Hyperbolic geometry and real moduli of five points on the line

Olivier de Gaay Fortman*

1 Introduction

Let $X \cong \mathbb{A}^6_{\mathbb{R}}$ be the affine space parametrizing homogeneous degree 5 polynomials $F \in \mathbb{R}[x,y]$. Let the variety $X_0 \subset X$ parametrize polynomials with distinct roots, and $X_s \subset X$ polynomials with roots of multiplicity at most two (i.e. stable in the sense of geometric invariant theory). The principal goal of this paper is to study the moduli space of stable *real binary quintics*

$$\overline{\mathscr{M}}_{\mathbb{R}} := \mathrm{GL}_2(\mathbb{R}) \setminus X_s(\mathbb{R}) \supset \mathrm{GL}_2(\mathbb{R}) \setminus X_0(\mathbb{R}) =: \mathscr{M}_{\mathbb{R}}.$$

If $P_s \subset \mathbb{P}^1(\mathbb{C})^5$ is the set 5-tuples $(x_1, \dots, x_5)$ such that no three $x_i \in \mathbb{P}^1(\mathbb{C})$ coincide (c.f. [MS72]), and $P_0 \subset P_s$ the subset of 5-tuples all whose coordinates are distinct, then

$$\mathscr{M}_{\mathbb{R}} \cong \mathrm{PGL}_2(\mathbb{R}) \setminus (P_0/\mathfrak{S}_5)(\mathbb{R}) \qquad \text{and} \qquad \overline{\mathscr{M}}_{\mathbb{R}} \cong \mathrm{PGL}_2(\mathbb{R}) \setminus (P_s/\mathfrak{S}_5)(\mathbb{R}).$$

In other words, $\mathscr{M}_{\mathbb{R}}$ is the space of subsets $S \subset \mathbb{P}^1(\mathbb{C})$ of cardinality $|S| = 5$ stable by complex conjugation modulo real projective transformations, and in $\overline{\mathscr{M}}_{\mathbb{R}}$ one or two pairs of points are allowed to collapse. For $i = 0, 1, 2$, we define $\mathscr{M}_i$ to be the connected component of $\mathscr{M}_{\mathbb{R}}$ parametrizing 5-tuples in $\mathbb{P}^1(\mathbb{C})$ with $2i$ complex and $5 - 2i$ real points.

There is a natural period map that defines an isomorphism between $\mathrm{GL}_2(\mathbb{C}) \setminus X_s(\mathbb{C})$ and a certain arithmetic ball quotient $P\Gamma \setminus \mathbb{C}H^2$ [Shi64], [DM86]. Moreover, one can prove that strictly stable quintics correspond to points in a hyperplane arrangement $\mathscr{H} \subset \mathbb{C}H^2$ (Proposition 5.10). Investigating the equivariance of the period map with respect to a suitable set of anti-holomorphic involutions $\alpha_i : \mathbb{C}H^2 \to \mathbb{C}H^2$, we obtain the following real analogue:

Theorem 1.1. *For each $i \in \{0, 1, 2\}$, the period map induces an isomorphism of real analytic orbifolds $\mathscr{M}_i \cong P\Gamma_i \setminus (\mathbb{R}H^2 - \mathscr{H}_i)$. Here $\mathbb{R}H^2$ is the real hyperbolic plane, $\mathscr{H}_i$ a union of geodesic subspaces in $\mathbb{R}H^2$ and $P\Gamma_i$ an arithmetic lattice in $\mathrm{PO}(2,1)$. Moreover, the $P\Gamma_i$ are projective orthogonal groups attached to explicit quadratic forms over $\mathbb{Z}[\zeta_5 + \zeta_5^{-1}]$, see (23).*

In particular, Theorem 1.1 endows each component $\mathscr{M}_i$ with a natural hyperbolic metric. Since one can deform the topological type of a $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$-stable five-element subset of $\mathbb{P}^1(\mathbb{C})$ by allowing two points to collide, the compactification $\overline{\mathscr{M}}_{\mathbb{R}} \supset \mathscr{M}_{\mathbb{R}}$ is connected. One may thus wonder whether the metrics on the components $\mathscr{M}_i$ extend to a metric on the whole of $\overline{\mathscr{M}}_{\mathbb{R}}$, and if so, what the resulting space looks like at the boundary. Our main result is the following:

Theorem 1.2. *There exists a complete hyperbolic metric on $\overline{\mathscr{M}}_{\mathbb{R}}$ that restricts to the metrics on $\mathscr{M}_i$ induced by Theorem 1.1. With respect to it, $\overline{\mathscr{M}}_{\mathbb{R}}$ is isometric to the hyperbolic triangle of angles $\pi/3, \pi/5, \pi/10$. Thus, if $P\Gamma_{\mathbb{R}} = \langle \alpha_1, \alpha_2, \alpha_3 | \alpha_i^2 = (\alpha_1\alpha_2)^3 = (\alpha_1\alpha_3)^5 = (\alpha_2\alpha_3)^{10} = 1 \rangle$, there is a homeomorphism $\overline{\mathscr{M}}_{\mathbb{R}} \cong P\Gamma_{\mathbb{R}} \setminus \mathbb{R}H^2$ extending the isomorphisms in Theorem 1.1.*

---

*École normale supérieure, 45 rue d'Ulm, Office T17, 75230 Paris, olivier.de.gaay.fortman@ens.fr.

**Date:* November 13, 2021 - This project has received funding from the European Union's Horizon 2020 research and innovation programme under the Marie Skłodowska-Curie grant agreement № 754362.

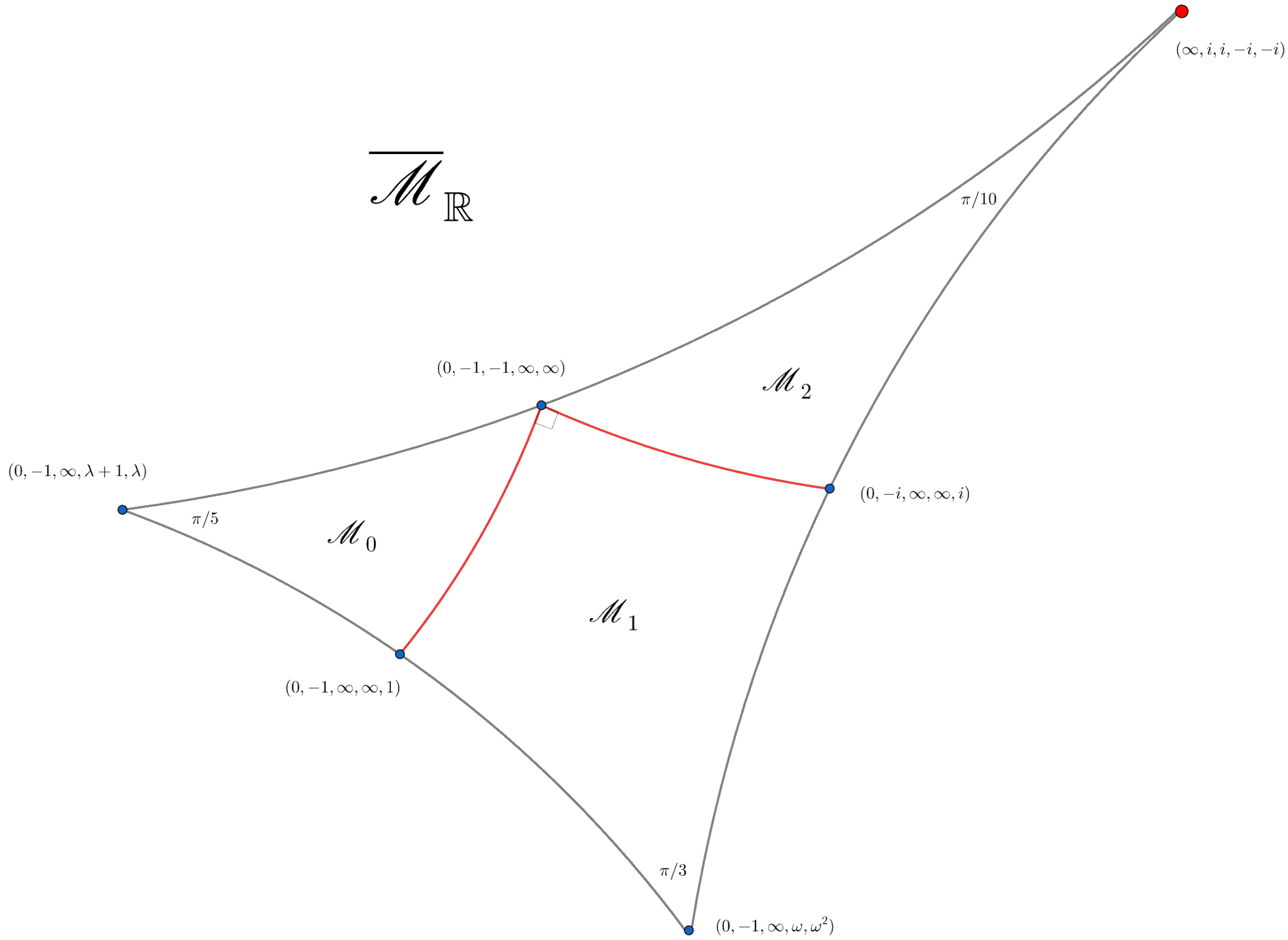


Figure 1: $\overline{\mathscr{M}}_{\mathbb{R}}$ *as the hyperbolic triangle* $\Delta_{3,5,10} \subset \mathbb{R}H^2$*. Here* $\lambda = \zeta_5 + \zeta_5^{-1}$ *and* $\omega = \zeta_3$*.*

Remark 1.3. The lattice $P\Gamma_{\mathbb{R}} \subset \mathrm{PO}(2,1)$ is *non-arithmetic*, as follows from [Tak77].

Equivariant period maps arise often in real algebraic geometry as a method to obtain real uniformization of the connected components of the moduli space of smooth varieties. For instance, this works for abelian varieties [GH81], algebraic curves [SS89], K3 surfaces [Nik80] and quartic curves [HR16]. Only recently, Allcock, Carlson and Toledo have shown that in the cases of cubic surfaces [ACT10] and binary sextics [ACT06], [ACT07], the real ball quotient components can be glued along the hyperplane arrangement in order to uniformize the moduli space of real stable varieties. Binary quintics provide the first new example of this phenomenon.

To prove Theorem 1.2, we construct a method of gluing together real arithmetic ball quotients, generalizing the work of Allcock, Carlson and Toledo cited above. The quotient spaces that we glue are real loci of various real structures on the complex orbifold ball quotient attached to a hermitian lattice of hyperbolic signature over the ring of integers of CM field. Let us outline this construction.

Let K be a CM field of degree $2g$ over $\mathbb{Q}$ with ring of integers $\mathcal{O}_K$, and let Λ be a finite free $\mathcal{O}_K$-module equipped with a hermitian form $h : \Lambda \times \Lambda \to \mathcal{O}_K$. Suppose that h has signature $(n, 1)$ with respect to a fixed embedding $\tau : K \to \mathbb{C}$ and is definite for the other infinite places of K. Let $\mathbb{C}H^n$ be the space of negative lines in $\Lambda \otimes_{\mathcal{O}_K,\tau} \mathbb{C}$ and $P\Gamma = \mathrm{Aut}(\Lambda, h)/\mu_K$ where $\mu_K \subset \mathcal{O}_K^*$ is the group of finite units $\zeta \in \mathcal{O}_K^*$. Define $P\mathscr{A}$ to be the quotient of the set of anti-unitary involutions $\alpha : \Lambda \to \Lambda$ by μ_K.

Consider the hyperplane arrangement $\mathscr{H} = \cup_{h(r,r)=1} \langle r_{\mathbb{C}} \rangle^{\perp} \subset \mathbb{C}H^n$ and assume $(*)$: different hyperplanes intersect orthogonally or not at all, c.f. [ACT02]. For example, this holds under a condition on the CM field K (see Theorem 4.8) satisfied when K is cyclotomic or quadratic (see Lemma 4.10). (In fact, condition $(*)$ is *always* satisfied if one is willing to adapt the definition of $\mathscr{H}$, see Remark 4.11.)

We claim that there is a canonical way to glue the different copies $\mathbb{R}H^n_\alpha := (\mathbb{C}H^n)^\alpha \subset \mathbb{C}H^n$ of the real hyperbolic space $\mathbb{R}H^n$ along the hyperplane arrangement $\mathscr{H}$. See Remark 3.4 for the precise formulation of the equivalence relation. This gives a topological space which we denote by Y, acted upon by $P\Gamma$. Define $P\Gamma_\alpha \subset P\Gamma$ to be stabilizer of $\mathbb{R}H^n_\alpha$. The result is as follows:

Theorem 1.4. *The topological space $P\Gamma \setminus Y$ admits a metric that makes it a complete path metric space such that $P\Gamma \setminus Y \to P\Gamma \setminus \mathbb{C}H^n$ is a local isometry. This metric induces a real hyperbolic orbifold structure on $P\Gamma \setminus Y$ such that $\coprod_{\alpha \in P\Gamma \setminus P\mathscr{A}} [P\Gamma_\alpha \setminus (\mathbb{R}H^n_\alpha - \mathscr{H})] \subset P\Gamma \setminus Y$ is an open suborbifold. Moreover, for each connected component $C \subset P\Gamma \setminus Y$ there exists a lattice $P\Gamma_C \subset \mathrm{PO}(n,1)$ and an isomorphism of real hyperbolic orbifolds $C \cong [P\Gamma_C \setminus \mathbb{R}H^n]$.*

For some moduli stacks of hypersurfaces $\mathcal{M}$ one can then apply Theorem 1.4 to the hermitian lattice Λ that arises as the cohomology of the cover of projective space ramified along a member of the moduli space. Let $\mathcal{M}_s$ be the stack of GIT stable hypersurfaces. If the discriminant $\Delta = \mathcal{M}_s(\mathbb{C}) - \mathcal{M}(\mathbb{C})$ is a normal crossings divisor and the period map induces an isomorphism of analytic spaces $\mathcal{M}_s(\mathbb{C}) \cong P\Gamma \setminus \mathbb{C}H^n$ identifying Δ with $P\Gamma \setminus \mathscr{H}$, then there is a real period homeomorphism $\mathcal{M}_s(\mathbb{R}) \cong P\Gamma \setminus Y$. For cubic surfaces and binary sextics, this is the content of [ACT10], [ACT06] and for binary quintics, this yields Theorem 1.2 (see Theorem 6.5).

Remark 1.5. The lattice $P\Gamma_C$ attached to a component $C \subset P\Gamma \setminus Y$ can be non-arithmetic. Indeed, such is the case for $K = \mathbb{Q}(\zeta_5)$ and $h = \mathrm{diag}(1, 1, \frac{1-\sqrt{5}}{2})$ by Remark 1.3 and Theorem 8.1, and for $K = \mathbb{Q}(\zeta_3)$ and $h = \mathrm{diag}(1, \dots, 1, -1)$ for $n = 3$ [ACT06] and $n = 4$ [ACT10].

Remark 1.6. Our gluing construction relies on condition $(*)$, saying that $\mathscr{H} \subset \mathbb{C}H^n$ is an *orthogonal arrangement* in the sense of [ACT02]. Such arrangements are interesting in their own right. Indeed, if $n > 1$ then the orbifold fundamental $\pi_1^{\mathrm{orb}}(P\Gamma \setminus (\mathbb{C}H^n - \mathscr{H}))$ is not a lattice in any Lie group with finitely many connected components [*loc. cit.*, Theorem 1.2]. In particular, neither $\pi_1(P_0/\mathfrak{S}_5)$ nor $\pi_1^{\mathrm{orb}}(\mathscr{M}_\mathbb{C})$ is a lattice in any Lie group with finitely many connected components. The hyperplane arrangement $\mathscr{H} \subset \mathbb{C}H^n$ is an orthogonal arrangement under the following condition $(**)$, satisfied by quadratic and cyclotomic CM fields: the different ideal $\mathfrak{D}_K \subset \mathcal{O}_K$ is generated by an element $\eta \in \mathcal{O}_K - \mathcal{O}_F$ such that $\eta^2 \in \mathcal{O}_F$. See Section 4.

