mathbb{R})/V$ be a period domain. Assume that $G(\mathbb{R})$ is not of hermitian type.

Theorem 13.13 (Griffiths, Robles, Toledo)**.** *Any quotient of the form $\Gamma\backslash G(\mathbb{R})/V$ such that $\Gamma \subset G(\mathbb{Z})$ is of finite index, cannot be equipped with an algebraic structure.*

In fact, it is not possible to compactify the quotient $\Gamma\backslash G(\mathbb{R})/V$ by a compact Kähler manifold.

Proof. We have

$$\Gamma \circlearrowright G(\mathbb{R}) \setminus V \xrightarrow{p} G(\mathbb{R})/K \circlearrowright \Gamma,$$

where $G(\mathbb{R})/V = \mathcal{D} \subset \hat{\mathcal{D}}$, $K \supset V$ a maximal compact in $G(\mathbb{R})$ ($V \subset K \subset G(\mathbb{R})$), $G(\mathbb{R})/K$ the associated symmetric space. Note that the fibers $p^{-1}(a) \cong K/V$ are smooth projective varieties. This map p lands in $\hat{\mathcal{D}}$, and therefore induces a cycle on $\hat{\mathcal{D}}$. This gives a point in the Hilbert scheme. In fact, $\hat{\mathcal{D}}$ is the real part of a component of the Hilbert scheme. One can show that there exists a family Z_v of complex subvarieties of $\mathcal{D}$, parameterised by a Stein manifold U, which are all deformations of fibers of p. In fact, $U \subseteq \mathrm{Hilb}$ contains G/K as a totally real submanifold.

Theorem 13.14 (Griffiths, Robles, Toledo)**.** *Any two points in $\mathcal{D}$ are joined by a finite chain of Z_v's (which are smooth subvarieties and Fano manifolds).*

If $\Gamma \setminus \mathcal{D} = X^{\mathrm{an}}$, with X smooth algebraic, then in particular X is rationally connected. So X^{an} is necessarily simply connected. By Shafarevich morphism (Kaplan, Kollár), rationally connected implies simply connected. But then p is a fibration whose codomain is contractible. As the fibers and the target are simply connected, the total space must be simply connected, which is a contradiction. $\square$

14 Exercises

1. Show that a totally ordered ring that admits an o-minimal structure expanding it is a real closed field.

2. Show that $\mathbb{C}_0^*$ and $\mathbb{G}_m^{\mathrm{def}}$ are not isomorphic definable topological spaces in $\mathbb{R}_{\mathrm{alg}}$.

3. Prove that $\mathrm{id} : \mathbb{C}_a^* \to \mathbb{C}_b^*$ is not definable, and that thus there exist no biholomorphic definable map between them.

4. Consider the polarised Hodge structures arising from elliptic curves. What is the underlying $\mathbb{Z}$-module and polarisation? What $\mathcal{D} \subset \hat{\mathcal{D}}$ in this case? The Hodge form?

5. Take a period domain $\mathcal{D} \subset \prod \mathrm{Grass}$ such that $T_x\mathcal{D} \subsetneq T\mathcal{D}$ (example?). Prove that most elements of $\mathcal{D}$ are not isomorphic to the polarised Hodge structure of the (primitive) cohomology of a smooth projective variety.

6. Let $\mathcal{X} \to S$ a smooth projective morphism of algebraic varieties over $\mathbb{C}$. Let $\mathcal{D}$ be the period domain and $\pi : S \to G(\mathbb{Z})\backslash\mathcal{D}$ be the holomorphic horizontal period map (horizontal means that the image of the tangent space at every point is contained in the tangent space of the flag manifold). Prove that $\overline{\pi(S)}$ has no interior points.

7. Calculate the Mumford-Tate group of question 4.

8. For $z \in \mathbb{H}$, show that
$$h = \frac{1}{y}\begin{pmatrix} 1 & -x \\ -x & |z|^2 \end{pmatrix}$$
where $z = x + iy$.

9. Calculate $G = \mathrm{Aut}(H, q)$, show that $G(\mathbb{C}) = \mathrm{SL}_2(\mathbb{C}) \circlearrowright \hat{\mathcal{D}}$ and $\mathrm{SL}_2(\mathbb{R}) \circlearrowright \mathcal{D} = \mathbb{H}$, and confirm that the latter action coincides with the usual action of $\mathrm{SL}_2(\mathbb{R})$ on $\mathbb{H}$.

10. Let $\mathcal{X}$ be a compact complex manifold. Show that there is a unique $\mathbb{R}_{\mathrm{an}}$ definable analytic space X such that $X^{\mathrm{an}} = \mathcal{X}$.

11. Let $\mathcal{X} \subset \overline{\mathcal{X}}$ be a log smooth compact complex manifold (this is the complement of a log smooth divisor). Use punctured polydiscs to make an $\mathbb{R}_{\mathrm{an}}$ definable analytic space structure on $\mathcal{X}$ that is *not* coming from $\overline{\mathcal{X}}$.

12. Prove that a geometric quotient is a categorical quotient.

13. Let $\mathcal{D}$ be the period domain associated to (H, q). Let L be an algebraic subgroup of $\mathrm{GL}(H)$. Prove that for a very general element φ of $\mathrm{NL}(L)$, the irreducible component of $\mathrm{NL}(L)$ at φ is equal to $\mathrm{MT}(\varphi)(\mathbb{R})^0 \cdot \varphi = \mathrm{NL}(\varphi)^0$.

14. Show that there is an equivalence of categories

(Elliptic curves $E \ni 0$) $\leftrightarrow$ (weight -1 Hodge structures on $\mathbb{Z}^2 : h^{-1,0} = h^{0,-1} = 1$).

15. Consider the inclusion of $\mathcal{D} = \mathbb{H}$ in $\hat{\mathcal{D}} = \mathbb{P}(H_{\mathbb{C}})$, where $G(\mathbb{R}) = \mathrm{SL}_2(\mathbb{R})$ acts on $\mathbb{H}$ and $G(\mathbb{C}) = \mathrm{SL}(2, \mathbb{C})$ acts on $\mathbb{P}(H_{\mathbb{C}})$. What is the quotient generic Mumford-Tate group?

16. When is it generic and when not? Use the fact that the only reductive $\mathbb{Q}$-subgroups of SL_2 are the 1-dimensional connected ones.