Remark 1.7. In fact, there is always a canonical orthogonal arrangement $\mathcal{H} \subset \mathbb{C}H^n$ attached to h in such a way that $\mathcal{H} = \mathscr{H}$ when condition $(**)$ holds, see Remark 4.11. Moreover, one can glue the different copies $\mathbb{R}H^n_\alpha$ of real hyperbolic n-space along the hyperplane arrangement $\mathcal{H}$ obtaining a complete hyperbolic orbifold as in Theorem 1.4, but we will not prove this.

Remark 1.8. The gluing construction of Gromov and Piatetski-Shapiro [GPS87] seems close to our construction. However, there are differences: see [ACT10, Section 13, Remark (1)]. In a different way, it might be interesting to compare Shimura's study of real points $V(\mathbb{R})$ of an arithmetic quotient $V(\mathbb{C})$ of a bounded symmetric domain, see [Shi75].

1.1 Acknowledgements

I thank my thesis advisor Olivier Benoist for his great guidance and support. Moreover, I thank Romain Branchereau, Samuel Bronstein and Nicolas Tholozan for answering questions on hyperbolic geometry. I thank Frans Oort for a useful discussion, and Nicolas Bergeron for pointing me to Takeuchi's paper on hyperbolic triangle groups which led to Remark 1.3. Finally, I would like to stress that the gluing construction, taking up the first part of this paper, is inspired by and based on work of Allcock-Carlson-Toledo [ACT06], [ACT07], [ACT10].

2 Outline of the paper

This paper is organized as follows. We start with the gluing construction in Section 3, which explains how to obtain the complete real hyperbolic orbifold $P\Gamma\backslash Y$ from a free hermitian lattice Λ over the ring of integers over a CM field K. Then, in Section 4, we prove that the complex ball quotient $P\Gamma \setminus \mathbb{C}H^n$ is a moduli space for abelian varieties, and that the abelian varieties in $P\Gamma \setminus \mathscr{H}$ contain a CM abelian subvariety. In Section 5, we define the period for complex binary quintics, prove its compatibility with [DM86] and show that the monodromy group is the group $P\Gamma$ attached to the hermitian lattice of item (5) in the list of [Shi64]. In Section 6 we define the real period map for stable binary quintics and prove that it is a homeomorphism. In Section 7, we calculate the orbifold structure of $\overline{\mathscr{M}}_{\mathbb{R}}$ and finish the proof of Theorem 1.2.

3 Gluing Real Hyperbolic Quotient Spaces

In this section we consider a hermitian form h on a finite free module over the ring of integers of a CM field K with hyperbolic signature for some embedding of K into $\mathbb{C}$. There is a complex ball quotient $P\Gamma \setminus \mathbb{C}H^n$ attached to h in a canonical way. Our goal is to prove that there is also a natural real ball quotient $P\Gamma_{\mathbb{R}} \setminus \mathbb{R}H^n$ attached to h (or rather a disjoint union of those). It is defined by considering the real hyperbolic spaces $(\mathbb{C}H^n)^{\alpha}$ attached to anti-unitary involutions $\alpha : \Lambda \to \Lambda$, gluing them along an orthogonal hyperplane arrangement $\mathscr{H} \subset \mathbb{C}H^n$, taking the quotient by $P\Gamma$, and defining a complete real hyperbolic orbifold structure on the result. As is the case over the complex numbers, $P\Gamma_{\mathbb{R}} \setminus \mathbb{R}H^n$ is sometimes a moduli space for real varieties.

3.1 Notation and assumptions

- Let K be a CM field of degree $2g$ over $\mathbb{Q}$ and let $F \subset K$ be its totally real subfield. Let $\mathcal{O}_K$ be the ring of integers of K and denote the non-trivial element in $\mathrm{Gal}(K/F)$ by σ. Fix a set of embeddings $\Psi = \{\tau_i : K \to \mathbb{C}\}_{1 \le i \le g}$ such that $\Psi \cup \Psi\sigma = \{\tau_i, \tau_i\sigma\}_{1 \le i \le g} = \mathrm{Hom}(K, \mathbb{C})$.

- Let Λ be a free $\mathcal{O}_K$-module of rank $n+1$ equipped with a hermitian form $h : \Lambda \times \Lambda \to \mathcal{O}_K$ of signature (r_i, s_i) with respect to τ_i. In other words, h is linear in its first argument and σ-linear in its second, and the complex vector space $\Lambda \otimes_{\mathcal{O}_K, \tau_i} \mathbb{C}$ admits a basis $\{e_i\}$ such that $(h^{\tau_i}(e_i, e_j))_{ij}$ is a diagonal matrix with r_i diagonal entries equal to 1 and s_i diagonal entries equal to -1. Here h^{τ_i} is the hermitian form on $\Lambda \otimes_{\mathcal{O}_K, \tau_i} \mathbb{C}$ induced by h.

- Consider $\tau = \tau_1 : K \to \mathbb{C}$ and define $V = \Lambda \otimes_{\mathcal{O}_K, \tau} \mathbb{C}$. We suppose that $r_1 = n$ and $r_2 = 1$.

- For $m \in \mathbb{Z}$, define $\zeta_m = e^{2\pi i/m} \in \mathbb{C}$. Denote by $\mu_K \subset \mathcal{O}_K^*$ be the group of roots of unity in K. If m is the largest positive integer for which the m-th cyclotomic field $\mathbb{Q}(\zeta_m)$ can be embedded in K, then $\mu_K = \langle \zeta \rangle$ for a primitive m-th root of unity $\zeta \in K$. Let $\Gamma = U(\Lambda)(\mathcal{O}_K) = \mathrm{Aut}_{\mathcal{O}_K}(\Lambda, h)$ and define $P\Gamma$ to be the quotient group Γ/μ_K.

- Define $\mathbb{C}H^n$ to be the space of negative lines in V.

- For $r \in V$ and $i \in \mathbb{Z}/m$, define an isometry $h_r^i : V \to V$ of as follows:

$$h_r^i(x) = x - (1 - \zeta^i)\frac{h(x, r)}{h(r, r)} r.$$

  Then $h_r^i \in \Gamma$ is called the ζ^i-*reflection* in r; we write $h_r = h_r^1$. A norm 1 vector of Λ is called a *short root*; then for $i \in (\mathbb{Z}/m)^*$, the ζ^i-reflections in short roots are isometries Λ.

- Let $\mathscr{R} \subset \Lambda$ be the set of short roots, define $H_r = \{x \in \mathbb{C}H^n : h(x,r) = 0\}$ for $r \in \mathscr{R}$, and

$$\mathscr{H} = \bigcup_{r \in \mathscr{R}} H_r \quad \subset \quad \mathbb{C}H^n.$$

- Remark that the family of hyperplanes $(H_r)_{r\in\mathscr{R}}$ is locally finite by [Bea09, Lemma 5.3], so that the hyperplane arrangement $\mathscr{H} \subset \mathbb{C}H^n$ is a divisor of $\mathbb{C}H^n$.

- Define a $\mathcal{O}_F$-linear map $\alpha : \Lambda \to \Lambda$ to be *anti-unitary* if for all $x, y \in \Lambda$ and $\lambda \in \mathcal{O}_K$, one has $\alpha(\lambda x) = \sigma(\lambda)\alpha(x)$ and $h(\alpha(x), \alpha(y)) = \sigma(h(x,y)) \in \mathcal{O}_K$. Let $\mathscr{A}$ be the set of anti-unitary involutions. It is easy to see that μ_K acts on $\mathscr{A}$ by multiplication; define $P\mathscr{A} = \mu_K \setminus \mathscr{A}$. Moreover, $P\Gamma$ acts on $P\mathscr{A}$ by conjugation; define $C\mathscr{A} = P\Gamma \setminus P\mathscr{A}$.

- Any $\alpha \in P\mathscr{A}$ defines an anti-holomorphic involution $\alpha : \mathbb{C}H^n \to \mathbb{C}H^n$. We define $\mathbb{R}H^n_\alpha \subset \mathbb{C}H^n$ to be its fixed-point set. If $\alpha \in \mathscr{A}$ is a representative for α, then the lattice V^α is hyperbolic and $\mathbb{R}H^n_\alpha = \mathbb{R}H(V^\alpha)$, the space of negative real lines in $V^\alpha = \Lambda^\alpha \otimes_{\mathcal{O}_F} \mathbb{R}$. Therefore $\mathbb{R}H^n_\alpha \subset \mathbb{C}H^n$ is isometric to the n-dimensional real hyperbolic space $\mathbb{R}H^n$.

- Finally, let $P\Gamma_\alpha = \mathrm{Stab}_{P\Gamma}(\mathbb{R}H^n_\alpha) \subset P\Gamma$ be the stabilizer of $\mathbb{R}H^n_\alpha$ in $P\Gamma$.

3.2 The gluing procedure

We assume that the following condition is satisfied:

Condition 3.1. *If $r, t \in \mathscr{R}$ are such that $H_r \neq H_t$ and $H_r \cap H_t \neq \emptyset$, then $h(r,t) = 0$.*

Example 3.2. Theorem 4.8 of Section 4 shows that Condition 3.1 holds if the following conditions are satisfied: (1) the different ideal $\mathfrak{D}_K \subset \mathcal{O}_K$ is generated by a single element $\eta \in \mathcal{O}_K$ such that $\sigma(\eta) = -\eta$ and $\Im(\tau_i(\eta)) > 0$ for every i, (2) $(r_i, s_i) = (n+1, 0)$ for $2 \leq i \leq g$, and (3) the CM type (K, Ψ) is primitive. Note that Condition (1) is satisfied for quadratic and cyclotomic CM fields K, see Lemma 4.10.

Let $x \in \mathscr{H} \subset \mathbb{C}H^n$ and $\mathbf{r} = (r_1, \dots, r_k) \in \mathscr{R}^k$ be such that $x \in H_{r_1} \cap \ldots \cap H_{r_k}$ with $H_{r_i} \neq H_{r_j}$ for all $r_i \neq r_j$. The hyperplanes are orthogonal by Condition 3.1 hence $H_{r_1} \cap \ldots \cap H_{r_k} \cong \mathbb{R}H^{n-k}$, a totally geodesic subspace of codimension k. The ζ-reflections $h_{r_i} \in P\Gamma$ define a group $\langle h_{r_1}, \dots, h_{r_k} \rangle \subset \Gamma$ which is isomorphic to $(\mathbb{Z}/m)^k$, and the composition $\langle h_{r_1}, \dots, h_{r_k} \rangle \to \Gamma \to P\Gamma$ is injective. Define $G(\mathbf{r}) = \langle h_{r_1}, \dots, h_{r_k} \rangle \subset P\Gamma$.

Definition 3.3. Set $\widetilde{Y} = \coprod_{\alpha \in P\mathscr{A}} \mathbb{R}H^n_\alpha$. Define a relation $R \subset \widetilde{Y} \times \widetilde{Y}$: an element $((x,\alpha),(y,\beta)) \in \widetilde{Y} \times \widetilde{Y}$ is an element of R if the following conditions are satisfied:

1. We have $x = y \in \mathbb{C}H^n$.
2. If $\alpha \neq \beta$, then $x = y \in \mathscr{H}$ and $\beta = \phi \circ \alpha$ with $\phi \in G(\mathbf{r})$ for some $\mathbf{r} = (r_i)_i \in \mathscr{R}^k$ such that $x \in \cap_{i=1}^k H_{r_i}$.

Remark 3.4. Conditions 1 and 2 in Definition 3.3 say that we are identifying points of $\mathbb{R}H^n_\alpha \cap \mathscr{H}$ and $\mathbb{R}H^n_\beta \cap \mathscr{H}$ that have the same image in $\mathbb{C}H^n$. But we do not glue all such points: the real structures α and β are required to differ by complex reflections in the hyperplanes through x. In fact, we will see below (see Lemma 3.11) that the gluing rules can be rephrased as follows: we glue $\mathbb{R}H^n_\alpha$ and $\mathbb{R}H^n_\beta$ along their intersection, provided that this intersection is contained in $\mathscr{H}$ in such a way that for some (equivalently, any) $x \in \mathbb{R}H^n_\alpha \cap \mathbb{R}H^n_\beta$, the real structures α and β differ by reflections in hyperplanes that pass through x.

Definition 3.5. Let $x \in \mathbb{C}H^n$. We say that a set of hyperplanes $\{H_{r_1}, \ldots, H_{r_k}\}$ attached to roots $r_i \in \mathscr{R}$ is a *maximal set of hyperplanes through* x if $x \in \cap_{i=1}^k H_{r_i}$, $H_{r_i} \neq H_{r_j}$ for $i \neq j$ and if whenever $t \in \mathscr{R}$ is such that $x \in H_t$, then $H_r = H_{r_i}$ for some i.

Lemma 3.6. *R is an equivalence relation.*

Proof. Let $(x, \alpha), (y, \beta), (z, \gamma) \in \widetilde{Y}$; the fact that $(x, \alpha) \sim (x, \alpha)$ is clear. Suppose that $(x, \alpha) \sim (y, \beta)$. If $\alpha = \beta$ then $(x, \alpha) = (y, \beta) \in \widetilde{Y}$ hence $(y, \beta) \sim (x, \alpha)$. If $\alpha \neq \beta$ then $x = y \in \mathscr{H} \subset \mathbb{C}H^n$, and $\beta = \phi \circ \alpha$ for $\phi \in G(\mathbf{r})$ as in Definition 3.3. Since $\alpha = \phi^{-1} \circ \beta$ with $\phi^{-1} \in G(\mathbf{r})$, this shows that $(y, \beta) \sim (x, \alpha)$. Finally, suppose that $(x, \alpha) \sim (y, \beta)$ and $(y, \beta) \sim (z, \gamma)$; we claim that $(x, \alpha) \sim (z, \gamma)$. We may and do assume that α, β and γ are different, which implies that $x = y = z \in \mathscr{H}$. Let $\phi, k, \mathbf{r}$ and $\psi, l, \mathbf{t}$ be as in Definition 3.3: we have $\beta = \phi \circ \alpha$ with $\phi \in G(\mathbf{r})$ for some $\mathbf{r} = (r_i)_i \in \mathscr{R}^k$ such that $x \in \cap H_{r_i}$, and $\gamma = \psi \circ \beta$ with $\psi \in G(\mathbf{t})$ for some $\mathbf{t} = (t_i)_i \in R^l$ such that $y \in \cap H_{t_i}$. But then $x = y \in (\cap_i H_{r_i}) \cap (\cap_i H_{t_i})$. We may assume that the sets of hyperplanes $\{H_{r_i}\}$ and $\{H_{t_j}\}$ through x and y are maximal, which implies that $k = l$, and there exists an element $\sigma \in S_k$ such that $H_{t_i} = H_{r_{\sigma(i)}}$ for each i. By Lemma 3.7 below, we have $G(\mathbf{r}) = G(\mathbf{t}) \subset P\Gamma$. Therefore, $\psi \circ \phi \in G(\mathbf{r})$, and we conclude that $(x, \alpha) \sim (z, \gamma)$. ☐

Lemma 3.7. *Let $\phi : \mathbb{C}H^n \to \mathbb{C}H^n$ be an isometry such that $\phi^m = \mathrm{id}$ and such that ϕ is the identity when restricted to $H_r \subset \mathbb{C}H^n$ for some $r \in \mathscr{R}$. Then $\phi = h_r^i$ for some $i \in \mathbb{Z}/m$.*

Proof. It suffices to show that $\phi \in G(\mathbf{r})$. Let $\mathbb{H}^n_{\mathbb{C}}$ be the hyperbolic space attached to the standard hermitian space $\mathbb{C}^{n,1}$ of dimension $n+1$. It is easy to see that

$$\mathrm{Stab}_{U(n,1)}(\mathbb{H}^{n-1}_{\mathbb{C}}) = U(n) \times U(1)$$

hence any element $f \in U(n, 1)$ that fixes $\mathbb{H}^{n-1}_{\mathbb{C}}$ pointwise is contained in the subgroup $U(1) = \{z \in \mathbb{C}^* : |z|^2 = 1\}$. If $f^m = \mathrm{id}$ then it is contained in the unique order m subgroup of $U(1)$. ☐

Definition 3.8. Define Y to be the quotient of $\widetilde{Y}$ by the equivalence relation R, and equip it with the quotient topology. We shall prove (Lemma 3.9) that the group $P\Gamma$ acts on Y. We call $P\Gamma \setminus Y$ the *hyperbolic gluing* of the hermitian $\mathcal{O}_K$-lattice (Λ, h).

Lemma 3.9. *The action of $P\Gamma$ on $\mathbb{C}H^n$ induces an action of $P\Gamma$ on $\widetilde{Y}$ which is compatible with the equivalence relation R, so that $P\Gamma$ acts on Y. Moreover, $P\Gamma \setminus \widetilde{Y} = \coprod_{\alpha \in C\mathscr{A}} P\Gamma \setminus \mathbb{R}H^n_\alpha$.*

Proof. If $\phi \in P\Gamma$, then $\phi\left(\mathbb{R}H^n_\alpha\right) = \mathbb{R}H^n_{\phi\alpha\phi^{-1}}$ hence $P\Gamma$ acts on $\widetilde{Y} = \coprod_{\alpha \in P\mathscr{A}} \mathbb{R}H^n_\alpha$, and

$$P\Gamma \setminus \widetilde{Y} = P\Gamma \setminus \coprod_{\alpha \in P\mathscr{A}} \mathbb{R}H^n_\alpha = \coprod_{\alpha \in C\mathscr{A}} P\Gamma \setminus \mathbb{R}H^n_\alpha.$$

Now suppose that $(x, \alpha) \sim (y, \beta) \in \widetilde{Y}$ and $f \in P\Gamma$. We claim that $(f(x), f\alpha f^{-1}) \sim (f(y), f\beta f^{-1})$. We may and do assume that $(x, \alpha) \neq (y, \beta)$, hence $x = y \in \mathscr{H} \subset \mathbb{C}H^n$. Let $r = (r_i)_i \in \mathscr{R}^k$ with $x \in \cap_i H_{r_i}$ and $\beta = \phi \circ \alpha$ as in Definition 3.3. Observe that $f(H_{r_i}) = H_{f(r_i)}$. Therefore $f(x) \in \cap_i H_{f(r_i)}$, and we have $f\beta f^{-1} = f\phi f^{-1} \circ f\alpha f^{-1}$ and $f\phi f^{-1} \in fG(\mathbf{r})f^{-1} = \langle f h_{r_i} f^{-1} \rangle = \langle h_{f(r_i)} \rangle = G(f\mathbf{r})$. ☐

We are now in position to state the main theorem of Section 3:

Theorem 3.10. *1. The hyperbolic gluing $P\Gamma \setminus Y$ admits a metric that makes it a complete path metric space. With respect to it, the natural map $P\Gamma \setminus Y \to P\Gamma \setminus \mathbb{C}H^n$ is a local isometry. Each point $x \in P\Gamma \setminus Y$ admits an open neighborhood $U \subset P\Gamma \setminus Y$ which is isometric to the quotient of an open subset $V \subset \mathbb{R}H^n$ by a finite group of isometries. Hence $P\Gamma \setminus Y$ has a real hyperbolic orbifold structure.*

2. *One has* $O := \coprod_{\alpha \in C\mathscr{A}} [P\Gamma_\alpha \setminus (\mathbb{R}H^n_\alpha - \mathscr{H})] \subset P\Gamma \setminus Y$ *as an open suborbifold.*

3. *The connected components of the real-hyperbolic orbifold* $P\Gamma \backslash Y$ *are uniformized by* $\mathbb{R}H^n$*: for each component* $C \subset P\Gamma \backslash Y$ *there exists a lattice* $P\Gamma_C \subset \mathrm{PO}(n,1)$ *and an isomorphism of real hyperbolic orbifolds* $C \cong [P\Gamma_C \setminus \mathbb{R}H^n]$*. In other words,*

$$P\Gamma \setminus Y \cong \coprod_{C \in \pi_0(P\Gamma \backslash K)} [P\Gamma_C \setminus \mathbb{R}H^n].$$

The remaining part of Section 3 will be devoted to the proof of Theorem 3.10. It may very well happen that $P\Gamma \backslash Y$ is connected: such is the case when $K = \mathbb{Q}(\zeta_3)$ and $h = \mathrm{diag}(1,1,1,1,-1)$, see [ACT10]. When $K = \mathbb{Q}(\zeta_3)$ and $h = \mathrm{diag}(1,1,1,-1)$, then $P\Gamma \backslash Y$ has two components, see [ACT07, Remark 6]. See also Section 6: for some d and n, there is a homeomorphism between $P\Gamma \setminus Y$ and the moduli space of GIT-stable hypersurfaces of degree d in $\mathbb{P}^n_{\mathbb{R}}$, restricting to an isomorphism of orbifolds between O and the moduli of smooth hypersurfaces (Theorem 6.5).

3.3 The path metric on the hyperbolic gluing

We start with a lemma. We will need it in the proof of Lemma 3.13 below, which will in turn be used to define a path metric on $P\Gamma \setminus Y$ making it locally isometric to quotients of $\mathbb{R}H^n$ by finite groups of isometries. But it can also serve as a sanity check: it says that once there exists an element $x \in \mathbb{R}H^n_\alpha \cap \mathbb{R}H^n_\beta$ such that $(x,\alpha) \sim (x,\beta)$, then one glues the entire copy $\mathbb{R}H^n_\alpha$ to the copy $\mathbb{R}H^n_\beta$ along their intersection in $\mathbb{C}H^n$.

Lemma 3.11. 1. *Let* $\phi = \prod_{i=1}^l h_{r_i}^{j_i} \in \Gamma$ *for some set* $\{r_i\} \subset \mathscr{R}$ *of mutually orthogonal short roots* r_i*, where* $j_i \neq 0 \bmod m$ *for each* i*. Then* $(\mathbb{C}H^n)^\phi \subset \cap_{i=1}^l H_{r_i}$*.*

2. *Let* $\alpha, \beta \in P\mathscr{A}$ *and* $x \in \mathbb{R}H^n_\alpha \cap \mathbb{R}H^n_\beta$ *such that* $(x,\alpha) \sim (x,\beta)$*. Then* $(y,\alpha) \sim (y,\beta)$ *for every* $y \in \mathbb{R}H^n_\alpha \cap \mathbb{R}H^n_\beta$*.*

3. *The natural map* $\widetilde{Y} \to \mathbb{C}H^n$ *descends to a* $P\Gamma$*-equivariant map* $\mathscr{P} : Y \to \mathbb{C}H^n$*.*

Proof. 1. Let $y \in V$ be representing an element in $(\mathbb{C}H^n)^\phi$. Since the r_i are orthogonal, and $\phi(y) = \lambda$ for some $\lambda \in \mathbb{C}^*$, we have

$$\phi(y) = \prod_{i=1}^l h_{r_i}^{j_i}(y) = y - \sum_{i=1}^l \left(1 - \zeta^{j_i}\right) h(y, r_i) r_i = \lambda y, \tag{1}$$

hence $(1-\lambda)y = \sum_{i=1}^l (1-\zeta^{j_i})\, h(y,r_i) r_i \in V$. But y spans a negative definite subspace of V while the r_i span a positive definite subspace, so that we must have $1 - \lambda = 0 = \sum_{i=1}^l (1-\zeta^{j_i})\, h(y,r_i) r_i$. Since the r_i are mutually orthogonal, they are linearly independent; since $\zeta^{j_i} \neq 1$ we find $h(y,r_i) = 0$ for each i.

2. Let $\mathbf{r} = (r_1, \dots, r_k) \in \mathscr{R}^k$ such that $x \in \cap_i H_{r_i}$ for a maximal set of hyperplanes $\{H_{r_i}\}$ through x. We have $\beta = \phi \circ \alpha$ for some $\phi \in G(\mathbf{r})$. Let $I \subset \{1, \dots, k\}$ be a subset such that $\phi = \prod_{i \in I} h_{r_i}^{j_i}$ with $j_i \neq 0 \bmod m$. If $I = \emptyset$ then there is nothing to prove, so suppose the contrary. Notice that $\phi(y) = y$. Part 1 implies that $y \in \cap_{i \in I} H_{r_i}$. Now let $\mathbf{t} = (t_1, \dots, t_l) \in \mathscr{R}^l$ such that $y \in \cap_j H_{t_j}$ as in Definition 3.3. Then for each i there is a j such that $H_{r_i} = H_{t_j}$. By Lemma 3.7, we have $h_{r_i} \in G(t_j)$, so that $\phi \in G(\mathbf{r}) \subset G(\mathbf{t})$.

3. If $(x,\alpha) \sim (y,\beta)$, then $x = y \in \mathbb{C}H^n$. □

Consequently, we obtain continuous maps $\mathscr{P} : K \to \mathbb{C}H^n$ and $\bar{\mathscr{P}} : P\Gamma \setminus Y \to P\Gamma \setminus \mathbb{C}H^n$. Our next goal is to prove that each point $x \in Y$ has a neighbourhood $V \subset Y$ that maps homeomorphically onto a finite union $\cup_{i=1}^{l} \mathbb{R}H^n_{\alpha_i} \subset \mathbb{C}H^n$. Hence x has an open neighourhood $x \in U \subset V$ that identifies with an open set in a union of copies of $\mathbb{R}H^n$ in $\mathbb{C}H^n$ under the map $\mathscr{P}$. This allows us to define a metric on Y by pulling back the metric on $\mathbb{C}H^n$.

Fix a point $f \in Y$ and a point $(x, \alpha) \in \widetilde{Y}$ lying above f. Let $\alpha_1, \dots, \alpha_l$ be the elements in $P\mathscr{A}$ such that $(x, \alpha_i) \sim (x, \alpha)$ for each i. Let $p : \widetilde{Y} \to Y$ be the quotient map. Define

$$Y_f = p\left(\coprod_{i=1}^{l} \mathbb{R}H^n_{\alpha_i}\right) \subset Y. \tag{2}$$

Lemma 3.12. *Each compact set $Z \subset \mathbb{C}H^n$ meets only finitely many $\mathbb{R}H^n_\alpha$, $\alpha \in P\mathscr{A}$.*

Proof. If $P\Gamma' \subset \mathrm{Isom}(\mathbb{C}H^n)$ is the group of all μ_K-orbits of isometries and anti-isometries $\phi : \Lambda \to \Lambda$, then $P\Gamma'$ acts properly discontinuously on $\mathbb{C}H^n$. So if S is the set of $\alpha \in P\mathscr{A}$ such that $\alpha Z \cap Z \neq \emptyset$, then S is finite. In particular, Z meets only finitely many $\mathbb{R}H^n_\alpha$. □

It follows that Y is locally isometric to opens in unions of real hyperbolic subspaces of $\mathbb{C}H^n$:

Lemma 3.13. *1. The set Y_f is closed in Y.*

2. We have $\mathscr{P}(Y_f) = \cup_{i=1}^{l} \mathbb{R}H^n_{\alpha_i} \subset \mathbb{C}H^n$, and the map

$$\mathscr{P}_f : Y_f \to \cup_{i=1}^{l} \mathbb{R}H^n_{\alpha_i}$$

induced by $\mathscr{P}$ is a homeomorphism.

3. The set $Y_f \subset Y$ contains an open neighborhood U_f of f in Y.

Proof. 1. Indeed, one has

$$p^{-1}(Y_f) = p^{-1}\left(p\left(\coprod_{i=1}^{l} \mathbb{R}H^n_{\alpha_i}\right)\right) = \bigcup_{i=1}^{l} p^{-1}\left(p\left(\mathbb{R}H^n_{\alpha_i}\right)\right)$$

so that it suffices to show that $p^{-1}\left(p\left(\mathbb{R}H^n_{\alpha_i}\right)\right)$ is closed in Y. But notice that $(x, \beta) \in p^{-1}(p(\mathbb{R}H^n_\alpha))$ if and only if $x \in \mathbb{R}H^n_\alpha$ and $(x, \alpha) \sim (x, \beta)$, which implies (Lemma 3.11) that $\mathbb{R}H^n_\alpha \cap \mathbb{R}H^n_\beta \subset p^{-1}(p(\mathbb{R}H^n_\alpha))$. Hence for any $\alpha \in P\mathscr{A}$, one has $p^{-1}(p(\mathbb{R}H^n_\alpha)) = \coprod_{\beta \sim \alpha} \mathbb{R}H^n_\alpha \cap \mathbb{R}H^n_\beta$, where $\beta \sim \alpha$ if and only if there exists $x \in \mathbb{R}H^n_\alpha \cap \mathbb{R}H^n_\beta$ such that $(x, \alpha) \sim (x, \beta)$. It follows that $p^{-1}(p(\mathbb{R}H^n_\alpha)) \cap \mathbb{R}H^n_\beta$ is closed in $\mathbb{R}H^n_\beta$ for every $\beta \in P\mathscr{A}$. But the $\mathbb{R}H^n_\beta$ are open in $\widetilde{Y}$ and cover $\widetilde{Y}$, so that $p^{-1}(p(\mathbb{R}H^n_\alpha))$ is closed in $\widetilde{Y}$.

2. We have

$$\mathscr{P}_f(Y_f) = \mathscr{P}\left(p\left(\coprod_{i=1}^{l} \mathbb{R}H^n_{\alpha_i}\right)\right) = \tilde{\mathscr{P}}\left(\coprod_{i=1}^{l} \mathbb{R}H^n_{\alpha_i}\right) = \cup_{i=1}^{l} \mathbb{R}H^n_{\alpha_i} \subset \mathbb{C}H^n.$$

To prove injectivity, let $(x, \alpha_i), (y, \alpha_j) \in \widetilde{Y}$ and suppose that $x = y \in \mathbb{C}H^n$. Then indeed, $(x, \alpha_i) \sim (y, \alpha_j)$ because $\sim$ is an equivalence relation by Lemma 3.6.

Let $Z \subset \mathbb{C}H^n$ be a compact set. Write $\tilde{\mathscr{P}} : \widetilde{Y} \to \mathbb{C}H^n$ for the canonical map. Remark that Z meets only finitely many of the $\mathbb{R}H^n_\alpha$ for $\alpha \in P\mathscr{A}$, see Lemma 3.12. Each $Z \cap \mathbb{R}H^n_\alpha$ is closed in Z since $\mathbb{R}H^n_\alpha$ is closed in $\mathbb{C}H^n$, so each $Z \cap \mathbb{R}H^n_\alpha$ is compact. We conclude that $\tilde{\mathscr{P}}^{-1}(Z) = \coprod Z \cap \mathbb{R}H^n_\alpha$ is compact. In particular, $\tilde{\mathscr{P}}$ is closed [Lee13, Theorem A.57].

Finally, we prove that $\mathscr{P}_f$ is closed. Let $Z \subset Y_f$ be a closed set. Then Z is closed in Y by part 1, hence $p^{-1}(Z)$ is closed in $\widetilde{Y}$, hence $\tilde{\mathscr{P}}\left(p^{-1}(Z)\right))$ is closed in $\mathbb{C}H^n$, so that

$$\mathscr{P}_f(Z) = \mathscr{P}(Z) = \tilde{\mathscr{P}}\left(p^{-1}(Z)\right) = \left(\tilde{\mathscr{P}}\left(p^{-1}(Z)\right)\right) \cap \left(\cup_{i=1}^{l} \mathbb{R}H^n_{\alpha_i}\right)$$

is closed in $\cup_{i=1}^{l} \mathbb{R}H^n_{\alpha_i}$.

3. Let $x = \mathscr{P}(f) \in \mathbb{C}H^n$. Since $\mathbb{C}H^n$ is locally compact, there exists a compact set $Z \subset \mathbb{C}H^n$ and an open set $U \subset \mathbb{C}H^n$ with $x \in U \subset Z$. Since Z is compact, it meets only finitely many of the $\mathbb{R}H^n_\beta \subset \mathbb{C}H^n$ (Lemma 3.12). Consequently, the same holds for U; define $V = \mathscr{P}^{-1}(U) \subset Y$. Let $\mathscr{B} = \{\beta \in P\mathscr{A} : U \cap \mathbb{R}H^n_\beta \neq \emptyset\}$. Also define, for $\beta \in P\mathscr{A}$, $H_\beta = p\left(\mathbb{R}H^n_\beta\right) \subset Y$. Then

$$f \in V \subset \bigcup_{\beta \in \mathscr{B}} H_\beta = \bigcup_{\substack{\beta \in \mathscr{B} \\ \beta(x) = x}} H_\beta \quad \bigcup_{\substack{\beta \in \mathscr{B} \\ \beta(x) \neq x}} H_\beta.$$

Since each H_β is closed in Y by the proof of part 1, there is an open subset $V' \subset V$ with

$$f \in V' \subset \bigcup_{\substack{\beta \in \mathscr{B} \\ \beta(x) = x}} H_\beta = \bigcup_{\substack{\beta \in \mathscr{B} \\ \beta(x) = x \\ (x,\beta) \sim (x,\alpha)}} H_\beta \quad \bigcup_{\substack{\beta \in \mathscr{B} \\ \beta(x) = x \\ (x,\beta) \not\sim (x,\alpha)}} H_\beta$$

Hence again there exists an open subset $V'' \subset V'$ with

$$f \in V'' \subset \bigcup_{\substack{\beta \in \mathscr{B} \\ \beta(x) = x \\ (x,\beta) \sim (x,\alpha)}} H_\beta \subset \bigcup_{\substack{\beta \in P\mathscr{A} \\ \beta(x) = x \\ (x,\beta) \sim (x,\alpha)}} H_\beta = Y_f.$$

Therefore, $U_f := V'' \subset Y$ satisfies the requirements. □

We need one further lemma:

Lemma 3.14. *The topological space Y is Hausdorff.*

Proof. Let $f, g \in Y$ be elements such that $f \neq g$. First suppose that $f \notin Y_g$. Since Y_g is closed in Y by Lemma 3.13, there is an open neighbourhood U of f such that $U \cap U_g \subset U \cap Y_g = \emptyset$. Next, suppose that $f \in Y_g$. Lift f and g to elements $(x, \alpha), (y, \beta) \in \widetilde{Y}$. Assume first that $x = y$. This means that $\mathscr{P}(f) = \mathscr{P}(g)$. Since $\mathscr{P} : Y_g \to \mathbb{C}H^n$ is injective, this implies that $f = g$, contradiction. So we have $x \neq y \in \mathbb{C}H^n$. But $\mathbb{C}H^n$ is Hausdorff, so there are open subsets $U \subset \mathbb{C}H^n$, $V \subset \mathbb{C}H^n$ such that $x \in U$, $y \in V$ and $U \cap V = \emptyset$. Then $\mathscr{P}^{-1}(U) \cap \mathscr{P}^{-1}(V) = \emptyset$. □

We then obtain:

Proposition 3.15. *Y is naturally a path metric space which is piecewise isometric to $\mathbb{R}H^n$.*

Proof. We conclude that for each $f \in Y$ there exists an open neighborhood $f \in U_f \subset Y$ such that $\mathscr{P}$ induces a homeomorphism $Y \supset U_f \xrightarrow{\sim} \mathscr{P}(U_f) \subset \mathbb{C}H^n$. Pull back the metric on $\mathscr{P}(U_f)$ to obtain a metric on U_f. Then define a metric on Y as the largest metric which is compatible with the metric on each open set U_f and which preserves the lengths of paths. □

Proposition 3.16. *The path metric on Y descends to a path metric on $P\Gamma \setminus Y$.*

Proof. The metric on Y descends in any case to a pseudo-metric on $P\Gamma \setminus Y$, which will be a metric in case $P\Gamma$ acts by isometries on Y with closed orbits [Gro07, Chapter 1]. The fact that $P\Gamma$ acts isometrically on Y comes from the $P\Gamma$-equivariance of $\mathscr{P} : Y \to \mathbb{C}H^n$ (Lemma 3.11) together with the construction of the metric on Y (Proposition 3.15). To check that the $P\Gamma$-orbits are closed in Y, let $f \in Y$ with representative $(x, \alpha) \in \widetilde{Y}$. By equivariance of $p : \widetilde{Y} \to Y$, we have $p^{-1}(P\Gamma \cdot f) = P\Gamma \cdot (p^{-1} f)$, so since p is a quotient map, it suffices to show that $P\Gamma \cdot (p^{-1} f) = P\Gamma \cdot \cup_{(x,\beta)\sim(x,\alpha)} (x, \beta) = \cup_{(x,\beta)\sim(x,\alpha)} P\Gamma \cdot (x, \beta)$ is closed in $\widetilde{Y}$, thus that each orbit $P\Gamma \cdot (x, \beta)$ is closed in $\widetilde{Y}$. Since $P\Gamma$ is discrete, it suffices to show that $P\Gamma$ acts properly on $\widetilde{Y}$. So let $Z \subset \widetilde{Y}$ be any compact set: we claim that $\{g \in P\Gamma : gZ \cap Z \neq \emptyset\}$ is a finite set. Indeed, for each $g \in P\Gamma$, one has $\tilde{\mathscr{P}}(gZ \cap Z) \subset g\tilde{\mathscr{P}}(Z) \cap \tilde{\mathscr{P}}(Z)$, and the latter is non-empty for only finitely many $g \in P\Gamma$, by properness of the action of $P\Gamma$ on $\mathbb{C}H^n$. Since the metric on Y is a path metric, so is the metric on $P\Gamma \setminus Y$ *[loc. cit.]*. □

3.4 The orbifold structure on the hyperbolic gluing

The next step is to prove that $P\Gamma \setminus Y$ is locally isometric to quotients of open sets in $\mathbb{R}H^n$ by finite groups of isometries.

Let $f \in Y$ with representative $(x, \alpha) \in \widetilde{Y}$ as before. Let $\mathbf{r} = (r_1, \ldots, r_k) \in \mathscr{R}^k$ be a vector of short roots such that $x \in \cap_{i=1}^k H_{r_i}$, $H_{r_i} \neq H_{r_j}$ for $i \neq j$ and k the order of x. We say that f *has k nodes*, and call the H_{r_i} the *nodes of f*. Note that the anti-unitary involution $\alpha : \mathbb{C}H^n \to \mathbb{C}H^n$ interchanges the hyperplanes H_{r_i} because $\alpha(x) = x$, k is the order of x and $\alpha H_{r_i} = H_{\alpha(r_i)}$. If $\alpha(H_{r_i}) = H_{r_i}$ then we call H_{r_i} a *real node of f*. If $\alpha(H_{r_i}) = H_{r_j}$ for some j with $H_{r_i} \neq H_{r_j}$ then we call the pair (H_{r_i}, H_{r_j}) a *a pair of complex conjugate nodes of f*. Write $k = 2a + b$, with a the number of pairs of complex conjugate nodes of f, and b the number of real nodes of f. We relabel the r_i so that they satisfy the following condition: $\alpha(H_{r_i}) = H_{r_{i+1}}$ for i odd and $i \leq 2a$, $\alpha(H_{r_i}) = H_{r_{i-1}}$ for i even and $i \leq 2a$, and $\alpha(H_{r_i}) = H_{r_i}$ for $i > 2a$.

Lemma 3.17. *Let $f \in Y$ with representative $(x, \alpha) \in \widetilde{Y}$ be as above.*

1. *If $\beta \in P\mathscr{A}$ is such that $(x, \beta) \sim (x, \alpha)$, then $\beta = \prod_{i=1}^a \left(h_{r_{2i-1}} \circ h_{r_{2i}}\right)^{j_i} \circ \prod_{i=2a+1}^k h_{r_i}^{j_i} \circ \alpha$ for some $j_1, \ldots, j_a, j_{2a+1}, \ldots, j_k \in \mathbb{Z}$. In particular, there are m^{a+b} such β.*

2. *There is an isometry $\mathbb{C}H^n \xrightarrow{\sim} \mathbb{B}^n(\mathbb{C})$ identifying x with the origin, h_{r_i} with*

$$\mathbb{B}^n(\mathbb{C}) \to \mathbb{B}^n(\mathbb{C}), \quad (t_1, \ldots, t_i, \ldots, t_n) \mapsto (t_1, \ldots, \zeta t_i, \ldots, t_n)$$

and α with the map defined by

$$t_i \mapsto \begin{cases} \bar{t}_{i+1} & \text{for } i \text{ odd and } i \leq 2a \\ \bar{t}_{i-1} & \text{for } i \text{ even and } i \leq 2a \\ \bar{t}_i & \text{for } i > 2a. \end{cases} \tag{3}$$

Proof. We have $\beta = \phi \circ \alpha$ for $\phi \in G(\mathbf{r})$ by Definition 3.3; write $\beta = \prod_{i=1}^k h_{r_i}^{n_i} \circ \alpha$. But then

$$\beta^2 = \left(\prod_{i=1}^k h_{r_i}^{n_i} \circ \alpha\right) \circ \left(\prod_{i=1}^k h_{r_i}^{n_i} \circ \alpha\right) = \prod_{i=1}^k h_{r_i}^{n_i} \circ \prod_{i=1}^k h_{\alpha(r_i)}^{-n_i} = \prod_{i=1}^k h_{r_i}^{n_i} \circ h_{\alpha(r_i)}^{-n_i} = \prod_{i=1}^{2a} h_{r_i}^{n_i} \circ h_{\alpha(r_i)}^{-n_i}$$

$$= \prod_{i=1,odd}^{2a} h_{r_i}^{n_i} \circ h_{r_{i+1}}^{-n_i} \prod_{i=2,even}^{2a} h_{r_i}^{n_i} \circ h_{r_{i-1}}^{-n_i} = \prod_{i=1,odd}^{2a} h_{r_i}^{n_i} \circ h_{r_i}^{-n_{i+1}} \circ \prod_{i=2,even}^{2a} h_{r_i}^{n_i} \circ h_{r_i}^{-n_{i-1}}$$

$$= \mathrm{id} \iff n_i = n_{i+1} \text{ for } i \leq 2a \text{ odd, and } n_i = n_{i-1} \text{ for } i \leq 2a \text{ even}$$

$\iff n_i = n_{i+1}$ for $i \leq 2a$ odd, which proves part 1. For part 2, since the H_{r_i} are orthogonal by Condition 3.1, and their intersection contains x, we can find coordinates $t_0, \ldots, t_n$ on V that identify (V, h) with $\mathbb{C}^{n+1}$ with the hermitian form $h(x,x) = \sum_{i=0}^{n-1} |x_i|^2 - |x_n|^2$, such that $H_{r_i} \subset V$ is identified with the hyperplane $\{t_i = 0\} \subset \mathbb{C}^{n+1}$ and $x \in \cap_i H_{r_i}$ with the point $(0, 0, \ldots, 0, 1)$. But the decomposition $V = \langle x \rangle \oplus_{i=1}^k H_{r_i} \oplus W$ where W is the orthogonal complement of the left hand side is such that $\alpha\left(\langle x \rangle \oplus_{i=1}^k H_{r_i}\right) = \left(\langle x \rangle \oplus_{i=1}^k H_{r_i}\right)$ hence $\alpha W = W$ as well. This implies that α is in the coordinate system t_i given by a matrix

$$\alpha = \begin{pmatrix} \alpha_0 & 0 & \ldots & 0 & & \\ 0 & 0 & \alpha_1 & 0 & & \\ 0 & \alpha_2 & 0 & 0 & & \\ 0 & 0 & 0 & 0 & & \\ \vdots & 0 & \ddots & \vdots & & \\ \vdots & \vdots & 0 & \vdots & \alpha_{n-1} & \\ 0 & 0 & \ldots & 0 & & \alpha_n \end{pmatrix}$$

where every α_i is an anti-holomorphic involution $\mathbb{C} \to \mathbb{C}$, and where $\alpha_i = \alpha_{i+1}$ for $i < 2a$. For $i = 0, \ldots, n$ we then choose $\phi_i \in \mathbb{C}^*$ such that $\phi_i \alpha_i \phi_i^{-1}$ = complex conjugation on $\mathbb{C}$; the matrix $\mathrm{diag}(\phi_i)$ then gives an isometry $\psi : \mathbb{C}^{n+1} \to \mathbb{C}^{n+1}$ such that $\psi \alpha \psi^{-1}$ is as in Equation (3). The end result is an isometry $V \xrightarrow{\sim} \mathbb{C}^{n,1}$ which induces the required $\mathbb{C}H^n \xrightarrow{\sim} \mathbb{B}^n(\mathbb{C})$. ☐

Define $A_f = \mathrm{Stab}_{P\Gamma}(f)$ to be the subgroup of $P\Gamma$ fixing $f \in Y$. This contains the group $G(\mathbf{r}) \cong (\mathbb{Z}/m)^k$. We shall also consider B_f, the subgroup of $G(\mathbf{r})$ generated by the order m complex reflections associated to the real nodes of f, rather than all the nodes. Hence $B_f = \langle h_{r_i} \rangle_{i > 2a} \cong (\mathbb{Z}/m)^b$. Observe that the following holds:

Lemma 3.18. *The stabilizer A_f of $f \in Y$ preserves the subset $Y_f \subset Y$.*

Proof. Let $\psi \in A_f$, with $f = [x, \alpha] \in Y$, $x \in \cap_i H_{r_i}$. Then $(\psi(x), \psi\alpha\psi^{-1}) \sim (x, \alpha) \in Y$. Now let $(y, \beta) \in Y_f$. Then $\beta(x) = x$ and $(x, \alpha) \sim (x, \beta)$. Hence $(x, \alpha) \sim \psi \cdot (x, \alpha) \sim \psi \cdot (x, \beta) = (\psi(x), \psi\beta\psi^{-1})$. This implies $\psi\beta\psi^{-1} \circ \alpha \in G(\mathbf{r})$, so that $(\psi(y), \psi\beta\psi^{-1}) \in Y_f$. ☐

We shall need the following lemma. Write $m = 2^a k$ with $k \neq 0 \bmod 2$.

Lemma 3.19. *Let $T = \{t \in \mathbb{C} : t^m \in \mathbb{R}\}$. Then $G = \langle \zeta_m \rangle$ acts on T by multiplication, and each element in T/G has a unique representative of the form $\zeta_{2^{a+1}}^\epsilon \cdot r$, with $r \geq 0$ and $\epsilon \in \{0, 1\}$.*

Proof. Since m is the largest integer such that $\mathbb{Q}(\zeta_m)$ embeds in K, we have $2|m$. Indeed, suppose the contrary: then $-\zeta_m = \zeta_{2m} \in K$, contradiction. Therefore, we have $a \geq 1$. Next, observe that $t = r\zeta_{2m}^j$ for some $j \in \mathbb{Z}$ and $r \in \mathbb{R}$ if and only if $t^m \in \mathbb{R}$. One easily shows that since $\gcd(2, k) = 1$, we have $\zeta_{2^{a+1}} \cdot \zeta_{2^a k} = (\zeta_{2^{a+1}k})^{k+2}$. Raising both sides to the power $b = (k+2)^{-1} \in (\mathbb{Z}/m)^*$ gives $\zeta_{2m} = \zeta_{2^{a+1}}^b \cdot \zeta_m^b$. Consequently, $t^m \in \mathbb{R}$ if and only if $t = r \cdot \zeta_{2^{a+1}}^{bj} \cdot \zeta_m^{bj}$ for some $r \in \mathbb{R}$. Finally, $\zeta_{2^{a+1}}^u \cdot \zeta_{2^a}^v = \zeta_{2^{a+1}}^{u+2v}$ hence $\langle \zeta_{2^{a+1}} \rangle / \langle \zeta_{2^a} \rangle \cong \mathbb{Z}/2$. ☐

We obtain:

Proposition 3.20. *Keep the above notations, and let $Y_f \subset Y$ be as in Equation (2).*

1. *If f has no nodes, then $G(\boldsymbol{r}) = B_f$ is trivial, and $Y_f = \mathbb{R}H^n_\alpha \cong \mathbb{B}^n(\mathbb{R})$.*

2. *If f has only real nodes, then $B_f \setminus Y_f$ is isometric to $\mathbb{B}^n(\mathbb{R})$.*

3. *If f has $k = 2a$ complex nodes and no other singularities, then $B_f \setminus Y_f = Y_f$ is the union of m^a copies of $\mathbb{B}^n(\mathbb{R})$, any two of which meet along a $\mathbb{B}^{2c}(\mathbb{R})$ for some integer c with $0 \leq c \leq a$.*

4. *If f has $2a$ complex conjugate nodes and b real nodes, then there is an isometry between $B_f \setminus Y_f$ and the union of m^a copies of $\mathbb{B}^n(\mathbb{R})$ identified along common $\mathbb{B}^{2c}(\mathbb{R})'s$, that is, the set Y_f of case 3 above.*

5. *In each case, A_f acts transitively on the indicated $\mathbb{B}^n(\mathbb{R})'s$. If $\mathbb{B}^n(\mathbb{R})$ is any one of them, and $\Gamma_f = (A_f/B_f)_{\mathbb{B}^n(\mathbb{R})}$ its stabilizer, then the natural map*

$$\Gamma_f \setminus \mathbb{B}^n(\mathbb{R}) \to (A_f/B_f) \setminus (B_f \setminus Y_f) = A_f \setminus Y_f$$

is an isometry of path metrics.

Proof. 1. This is clear.

2. Suppose then that f has k real nodes. Then in the local coordinates t_i of Lemma 3.17.2, we have that $\alpha : \mathbb{B}^n(\mathbb{C}) \to \mathbb{B}^n(\mathbb{C})$ is defined by $\alpha(t_i) = \bar{t}_i$. Part 1 of the same lemma shows that any $\beta \in P\mathscr{A}$ fixing x such that $(x, \alpha) \sim (x, \beta)$ is of the form

$$\mathbb{B}^n(\mathbb{C}) \to \mathbb{B}^n(\mathbb{C}), \quad (t_1, \ldots, t_i, \ldots, t_n) \mapsto (\bar{t}_1\zeta^{j_1}, \ldots, \bar{t}_k\zeta^{j_k}, \bar{t}_{k+1}, \ldots, \bar{t}_n).$$

Since f has k real nodes and no complex conjugate nodes, we have (writing $j = (j_1, \ldots, j_k)$ and $\alpha_j = \prod_{i=1}^k h_{r_i}^{j_i} \circ \alpha$):

$$Y_f \cong \bigcup_{j_1,\ldots,j_k=1}^{m} \mathbb{R}H^n_{\alpha_j} \cong \{(t_1, \ldots, t_n) \in \mathbb{B}^n(\mathbb{C}) : t_1^m, \ldots, t_k^m, t_{k+1}, \ldots, t_n \in \mathbb{R}\}\,.$$

Each of the 2^k subsets

$$K_{f,\epsilon_1,\ldots,\epsilon_k} = \left\{(t_1, \ldots, t_n) \in \mathbb{B}^n(\mathbb{C}) : \zeta_{2^{a+1}}^{-\epsilon_1} t_1, \ldots, \zeta_{2^{a+1}}^{-\epsilon_k} t_k \in \mathbb{R}_{\geq 0} \text{ and } t_{k+1}, \ldots, t_n \in \mathbb{R}\right\},$$

indexed by $\epsilon_1, \ldots, \epsilon_k \in \{0, 1\}$, is isometric to the closed region in $\mathbb{B}^n(\mathbb{R})$ bounded by k mutually orthogonal hyperplanes. By Lemma 3.19, their union U is a fundamental domain for B_f in the sense that it maps homeomorphically and piecewise-isometrically onto $B_f \setminus Y_f$. Under its path metric, $U = \cup K_{f,\epsilon_1,\ldots,\epsilon_k}$ is isometric to $\mathbb{B}^n(\mathbb{R})$ by the following map:

$$U \to \mathbb{B}^n(\mathbb{R}), \quad (t_1, \ldots, t_k) \mapsto \left((-\zeta_{2^{a+1}})^{-\epsilon_1} t_1, \ldots, (-\zeta_{2^{a+1}})^{-\epsilon_k} t_k, t_{k+1}, \ldots, t_n\right).$$

This identifies $B_f \setminus Y_f$ with the standard $\mathbb{B}^n(\mathbb{R}) \subset \mathbb{B}^n(\mathbb{C})$.

3. Now suppose f has $k = 2a$ nodes $H_{r_1}, \ldots, H_{r_{2a}}$. There are now m^a anti-isometric involutions α_{j_i} fixing x and such that $(x, \alpha_{j_i}) \sim (x, \alpha)$: they are given in the coordinates t_i as follows, taking $j = (j_1, \ldots, j_a) \in (\mathbb{Z}/m)^a$:

$$\alpha_j : (t_1, \ldots, t_n) \mapsto (\bar{t}_2\zeta^{j_1}, \bar{t}_1\zeta^{j_1}, \ldots, \bar{t}_{2a}\zeta^{j_a}, \bar{t}_{2a-1}\zeta^{j_a}, \bar{t}_{2a+1}, \ldots, \bar{t}_n).$$

So any fixed-point set $\mathbb{R}H^n_{\alpha_j}$ is identified with

$$\mathbb{B}^n(\mathbb{R})_{\alpha_j} = \left\{(t_1, \ldots, t_n) \in \mathbb{B}^n(\mathbb{C}) : t_i = \bar{t}_{i-1}\zeta^{j_i} \text{ for } 1 \leq i \leq 2a \text{ even, } t_i \in \mathbb{R} \text{ for } i > 2a.\right\}$$

All these m^a copies of $\mathbb{B}^n(\mathbb{R})$ meet at the origin of $\mathbb{B}^n(\mathbb{C})$; in fact, for $j \neq j'$, the space $\mathbb{B}^n(\mathbb{R})_{\alpha_j}$ meets the space $\mathbb{B}^n(\mathbb{R})_{\alpha_{j'}}$ in a $\mathbb{B}^{2c}(\mathbb{R})$ if c is the number of pairs (j_i, j'_i) with $j_i = j'_i$.

4. Now we treat the general case. In the local coordinates t_i, any anti-unitary involutions fixing x and equivalent to α is of the form

$$\alpha_j : (t_1, \ldots, t_n) \mapsto (\bar{t}_2\zeta^{j_1}, \bar{t}_1\zeta^{j_1}, \ldots, \bar{t}_{2a}\zeta^{j_a}, \bar{t}_{2a-1}\zeta^{j_a}, \bar{t}_{2a+1}\zeta^{j_{2a+1}}, \ldots, \bar{t}_k\zeta^{j_k}, \bar{t}_{k+1}, \ldots, \bar{t}_n)$$

for some $j = (j_1, \ldots, j_a, j_{2a+1}, \ldots, j_k) \in (\mathbb{Z}/m)^{a+b}$. We now have $B_f \cong (\mathbb{Z}/m)^b$ acting by multiplying the t_i for $2a+1 \leq i \leq k$ by powers of ζ, and there are m^{a+b} anti-unitary involutions α_j. We have $Y_f \cong$

$$\bigcup_{j_1, \ldots, j_k = 1}^{m} \mathbb{R}H^n_{\alpha_j} \cong \left\{(t_1, \ldots, t_n) \in \mathbb{B}^n(\mathbb{C}) : t_2^m = \bar{t}_1^m, \ldots, t_{2a}^m = \bar{t}_{2a-1}^m, t_{2a+1}^m, \ldots, t_k^m, t_{k+1}, \ldots, t_n \in \mathbb{R}\right\}.$$

We look at subsets $K_{f,\epsilon_1,\ldots,\epsilon_k} \subset Y_f$ again, this time defined as $K_{f,\epsilon} = K_{f,\epsilon_1,\ldots,\epsilon_k} =$

$$\left\{(t_1, \ldots, t_n) \in \mathbb{B}^n(\mathbb{C}) : t_i^m = \bar{t}_{i-1}^m \; i \leq 2a \text{ even, } \zeta_{2^{a+1}}^{-\epsilon_i} t_i \in \mathbb{R}_{\geq 0} \; 2a < i \leq k, \; t_i \in \mathbb{R}, i > k\right\}.$$

As before, we have that the natural map $U := \bigcup_\epsilon K_{f,\epsilon} \to B_f \setminus Y_f$ is an isometry. Define

$$\widetilde{Y}_f = \left\{(t_1, \ldots, t_n) \in \mathbb{B}^n(\mathbb{C}) : t_i^m = \bar{t}_{i-1}^m \text{ for } i \leq 2a \text{ even, } t_i \in \mathbb{R}, \text{ for } i > 2a\right\}.$$

Under its path metric, $U = \cup_\epsilon K_{f,\epsilon_1,\ldots,\epsilon_k}$ is isometric to $\widetilde{Y}_f$ by the following map:

$$U \to \widetilde{Y}_f, \quad (t_1, \ldots, t_k) \mapsto \left(t_1, \ldots, t_{2a}, (-\zeta_{2^{a+1}})^{-\epsilon_1} t_{2a+1}, \ldots, (-\zeta_{2^{a+1}})^{-\epsilon_k} t_k, t_{k+1}, \ldots, t_n\right).$$

Hence $B_f \setminus Y_f \cong \widetilde{Y}_f$; but since $\widetilde{Y}_f$ is what Y_f was in case 3, we are done.

5. The transitivity of A_f on the $\mathbb{B}^n(\mathbb{R})'s$ follows from the fact that $G(\mathbf{r}) \subset A_f$ contains transformations multiplying $t_1, \ldots, t_{2a}$ by powers of ζ, hence $t_i \mapsto \zeta^u t_i, t_{i-1} \mapsto t_{i-1}$ maps those t_{i-1}, t_i with $t_i = \bar{t}_{i-1}\zeta^{j_i}$ to those t_{i-1}, t_i with $t_i = \bar{t}_{i-1}\zeta^{j_i+u}$. So if B is any one of the $\mathbb{B}^n(\mathbb{R})'s$ and $G = (A_f/B_f)_H$ its stabilizer, then it remains to prove that $G \setminus B \to A_f \setminus Y_f$ is an isometry. Surjectivity follows from the transitivity of A_f on the $\mathbb{B}^n(\mathbb{R})'s$. It is a piecewise isometry so we only need to prove injectivity.

Lemma 3.21. *Let X be a set on which a group G acts, let Y be a set and $\phi_i : Y \hookrightarrow X$ embeddings, write $Y_i = \phi_i(Y)$ and suppose that $X = \cup_i Y_i$. Let $Y_0 \subset X$ be any one of the $Y_i's$. Let $H \subset G$ be the stabilizer of Y_0. Suppose that for all $y \in X$, the stabilizer of y in G acts transitively on the $Y_i's$ containing y. Then $H \setminus Y_0 \to G \setminus X$ is injective.*

Proof. Let $x, y \in Y_0$ and $g \in G$ such that $g \cdot x = y$. Then $y = gx \in gY_0$. Since also $y \in Y_0$, there is an element $h \in \mathrm{Stab}_G(y)$ such that $hgY_0 = Y_0$ and $hg(x) = h(y) = y$. Let $f = hg$; then $f \in H$ and $f \cdot x = y$, which proves what we want. □

Now let us use the lemma: suppose that $y \in B_f \setminus Y_f$. We need to prove that $\mathrm{Stab}_{A_f/B_f}(y)$ acts transtivitely on the $\mathbb{B}^n(\mathbb{R})'s$ containing y. There exists a tuple

$$j = (j_1, \ldots, j_a, j_{2a+1}, \ldots, j_k) \in (\mathbb{Z}/m)^{a+b}$$

such that $y = (t_1, \ldots, t_n)$ with $t_i = \bar{t}_{i-1}\zeta^{j_i}$ for $i \leq 2a$ even, $t_i = \bar{t}_{i-1}\zeta^{j_i}$ for $2a < i \leq k$, and $t_i \in \mathbb{R}$ for $i > k$. If all t_i are non-zero, then $y \in \cup_{j'} \mathbb{R}H^n_{\alpha_{j'}}$ is only contained in $\mathbb{R}H^n_{\alpha_j}$, so there is nothing to prove. Let us suppose that $t_1 = t_2 = 0$ and the other t_i are non-zero. Then y is contained in all the $\mathbb{R}H^n_{\alpha_{j'}}$ with $j'_i = j_i$ for $i \geq 2$; there are m of them. The stabilizer of y multiplies t_1 and t_2 by powers of ζ and leaves the other t_i invariant; it acts transitively on the $\mathbb{R}H^n_{\alpha_{j'}}$ containing y for if $t_2 = \bar{t}_1\zeta^{j'_1}$ then $\zeta^{(j''_1 - j'_1)}t_2 = \bar{t}_1\zeta^{j''_1}$. The general case is similar. □

Finally, we can prove Theorem 3.10.

Proof of Theorem 3.10. 1. The path metric on $P\Gamma \setminus Y$ is given by Proposition 3.16. Note that the map $\mathscr{P} : Y \to \mathbb{C}H^n$ is a local embedding by Lemma 3.13, which was used to define the metric on Y (Proposition 3.15) - thus almost by definition, $\mathscr{P}$ is a local isometry. For each $f \in Y$ we can find a $P\Gamma_f$-invariant open neighborhood $U_f \subset Y_f \subset Y$ such that $P\Gamma_f \setminus U_f \subset P\Gamma \setminus Y$, with U_f mapping bijectively onto an open subset V_f in the closed subset $\mathscr{P}(Y_f) = \cup_i \mathbb{R}H^n_{\alpha_i} \subset \mathbb{C}H^n$. By $P\Gamma$-equivariance of $\mathscr{P}$, the set V_f is $P\Gamma_f$-invariant, and we have $P\Gamma_f \setminus V_f \subset P\Gamma \setminus \mathbb{C}H^n$. Thus $\bar{\mathscr{P}} : P\Gamma \setminus Y \to P\Gamma \setminus \mathbb{C}H^n$ is also a local isometry.

Note that the map $\mathscr{P} : Y \to \mathbb{C}H^n$ is proper because any compact set in $\mathbb{C}H^n$ meets only finitely many $\mathbb{R}H^{n'}_\alpha s$, $\alpha \in P\mathscr{A}$ (Lemma 3.12), and $\mathscr{P}$ carries each $H_\alpha = p(\mathbb{R}H^n_\alpha)$ homeomorphically onto $\mathbb{R}H^n_\alpha$. Since $P\Gamma \setminus \mathbb{C}H^n$ is complete, the space $P\Gamma \setminus Y$ is complete as well.

Finally, each point $[f] \in P\Gamma \setminus Y$ has an open neighborhood isometric to the quotient of an open set W in $\mathbb{R}H^n$ by a finite group of isometries Γ_f. Indeed, take $Y_f \subset Y$ as in Equation (2), and $f \in U_f \subset Y_f$ as in Lemma 3.13.2. We let $A_f = P\Gamma_f$ be the stabilizer of f in $P\Gamma$ as before, and take an A_f-equivariant open neighborhood $V_f \subset U_f$ such that $A_f \setminus V_f \subset P\Gamma \setminus Y$. By Proposition 3.20.5, we know that $A_f \setminus Y_f$ is isometric to $\Gamma_f \setminus \mathbb{R}H^n$ for some finite group of isometries of $\mathbb{R}H^n$. This implies that $A_f \setminus V_f$ is isometric to some open set W' in $\Gamma_f \setminus \mathbb{R}H^n$. Take $W \subset \mathbb{R}H^n$ to be the preimage of W'. But for any path metric space X locally isometric to quotients of $\mathbb{R}H^n$ by finite groups of isometries, there is a unique real-hyperbolic orbifold structure on X whose path metric is the given one, for if U and U' are connected open subsets of $\mathbb{R}H^n$ and Γ and Γ' finite groups of isometries of $\mathbb{R}H^n$ preserving U and U' respectively, then any isometry $\bar{\phi} : \Gamma \setminus U \to \Gamma' \setminus U'$ extends to an isometry $\phi : \mathbb{R}H^n \to \mathbb{R}H^n$ such that $\phi(U) = U'$ and $\phi\Gamma\phi^{-1} = \Gamma' \subset \mathrm{Isom}(\mathbb{R}H^n)$. We conclude that $P\Gamma \setminus Y$ is naturally a real hyperbolic orbifold.

2. Let us show that $O = \coprod_{\alpha \in C\mathscr{A}} [P\Gamma_\alpha \setminus (\mathbb{R}H^n_\alpha - \mathscr{H})] \subset P\Gamma \setminus Y$ as a hyperbolic suborbifold. It suffices to show that for those $f = [x, \alpha] \in Y$ that have no nodes, the stabilizer $A_f = P\Gamma_f \subset P\Gamma$ of $f \in Y$ and the stabilizer $P\Gamma_{\alpha,x} \subset P\Gamma_\alpha$ of x in $\mathbb{R}H^n_\alpha$ agree as subgroups of $P\Gamma$. Indeed, we have $Y_f = \mathbb{R}H^n_\alpha$, hence $A_f \setminus \mathbb{R}H^n_\alpha = A_f \setminus Y_f = \Gamma_f \setminus \mathbb{R}H^n$ with $\Gamma_f = A_f \setminus B_f = A_f$; by construction the orbifold chart will then be $W \to A_f \setminus W \subset P\Gamma_\alpha \setminus \mathbb{R}H^n_\alpha \subset Y$ for an invariant open subset W of $\mathbb{R}H^n_\alpha$ containing x, which, if $A_f = P\Gamma_{\alpha,x}$, is also an orbifold chart for O at the point (x, α).

To prove that $A_f = P\Gamma_{\alpha,x}$, we first observe that $p : \widetilde{Y} \to Y$ induces an isomorphism between $P\Gamma_{(x,\alpha)}$, the stabilizer of $(x,\alpha) \in \widetilde{Y}$ and $P\Gamma_f$, the stabilizer of $f = [x,\alpha] \in Y$. So it suffices to show that $P\Gamma_{(x,\alpha)} = P\Gamma_{\alpha,x}$. For this we use that the normalizer $N_{P\Gamma}(\alpha)$ and the stabilizer $P\Gamma_\alpha \subset P\Gamma$ of α in $P\Gamma$ are equal, which implies that $P\Gamma_{\alpha,x} = P\Gamma_{(x,\alpha)}$ because

$$\{g \in P\Gamma_\alpha : gx = x\} = \{g \in N_{P\Gamma}(\alpha) : gx = x\} = \left\{g \in P\Gamma : g \cdot (x,\alpha) = (g(x), g\alpha g^{-1}) = (x,\alpha)\right\}.$$

3. The real-hyperbolic orbifold $P\Gamma \setminus Y$ is complete by part 1, so the uniformization of the connected components of $P\Gamma \setminus Y$ follows from the uniformization theorem for (G,X)-orbifolds, see [Thu80, Proposition 13.3.2]. This concludes the proof of Theorem 3.10. ☐

4 Unitary Shimura Varieties

The goal of this section is to prove Proposition 4.5, which describes (in case the signature of h is hyperbolic for one place of K and definite for all others) our ball quotient $P\Gamma \setminus \mathbb{C}H^n$ in terms of moduli of abelian varieties with $\mathcal{O}_K$-action of hyperbolic signature, and Proposition 4.7, which interprets the divisor $P\Gamma \setminus \mathscr{H}$ as the locus of abelian varieties A that admit a homomorphism $\mathbb{C}^g/\Psi(\mathcal{O}_K) \to A$. This has two applications. Firstly, let $\mathscr{X} \to P \to S$ be a relative uniform cyclic cover [AV04] where $P = \mathbb{P}^1_S$ (resp. $\mathbb{P}^3_S$), the fibers of $\mathscr{X} \to S$ are curves (resp. threefolds with $H^{0,3} = 0$) and the induced hermitian form on middle cohomology satisfies the above signature condition. Since the image $\mathfrak{J} \subset P\Gamma \setminus \mathbb{C}H^n$ of the period map $S(\mathbb{C}) \to P\Gamma \setminus \mathbb{C}H^n$ is contained in the locus of abelian varieties whose theta divisor is irreducible, one has $\mathfrak{J} \subset P\Gamma \setminus (\mathbb{C}H^n - \mathscr{H})$. Secondly, if $\mathfrak{D}_K = (\eta)$ for some $\eta \in \mathcal{O}_K - \mathcal{O}_F$ such that $\eta^2 \in \mathcal{O}_K$, then the hyperplanes in the arrangement $\mathscr{H} \subset \mathbb{C}H^n$ are orthogonal along their intersection.

4.1 Alternating and hermitian forms on Λ

The goal of this subsection is to prove two lemma's that will be used in Section 4.2 to show that $P\Gamma \setminus \mathbb{C}H^n$ is a moduli space of abelian varieties and interprete the divisor $P\Gamma \setminus \mathscr{H} \subset P\Gamma \setminus \mathbb{C}H^n$.

Lemma 4.1. *The assignment $T \mapsto \mathrm{Tr}_{K/\mathbb{Q}} \circ T$ defines a bijection between:*

1. *The set of skew-hermitian forms $T : \Lambda_\mathbb{Q} \times \Lambda_\mathbb{Q} \to K$.*
2. *The set of alternating forms $E : \Lambda_\mathbb{Q} \times \Lambda_\mathbb{Q} \to \mathbb{Q}$ such that $E(a \cdot x, y) = E(x, a^\sigma \cdot y)$.*

Under this correspondence, $T(\Lambda,\Lambda) \subset \mathfrak{D}_K^{-1}$ if and only if $E(\Lambda,\Lambda) \subset \mathbb{Z}$.

Proof. Let $T : \Lambda_\mathbb{Q} \times \Lambda_\mathbb{Q} \to K$ be as in 1. Define $E_T = \mathrm{Tr}_{K/\mathbb{Q}} \circ T$. Since T is skew-hermitian, $\mathrm{Tr}_{K/\mathbb{Q}} T(x,y) = -\mathrm{Tr}_{K/\mathbb{Q}} \overline{T(y,x)}$. Since $K/\mathbb{Q}$ is separable, for any $x \in K$, we have $\mathrm{Tr}_{K/\mathbb{Q}}(x) = \sum_{1 \le i \le g} (\tau_i(x) + \tau_i\sigma(x))$ hence $\mathrm{Tr}_{K/\mathbb{Q}}(\sigma(x)) = \mathrm{Tr}_{K/\mathbb{Q}}(x)$, so that $E_T(x,y) = -E_T(y,x)$ for any $x, y \in \Lambda_\mathbb{Q}$. The property in 2 is easily checked. Conversely, let $E : \Lambda_\mathbb{Q} \times \Lambda_\mathbb{Q} \to \mathbb{Q}$ be as in 2. Choose a basis $\{b_1, \ldots, b_{n+1}\} \subset \Lambda$ for Λ over $\mathcal{O}_K$. Define Q to be the induced map $K^{n+1} \times K^{n+1} \to \mathbb{Q}$ and consider the map $K \to \mathbb{Q}$, $a \mapsto Q(a \cdot e_i, e_j)$. Since the trace pairing $K \times K \to \mathbb{Q}$, $(x,y) \mapsto \mathrm{Tr}_{K/\mathbb{Q}}(xy)$ is non-degenerate [Stacks, Tag 0BIE], there exists a unique $t_{ij} \in K$ such that $Q(a \cdot e_i, e_j) = \mathrm{Tr}_{K/\mathbb{Q}}(a \cdot t_{ij})$ for every $a \in K$. This gives a matrix $(t_{ij})_{ij} \in M_{n+1}(K)$ such that $\sigma(t_{ij}) = -t_{ji}$, and the basis $\{b_i\}$ induces a skew-hermitian form $T_E : \Lambda_\mathbb{Q} \times \Lambda_\mathbb{Q} \to K$. The last claim follows by definition of $\mathfrak{D}_K^{-1}$ as the trace dual of $\mathcal{O}_K$. ☐

Examples 4.2. 1. Suppose $K = \mathbb{Q}(\sqrt{\Delta})$ is imaginary quadratic with discriminant Δ and non-trivial Galois automorphism $a \mapsto a^\sigma$. Let $E : \Lambda \times \Lambda \to \mathbb{Z}$ be an alternating form with $E(a \cdot x, y) = E(x, a^\sigma \cdot y)$. The form $T : \Lambda \times \Lambda \to \mathfrak{D}_K^{-1} = (\sqrt{\Delta})^{-1}$ is defined as

$$T(x,y) = \frac{E(\sqrt{\Delta} \cdot x, y) + E(x,y)\sqrt{\Delta}}{2\sqrt{\Delta}}.$$

2. Let $K = \mathbb{Q}(\zeta)$ where $\zeta = \zeta_p = e^{2\pi i/p} \in \mathbb{C}$ for some prime number $p > 2$. Let $E : \Lambda \times \Lambda \to \mathbb{Z}$ be an alternating form with $E(a \cdot x, y) = E(x, a^\sigma \cdot y)$. Then $\mathfrak{D}_K = (p/(\zeta - \zeta^{-1}))$ and

$$T : \Lambda \times \Lambda \to \mathfrak{D}_K^{-1}, \quad T(x,y) = \frac{1}{p}\sum_{j=0}^{p-1} \zeta^j E\left(x, \zeta^j \cdot y\right).$$

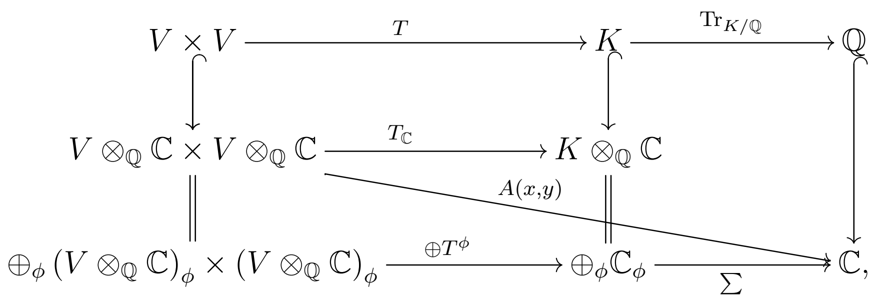
Now consider $E : \Lambda_\mathbb{Q} \times \Lambda_\mathbb{Q} \to \mathbb{Q}$ and $T : \Lambda_\mathbb{Q} \times \Lambda_\mathbb{Q} \to K$ as in Lemma 4.1, and suppose that E is non-degenerate. Let $\varphi : K \to \mathbb{C}$ be an embedding. Define a skew-hermitian form T^φ as

$$T^\varphi : \Lambda \otimes_{\mathcal{O}_K,\varphi} \mathbb{C} \times \Lambda \otimes_{\mathcal{O}_K,\varphi} \mathbb{C} \to \mathbb{C}, \quad T^\varphi(\sum_i x_i \otimes \lambda_i, \sum_j y_j \otimes \mu_j) = \sum_{ij} \lambda_i \overline{\mu_j} \cdot \varphi\left(T(x_i, y_j)\right). \quad (4)$$

On $\Lambda_\mathbb{C}$, we also have the skew-hermitian form $A(x,y) = E_\mathbb{C}(x, \bar{y})$. The composition $(\Lambda \otimes_\mathbb{Z} \mathbb{C})_\varphi \to \Lambda \otimes_\mathbb{Z} \mathbb{C} \to \Lambda \otimes_{\mathcal{O}_K,\varphi} \mathbb{C}$ is an isomorphism. Define A^φ to be the restriction of A to the subspace $(\Lambda \otimes_\mathbb{Z} \mathbb{C})_\varphi = \Lambda \otimes_{\mathcal{O}_K,\varphi} \mathbb{C} \subset \Lambda_\mathbb{C}$. Note that $\Lambda \otimes_\mathbb{Z} \mathbb{C} \cong \oplus_{\phi:K\to\mathbb{C}} (\Lambda \otimes_\mathbb{Z} \mathbb{C})_\phi$; for $x \in \Lambda \otimes_\mathbb{Z} \mathbb{C}$, denote by x^ϕ the image of x under the projection $\Lambda \otimes_\mathbb{Z} \mathbb{C} \to (\Lambda \otimes_\mathbb{Z} \mathbb{C})_\phi$.

Lemma 4.3. *We have $T^\varphi = A^\varphi$ as skew-hermitian forms $(\Lambda \otimes_\mathbb{Z} \mathbb{C})_\varphi \times (\Lambda \otimes_\mathbb{Z} \mathbb{C})_\varphi \to \mathbb{C}$. More precisely, we have $A(x,y) = \sum_{\phi:K\to\mathbb{C}} T^\phi(x^\phi, y^\phi)$ for every $x, y \in \Lambda \otimes_\mathbb{Z} \mathbb{C}$.*

Proof. Write $V = \Lambda_\mathbb{Q}$. The lemma follows from the fact that the following diagram commutes:



where ϕ ranges over the set of embeddings $K \to \mathbb{C}$, $\mathbb{C}_\phi$ is the K-module $\mathbb{C}$ where K acts via ϕ, and $T_\mathbb{C} : V \otimes_\mathbb{Q} \mathbb{C} \times V \otimes_\mathbb{Q} \mathbb{C} \to K \otimes_\mathbb{Q} \mathbb{C}$ is the map that sends $(v \otimes \lambda, x \otimes \mu)$ to $\lambda\bar{\mu}T(v,w)$. ☐

4.2 Moduli of abelian varieties with $\mathcal{O}_K$-action

Notation 4.4. In the rest of Section 4, we fix a non-degenerate hermitian form $\mathfrak{h} : \Lambda \times \Lambda \to \mathfrak{D}_K^{-1}$. We also fix an element $\xi \in \mathfrak{D}_K^{-1}$ such that $\sigma(\xi) = -\xi$ and $\Im(\tau_i(\xi)) < 0$ for $1 \le i \le g$ and write $\eta = \xi^{-1}$. This defines a skew-hermitian form $T : \Lambda \times \Lambda \to \mathfrak{D}_K^{-1}$ where $T = \xi \cdot \mathfrak{h}$, in turn attached to a symplectic form $E : \Lambda \times \Lambda \to \mathbb{Z}$ such that $E(ax, y) = E(x, a^\sigma y)$ for all $a \in \mathcal{O}_K$, $x, y \in \Lambda$, see Lemma 4.1. Write $V_i = \Lambda_\mathbb{Q} \otimes_{K,\tau_i} \mathbb{C}$ and define $\mathfrak{h}^{\tau_i} : V_i \times V_i \to \mathbb{C}$ to be the hermitian form restricting to $\tau_i \circ \mathfrak{h}$ on Λ. Let (r_i, s_i) be the signature of the hermitian form $\mathfrak{h}^{\tau_i}$.

Consider a complex abelian variety A equipped with a homomorphism $\iota : \mathcal{O}_K \to \mathrm{End}(A)$ and a polarization $\lambda : A \to A^\vee$, satisfying the following conditions:

1. $\iota(a)^\dagger = i(a^\sigma)$ for the corresponding Rosati involution $\dagger : \mathrm{End}(A)_\mathbb{Q} \to \mathrm{End}(A)_\mathbb{Q}$,
2. $\mathrm{char}(t, \iota(a)|\mathrm{Lie}(A)) = \prod_{\nu=1}^{g}(t - a^{\tau_i})^{r_i} \cdot (t - a^{\tau_i\sigma})^{s_i} \quad \in \quad \mathbb{C}[t]$.

Note that $\dim A = g(n+1)$. Define $E_A : H_1(A,\mathbb{Z}) \times H_1(A,\mathbb{Z}) \to \mathbb{Z}$ to be the alternating form corresponding to λ. The condition on the Rosati involution implies that $E_A(\iota(a)x, y) = E_A(x, \iota(a^\sigma)y)$ for $x, y \in H_1(A,\mathbb{Q})$. Let $\mathfrak{h}_A$ be the hermitian form $\mathfrak{h}_A = \eta T_A$ on $H_1(A,\mathbb{Z})$, where $T_A : H_1(A,\mathbb{Z}) \times H_1(A,\mathbb{Z}) \to \mathfrak{D}_K^{-1}$ is the skew-hermitian form attached to E_A via Lemma 4.1.

Let $\widetilde{\mathrm{Sh}}_K(\mathfrak{h})$ be the set of isomorphism classes of four-tuples (A, i, λ, j) where (A, i, λ) is as above and where $j : H_1(A,\mathbb{Z}) \to \Lambda$ is a symplectic isomorphism of $\mathcal{O}_K$-modules. Finally, define $\mathbb{D}(V_i)$ to be the space of negative s_i-planes in the hermitian space $(V_i, \mathfrak{h}^{\tau_i})$. We then have the following proposition which is due to Shimura, see [Shi63, Theorem 2] or [Shi64, §1]. We give a different proof since it will imply Proposition 4.7 below, whereas we did not know how to deduce Proposition 4.7 from *loc. cit.* We remark that Shimura assumes Λ to be an R-module for any order $R \subset \mathcal{O}_K$; our proof carries over but we don't need this generalization.

Proposition 4.5. *There is a canonical bijection* $\widetilde{\mathrm{Sh}}_K(\mathfrak{h}) \cong \mathbb{D}(V_1) \times \ldots \times \mathbb{D}(V_g)$.

Proof. Let (A, i, λ, j) be a representative of an isomorphism class in $\widetilde{\mathrm{Sh}}_K(\mathfrak{h})$. Let $H_1(A,\mathbb{C}) = H^{-1,0} \oplus H^{0,-1}$ be the Hodge decomposition of A. For $1 \le i \le g$ there is a decomposition

$$H_1(A,\mathbb{C})_{\tau_i} = H^{-1,0}_{\tau_i} \oplus H^{0,-1}_{\tau_i} \tag{5}$$

where $\dim H^{-1,0}_{\tau_i} = r_i$ and $\dim H^{0,-1}_{\tau_i} = s_i$, where the latter holds because $\overline{H^{-1,0}_{\tau_i\sigma}} = H^{0,-1}_{\tau_i}$. By Lemma 4.3, $\tau_i(\eta)E_{A,\mathbb{C}}(x,\bar{y})$ and $\mathfrak{h}^{\tau_i}_{A,\mathbb{C}}(x,y)$ agree as hermitian forms on $H_1(A,\mathbb{Z}) \otimes_{\mathcal{O}_K,\tau_i} \mathbb{C}$. Since $\Im \tau_i(\eta) > 0$ for every i, the decomposition of $H_1(A,\mathbb{C})_{\tau_i}$ in (5) is a decomposition into a positive definite r_i-dimensional subspace and a negative definite s_i-dimensional subspace. The isomorphism $j : H_1(A,\mathbb{Q}) \to \Lambda_\mathbb{Q}$ induces an isometry $j_i : H_1(A,\mathbb{C})_{\tau_i} \to V_i$ for every i, and so we obtain a negative s_i-plane $j(H^{0,-1}_{\tau_i})$ in the hermitian space V_i for all i. Reversing the argument shows that given a negative s_i-plane $X_i \subset V_i$ for every i, there is a canonical polarized abelian variety $A = H^{-1,0}/\Lambda$, acted upon by $\mathcal{O}_K$ and inducing the planes $X_i \subset V_i$. □

Let $\mathrm{Sh}_K(\mathfrak{h})$ be the set of isomorphism classes of polarized $\mathcal{O}_K$-linear abelian varieties (A, i, λ) such that $H_1(A,\mathbb{Z})$ is isometric to Λ as hermitian $\mathcal{O}_K$-modules. Let $\Gamma(\mathfrak{h}) = \mathrm{Aut}_{\mathcal{O}_K}(\Lambda, \mathfrak{h})$: this is the group of $\mathcal{O}_K$-linear automorphisms of Λ preserving our form $\mathfrak{h} : \Lambda \times \Lambda \to \mathfrak{D}_K^{-1}$. The bijection in Proposition 4.5 being $\Gamma(\mathfrak{h})$-equivariant, we obtain the following:

Corollary 4.6. *There is a canonical bijection* $\mathrm{Sh}_K(\mathfrak{h}) \cong \Gamma(\mathfrak{h}) \backslash \prod_{i=1}^{g} \mathbb{D}(V_i)$. □

4.3 Abelian varieties in the hyperplane arrangement

The set of embeddings Ψ defines a map $\Psi : \mathcal{O}_K \to \mathbb{C}^g$ giving a complex torus $\mathbb{C}^g/\Psi(\mathcal{O}_K)$. The map $Q : K \times K \to \mathbb{Q}$, $Q(x,y) = \mathrm{Tr}_{K/\mathbb{Q}}(\xi x \bar{y})$ is a non-degenerate $\mathbb{Q}$-bilinear form such that $Q(ax, y) = Q(x, a^\sigma y)$ for every $a, x, y \in K$, and $Q(\mathcal{O}_K, \mathcal{O}_K) \subset \mathbb{Z}$ because $\xi \in \mathfrak{D}_K^{-1}$. By [Mil20, Example 2.9 & Footnote 16], Q defines a Riemann form on the complex torus $\mathbb{C}^g/\Psi(\mathcal{O}_K)$.

Suppose that $(r_1, s_1) = (n, 1)$ and $(r_i, s_i) = (n+1, 0)$ for $2 \le i \le g$. As in Section 3, let $\mathbb{C}H^n$ be the set of negative lines in $\Lambda \otimes_{\mathcal{O}_K,\tau_1} \mathbb{C}$, and define $\mathscr{H} = \cup_{\mathfrak{h}(r,r)=1} r^\perp_\mathbb{C} \subset \mathbb{C}H^n$.

Proposition 4.7. *Under the bijection $\widetilde{\mathrm{Sh}}_K(\mathfrak{h}) \cong \mathbb{C}H^n$ of Proposition 4.5, the subset $\mathscr{H} \subset \mathbb{C}H^n$ corresponds to the isomorphism classes of those polarized marked $\mathcal{O}_K$-linear abelian varieties A that admit a $\mathcal{O}_K$-linear homomorphism $\mathbb{C}^g/\Psi(\mathcal{O}_K) \to A$ of polarized abelian varieties.*

Proof. Let $[A, i, \lambda, y] \in \widetilde{\mathrm{Sh}}_K(\mathfrak{h})$ correspond to $[x] \in \mathbb{C}H^n$. We may assume that $A = H^{-1,0}/\Lambda$ with $\Lambda \otimes_{\mathbb{Z}} \mathbb{C} = H^{-1,0} \oplus H^{0,-1}$, and that $T_A = T$. Let $\phi : \mathbb{C}^g/\Psi(\mathcal{O}_K) \to A$ be such a homomorphism. We obtain a homomorphism $\mathcal{O}_K \to \Psi(\mathcal{O}_K) \to H_1(A, \mathbb{Z}) = \Lambda$ which, for simplicity, we also denote by $\phi : \mathcal{O}_K \to \Lambda$. Let $r \in \Lambda$ be the image of $1 \in \mathcal{O}_K$. The fact that $Q = \phi^* E_A$ implies that $T_Q = \phi^* T_A = \phi^* T$. Therefore, we have

$$\eta^{-1} = T_Q(1,1) = T_A(\phi(1), \phi(1)) = T(\phi(1), \phi(1)) = T(r, r),$$

so that $\mathfrak{h}(r, r) = \eta \cdot T(r, r) = 1$. Next, we claim that $\mathfrak{h}(x, r_\tau) = 0$, where $r_\tau \in (\Lambda \otimes_{\mathbb{Z}} \mathbb{C})_\tau = H_\tau$ is the image of $r \in \Lambda$. Write $\Psi(\mathcal{O}_K) = L$, $L \otimes \mathbb{C} = W^{-1,0} \oplus W^{0,-1}$, and let $\alpha \in L$ correspond to $1 \in \mathcal{O}_K$. Notice that $(L \otimes_{\mathbb{Z}} \mathbb{C})_\tau = W_\tau^{-1,0}$. Consequently, since the composition

$$W_\tau^{-1,0} = (L \otimes_{\mathbb{Z}} \mathbb{C})_\tau \to (\Lambda \otimes_{\mathbb{Z}} \mathbb{C})_\tau = H_\tau^{-1,0} \oplus H_\tau^{0,-1}$$

factors through the inclusion of $H_\tau^{-1,0}$ into $(L \otimes_{\mathbb{Z}} \mathbb{C})_\tau$, we see that $r_\tau = r_\tau^{-1,0} \in H_\tau^{-1,0} = (H_\tau^{0,-1})^\perp = \langle x \rangle^\perp$, and the claim follows. Conversely, let $[x] \in r_{\mathbb{C}}^\perp \subset \mathscr{H}$ with $r \in \Lambda$ such that $\mathfrak{h}(r, r) = 1$ and consider the marked abelian variety $A = H^{-1,0}/\Lambda$ corresponding to $[x]$. Define a homomorphism $\phi : \mathcal{O}_K \to \Lambda$ by $\phi(1) = r$. Then ϕ can easily be shown to be is a morphism of Hodge structures using the fact that its $\mathbb{C}$-linear extension preserves the eigenspace decompositions. We obtain an $\mathcal{O}_K$-linear homorphism $\phi : \mathbb{C}^g/\Psi(\mathcal{O}_K) \to A$ it is easily shown that the fact that $\mathfrak{h}(r, r) = 1$ implies that ϕ preserves the polarizations on both sides. ☐

Observe that if the different $\mathfrak{D}_K \subset \mathcal{O}_K$ is a principal ideal $(\eta) \subset \mathcal{O}_K$, then we have

$$\{x \in K : \mathrm{Tr}_{K/\mathbb{Q}}\left(x\eta^{-1}\mathcal{O}_K\right) \subset \mathbb{Z}\} = \{x \in K : x \cdot \eta^{-1}\mathcal{O}_K \subset \eta^{-1}\mathcal{O}_K\} = \{x \in K : x\mathcal{O}_K \subset \mathcal{O}_K\} = \mathcal{O}_K,$$

hence $Q : \Psi(\mathcal{O}_K) \times \Psi(\mathcal{O}_K) \to \mathbb{Z}$ defines a principal polarization on $\mathbb{C}^g/\Psi(\mathcal{O}_K)$. In fact, for $\beta \in K$, the rational Riemann form $\Psi(K) \times \Psi(K) \to \mathbb{Q}$, $(\Psi(x), \Psi(y)) \mapsto \mathrm{Tr}_{K/\mathbb{Q}}(\beta^{-1} x \bar{y})$ defines a principal polarization on $\mathbb{C}^g/\Psi(\mathcal{O}_K)$ if and only if β generates the different ideal $\mathfrak{D}_K$, $\sigma(\beta) = -\beta$ and $\Im(\varphi(\beta)) > 0$ for every $\varphi \in \Psi$. This follows from the above; see also [Wam99].

Theorem 4.8. *Suppose that the CM type (K, Ψ) is primitive, $\mathfrak{D}_K = (\eta)$ for $\eta \in \mathcal{O}_K$ such that $\sigma(\eta) = -\eta$, and the signature of $\mathfrak{h}^{\tau_i}$ is $(n, 1)$ for $i = 1$ and $(n+1, 0)$ for $i \neq 1$. Let $r_1, r_2 \in \Lambda$ satisfy $H_{r_1} \cap H_{r_2} \neq \emptyset$ and $H_{r_1} \neq H_{r_2} \subset \mathbb{C}H^n$ for $H_{r_1} = (r_i)_{\mathbb{C}}^\perp \subset \mathbb{C}H^n$. Then $\mathfrak{h}(r_1, r_2) = 0$.*

Proof. Let $[x] \in H_r \cap H_t \subset \mathbb{C}H^n(V)$ and let A be the abelian variety attached to $[x]$. Define B to be the principally polarized abelian variety $\mathbb{C}^g/\Psi(\mathcal{O}_K)$. By Proposition 4.7, the roots r and t induce $\mathcal{O}_K$-linear embeddings $\phi_1 : B \hookrightarrow A$ and $\phi_2 : B \hookrightarrow A$ of polarized abelian varieties. By Lemma 4.9 below, each ϕ induces a decomposition $A \cong B \times C_i$ as polarized abelian varieties. Note that B is non-decomposable as an abelian variety because $\mathrm{End}(B) \otimes_{\mathbb{Z}} \mathbb{Q} = K$ is a field (here we use that the CM type (K, Ψ) is primitive). By [Deb96], the decomposition of (A, λ) into non-decomposable polarized abelian subvarieties is unique, in the strong sense that if (A_i, λ_i), $i \in \{1, \dots, r\}$ and (B_j, μ_j), $j \in \{1, \dots, m\}$ are polarized abelian subvarieties such that the natural homomorphisms $\prod_i (A_i, \lambda_i) \to (A, \lambda)$ and $\prod_j (B_j, \lambda_j) \to (A, \lambda)$ are isomorphisms, then $r = m$ and there exists a permutation σ on $\{1, \dots, r\}$ such that B_j and $A_{\sigma(j)}$ are *equal* as polarized abelian subvarieties of (A, λ), for every $j \in \{1, \dots, r\}$. Consequently, if we write $B_i = \phi_i(B) \subset A$, then either $B_1 = B_2 \subset A$ or $B_1 \cap B_2 = \{0\}$. Suppose first that $B_1 = B_2$.

Then $\mathcal{O}_K \cdot r = \phi_1(\mathcal{O}_K) = \phi_2(\mathcal{O}_K) = \mathcal{O}_K \cdot t \subset \Lambda$. Consequently, $r = \lambda t$ for some $\lambda \in \mathcal{O}_K^*$; but then $H_r = H_t$ which is absurd. Therefore, we must have $A \cong B_1 \times B_2 \times C$ as polarized abelian varieties for some polarized abelian subvariety C of A. This implies that $H^{-1,0} = \mathrm{Lie}(A) \cong \mathrm{Lie}(B_1) \times \mathrm{Lie}(B_2) \times \mathrm{Lie}(C)$, which is orthogonal for the positive definite hermitian form $iE_{\mathbb{C}}(x, \bar{y})$ on $H^{-1,0}$. On the other hand, we have that $r_\tau = r_\tau^{-1,0} \in H_\tau^{-1,0}$ and $t_\tau = t_\tau^{-1,0} \in H_\tau^{-1,0}$: see the proof of Proposition 4.7. Moreover, by Lemma 4.3 we have

$$\mathfrak{h}(r,t) = \mathfrak{h}^\tau(r_\tau, t_\tau) = \tau(\eta) \cdot T_{\mathbb{C}}^\tau(r_\tau, t_\tau) = \tau(\eta) \cdot E_{\mathbb{C}}(r_\tau, \bar{t_\tau}) = \tau(\eta) \cdot E_{\mathbb{C}}(r_\tau^{-1,0}, \overline{t_\tau^{-1,0}})$$

so it suffices to show that $iE_{\mathbb{C}}(r_\tau^{-1,0}, \overline{t_\tau^{-1,0}}) = 0$. But $r_\tau^{-1,0} \in \mathrm{Lie}(B_1)$ and $t_\tau^{-1,0} \in \mathrm{Lie}(B_2)$. □

Lemma 4.9. *Let A be an abelian variety over a field k, with polarization $\lambda : A \to \widehat{A}$. Let $B \subset A$ be an abelian subvariety such that the polarization $\mu = \lambda|_B$ is principal. There exists a polarized abelian subvariety $Z \subset A$ such that $A \cong B \times Z$ as polarized abelian varieties.*

Proof. Let $W = \mathrm{Ker}(A \xrightarrow{\lambda} \widehat{A} \to \widehat{B})$. Let $Z = W^0_{\mathrm{red}}$. Then Z is an abelian subvariety of A, and has dimension $\dim(A) - \dim(B)$. The kernel of the natural homomorphism $B \times Z \to A$ is contained in $(B \cap Z) \times (B \cap Z)$; but $B \cap Z \subset B \cap W = \{0\}$ because $\mu : B \to \widehat{B}$ is an isomorphism. Therefore the natural homomorphism $B \times Z \to A$ is an isomorphism. □

Finally, we remark that the condition on the different $\mathfrak{D}_K$ in Theorem 4.8 is quite often satisfied:

Lemma 4.10. *Suppose that $K/\mathbb{Q}$ is imaginary quadratic or that $K = \mathbb{Q}(\zeta_n)$ for some integer $n \geq 3$. Then $\mathfrak{D}_K = (\eta) \subset \mathcal{O}_K$ for some element $\eta \in \mathcal{O}_K$ such that $F = K(\eta)$ and $\sigma(\eta) = -\eta$.*

Proof. If $K/\mathbb{Q}$ is imaginary quadratic with discriminant Δ, then $\mathfrak{D}_K = (\sqrt{\Delta})$ and the assertion is immediate. Let $n \geq 3$ be an integer and consider $K = \mathbb{Q}(\zeta_n)$ and $F = \mathbb{Q}(\alpha)$ with $\alpha = \zeta_n + \zeta_n^{-1}$. Since $\mathcal{O}_K = \mathbb{Z}[\zeta_n]$ by [Neu99, I, Proposition 10.2], we have $\mathcal{O}_K = \mathcal{O}_F[\zeta_n]$. Notice that $f(x) = x^2 - \alpha x + 1 \in \mathcal{O}_F[x]$ is the minimal polynomial of ζ_n over F. We have $f'(\zeta_n) = 2\zeta_n - \alpha\zeta_n = \zeta_n - \zeta_n^{-1}$. Therefore, $\mathfrak{D}_{K/F} = (f'(\zeta_n)) = (\zeta_n - \zeta_n^{-1})$ [Neu99, III, Proposition 2.4]. By [Lia76], we know that $\mathcal{O}_F = \mathbb{Z}[\alpha]$. If $g(x) \in \mathbb{Z}[x]$ is the minimal polynomial of α over $\mathbb{Q}$, then $\mathfrak{D}_{F/\mathbb{Q}} = (g'(\alpha))$. Since $\mathfrak{D}_{K/\mathbb{Q}} = \mathfrak{D}_{K/F}\mathfrak{D}_{F/\mathbb{Q}}$ [Neu99, III, Proposition 2.2], we obtain:

$$\mathfrak{D}_{K/\mathbb{Q}} = \mathfrak{D}_{K/F}\mathfrak{D}_{F/\mathbb{Q}} = \left(\zeta_n - \zeta_n^{-1}\right) \cdot (g'(\alpha)) = \left((\zeta_n - \zeta_n^{-1})g'(\alpha)\right).$$

□

Remark 4.11. 100 It would be more natural to attach an orthogonal hyperplane arrangement $\mathcal{H} \subset \mathbb{C}H^n$ to every primitive CM field K and integral hermitian form $\mathfrak{h}$ of hyperbolic signature in such a way that $\mathcal{H} = \mathscr{H} = \cup_{\mathfrak{h}(r,r)=1} r_{\mathbb{C}}^{\perp}$ in case $\mathfrak{D}_K = (\eta)$ for some $\eta \in \mathcal{O}_K$ such that $\sigma(\eta) = -\eta$. It appears that this can be done. The idea is as follows. Consider our CM field K. Choose $\beta \in \mathcal{O}_K - \mathcal{O}_F$ such that $\beta^2 \in \mathcal{O}_F$; then choose a CM type $\Psi = \{\tau_i : K \to \mathbb{C}\}_{1 \leq i \leq g}$ such that $\Im(\tau_i(\beta)) > 0$ for all i. Let $\mathfrak{h}$ be a non-degenerate hermitian form $\Lambda \times \Lambda \to \mathfrak{D}_K^{-1}$ such that $\mathrm{sign}(\mathfrak{h}^\tau) = (n, 1)$ and $\mathrm{sign}(\mathfrak{h}^{\tau_i}) = (n+1, 0)$ for $i \neq 1$. Let $\mathscr{S}$ be the set of fractional ideals $\mathfrak{a} \subset K$ for which there exist an element $b \in \mathcal{O}_F$ such that $\mathfrak{D}_K \mathfrak{a}\bar{\mathfrak{a}} = (b\beta)$. By [Wam99, Theorem 4], $\mathscr{S}$ is not empty. For $\mathfrak{a} \in \mathscr{S}$, define $\eta = b\beta \in \mathcal{O}_K$ and consider the complex torus $B = \mathbb{C}^g/\Psi(\mathfrak{a})$. It is equipped with the Riemann form $Q : \Psi(\mathfrak{a}) \times \Psi(\mathfrak{a}) \to \mathbb{Z}$, $(x, y) \mapsto \mathrm{Tr}_{K/\mathbb{Q}}(\eta^{-1} x\bar{y})$, and Q defines a principal polarization on B [Wam99, Theorem 3]. Let $\mathscr{R}$ be the set of embeddings $\phi : \mathfrak{a} \to \Lambda$, $\mathfrak{a} \in \mathscr{S}$, such that $\mathfrak{h}(\phi(x), \phi(y)) = x\bar{y}$ for all $x, y \in \mathfrak{a}$. For $\phi \in \mathscr{R}$, one obtains a hyperplane $H_\phi = \{x \in \mathbb{C}H^n : \mathfrak{h}^\tau(x, \phi(\mathfrak{a})) = 0\} \subset \mathbb{C}H^n$. The sought-for hyperplane arrangement $\mathcal{H} \subset \mathbb{C}H^n$ is defined as $\mathcal{H} = \cup_{\phi \in \mathscr{R}} H_\phi$. Indeed, if the CM type (K, Ψ) is primitive, then $\mathcal{H}$ is an orthogonal arrangement by arguments similar to those used to prove Proposition 4.7 and Theorem 4.8.

5 The Moduli Space of Complex Binary Quintics as a Ball Quotient

Recall from Section 1 that $X \cong \mathbb{A}^6_{\mathbb{R}}$ is the real affine space of homogeneous degree 5 polynomials $F \in \mathbb{R}[x,y]$, X_0 the subvariety of polynomials with distinct roots, and $X_s \subset X$ the subvariety of polynomials with roots of multiplicity at most two, i.e. those non-zero polynomials whose class in the associated projective space is stable in the sense of geometric invariant theory for the action of $\mathrm{SL}_{2,\mathbb{R}}$ on it. In the remaining Sections 5, 6, 7 and 8, the CM field K will be the cyclotomic field $\mathbb{Q}(\zeta)$ with $\zeta = \zeta_5 = e^{2\pi i/5} \in \mathbb{C}$. Recall that $\mathcal{O}_K = \mathbb{Z}[\zeta]$, the ring of integers of K.

The goal of this section will be to prove that there exists a hermitian $\mathcal{O}_K$-lattice Λ of rank 3 and an isomorphism of complex analytic spaces $G \setminus X_s(\mathbb{C}) \cong P\Gamma \setminus \mathbb{C}H^2$ restricting to an orbifold isomorphism $G \setminus X_0(\mathbb{C}) \cong P\Gamma \setminus (\mathbb{C}H^2 - \mathscr{H})$, where $\Gamma = \mathrm{Aut}(\Lambda)$, $G = \mathrm{GL}_2(\mathbb{C})/\mu_K$, and $\mathscr{H} \subset \mathbb{C}H^n$ is the hyperplane arrangement defined by the norm 1 vectors in Λ.

5.1 The Jacobian of a cyclic quintic cover of $\mathbb{P}^1$

We begin with the following:

Lemma 5.1. *Let $Z \subset \mathbb{P}^1_{\mathbb{C}}$ be a smooth quintic hypersurface. Let $\mathbb{P}^2_{\mathbb{C}} \supset C \to \mathbb{P}^1_{\mathbb{C}}$ be the quintic cover of $\mathbb{P}^1$ ramified along Z. Then C has the following refined Hodge numbers:*

$$h^{1,0}(C)_\zeta = 3, \quad h^{1,0}(C)_{\zeta^2} = 2, \quad h^{1,0}(C)_{\zeta^3} = 1, \quad h^{1,0}(C)_{\zeta^4} = 0 \tag{6}$$
$$h^{0,1}(C)_\zeta = 0, \quad h^{0,1}(C)_{\zeta^2} = 1, \quad h^{0,1}(C)_{\zeta^3} = 2, \quad h^{0,1}(C)_{\zeta^4} = 3. \tag{7}$$

Proof. This follows from the Hurwitz-Chevalley-Weil formula, see [MO11, Proposition 5.9]. Alternatively, see [CT99, Section 5]. □

Now fix a point $F_0 \in X_0(\mathbb{C})$ and let

$$C = \{z^5 = F_0(x,y)\} \subset \mathbb{P}^2_{\mathbb{C}} \tag{8}$$

be the corresponding cyclic cover of $\mathbb{P}^1_{\mathbb{C}}$. Let $A = J(C) = \mathrm{Pic}^0(C)$ be the Jacobian of C. Then A is a principally polarized abelian variety of dimension 6 equipped with a homomorphism $\iota : \mathcal{O}_K = \mathbb{Z}[\zeta] \to \mathrm{End}(A)$. Write $\Lambda = H_1(A(\mathbb{C}), \mathbb{Z})$ and consider the Hodge decomposition $\Lambda \otimes_{\mathbb{Z}} \mathbb{C} = H^{-1,0} \oplus H^{0,-1}$. Define $\tau_i : K \to \mathbb{C}, i \in \{1,2\}$ by $\tau_1(\zeta) = \zeta^3, \tau_2(\zeta) = \zeta^4$. Since $H^{-1,0} = \mathrm{Lie}(A) = H^1(C, \mathcal{O}_C) = H^{0,1}(C)$, Lemma 5.1 implies that

$$\dim_{\mathbb{C}} H^{-1,0}_{\tau_1} = 2, \quad \dim_{\mathbb{C}} H^{-1,0}_{\tau_1\sigma} = 1, \quad \dim_{\mathbb{C}} H^{-1,0}_{\tau_2} = 3, \quad \dim_{\mathbb{C}} H^{-1,0}_{\tau_2\sigma} = 0. \tag{9}$$

Define $\eta = 5/(\zeta - \zeta^{-1})$. Then $\mathfrak{D}_K = (\eta)$ (see Lemma 4.10). Let $E : \Lambda \times \Lambda \to \mathbb{Z}$ be the alternating form corresponding to the polarization of the abelian variety A. Then for $a \in \mathcal{O}_K$ and $x, y \in \Lambda$, we have $E(\iota(a)x, y) = E(x, \iota(a^\sigma)y)$. Let $T : \Lambda \times \Lambda \to \mathfrak{D}_K^{-1}$ be the skew-hermitian form corresponding to E via Lemma 4.1. Then $T(x,y) = \frac{1}{5}\sum_{j=0}^4 \zeta^j E\left(x, \iota(\zeta)^j y\right)$ by Example 4.2.2. Putting this together, we obtain a hermitian form on the free $\mathcal{O}_K$-module Λ as follows:

$$\mathfrak{h} : \Lambda \times \Lambda \to \mathcal{O}_K, \quad \mathfrak{h}(x,y) = \eta T(x,y) = (\zeta - \zeta^{-1})^{-1} \sum_{j=0}^{4} \zeta^j E\left(x, \iota(\zeta)^j y\right) \tag{10}$$

Observe that the hermitian lattice $(\Lambda, \mathfrak{h})$ is unimodular because (Λ, E) is unimodular:

$$\{x \in \Lambda_{\mathbb{Q}} : \mathfrak{h}(x, \Lambda) \subset \mathcal{O}_K\} = \{x \in \Lambda_{\mathbb{Q}} : T(x, \Lambda) \subset \mathfrak{D}_K^{-1}\} = \{x \in \Lambda_{\mathbb{Q}} : E(x, \Lambda) \subset \mathbb{Z}\} = \Lambda.$$

For each embedding $\varphi : K \to \mathbb{C}$, the restriction of the hermitian form $\varphi(\eta) \cdot E_{\mathbb{C}}(x, \bar{y})$ on $\Lambda_{\mathbb{C}}$ to $(\Lambda_{\mathbb{C}})_\varphi \subset \Lambda_{\mathbb{C}}$ coincides with $\mathfrak{h}^\varphi$ by Lemma 4.3. Since $\Im(\tau_i(\zeta - \zeta^{-1})) < 0$ for $i = 1, 2$, the signature of $\mathfrak{h}^{\tau_i}$ is $(h^{-1,0}_{\tau_1}, h^{0,-1}_{\tau_1}) = (2,1)$ for $i = 1$ and $(h^{-1,0}_{\tau_2}, h^{0,-1}_{\tau_2}) = (3,0)$ for $i = 2$.

5.2 *Marked binary quintics*

Let $\pi : \mathscr{C} \to X_0$ be the universal family of cyclic covers $C \to \mathbb{P}^1$ ramified along a smooth binary quintic $\{F = 0\} \subset \mathbb{P}^1$. Let $\phi : J \to X_0$ be the relative Jacobian of π; it is an abelian scheme of relative dimension 6 over X_0 with $\mathcal{O}_K$-action of signature $\{(2,1),(3,0)\}$ with respect to $\Psi = \{\tau_1, \tau_2\}$. Let $\mathbb{V}$ be the local system of hermitian $\mathcal{O}_K$-modules $R^1\pi_*\mathbb{Z}$ underlying the abelian scheme J/X_0. It corresponds to a monodromy representation $\rho' : \pi_1(X_0(\mathbb{C}), F_0) \to \Gamma$, where $\Gamma = \mathrm{Aut}_{\mathcal{O}_K}(\Lambda, \mathfrak{h})$, whose with the quotient map $\Gamma \to P\Gamma = \Gamma/\mu_K$ defines a homomorphism

$$\rho : \pi_1(X_0(\mathbb{C}), F_0) \to P\Gamma. \tag{11}$$

We shall see that ρ is surjective, see Corollary 5.3 below.

Let $F \in X_0(\mathbb{C})$ and let $Z_F = \{F = 0\} \subset \mathbb{P}^1_{\mathbb{C}}$ be the associated hypersurface. A *marking* of F is a ring isomorphism $m : H^0(Z_F(\mathbb{C}), \mathbb{Z}) \xrightarrow{\sim} \mathbb{Z}^5$. Clearly, this is the same thing as a labelling of the points $p \in Z_F(\mathbb{C})$. Let $\mathcal{M}_0$ be the space of marked binary quintics. Let $\psi : \mathcal{Z} \to X_0(\mathbb{C})$ be the universal complex binary quintic, and consider the local system $H = \psi_*\mathbb{Z}$ of stalk $H_F = H^0(Z_F(\mathbb{C}), \mathbb{Z})$ for $F \in X_0(\mathbb{C})$. Then H corresponds to a monodromy representation

$$\tau : \pi_1(X_0(\mathbb{C}), F_0) \to \mathfrak{S}_5. \tag{12}$$

It can be shown that τ is surjective using the results in [Bea86]. This implies that $\mathcal{M}_0 \to X_0(\mathbb{C})$ is covering space, i.e. $\mathcal{M}_0$ is connected. If we choose a marking $m_0 : H^0(Z_{F_0}(\mathbb{C}), \mathbb{Z}) \cong \mathbb{Z}^5$ lying over our base point $F_0 \in X_0(\mathbb{C})$, we obtain an embedding $\pi_1(\mathcal{M}_0, m_0) \hookrightarrow \pi_1(X_0(\mathbb{C}), F_0)$, whose composition with ρ in (11) defines a homomorphism

$$\mu : \pi_1(\mathcal{M}_0, m_0) \to P\Gamma. \tag{13}$$

Define $\theta = \zeta - \zeta^{-1}$ and consider the 3-dimensional $\mathbb{F}_5$ vector space $\Lambda/\theta\Lambda$ and the quadratic space $W := (\Lambda/\theta\Lambda, q)$ where q is the quadratic form obtained by reducing $\mathfrak{h}$ modulo $\theta\Lambda$. Let Γ_θ be the kernel of $\Gamma \to \mathrm{Aut}(W)$; then $P\Gamma_\theta = \mathrm{Ker}\,(P\Gamma \to P\mathrm{Aut}(W)) \subset \mathrm{PU}(2,1)$. Remark that the composition $\mathcal{M}_0 \to X_0(\mathbb{C}) \to X_s(\mathbb{C})$ admits an essentially unique *completion* $\mathcal{M}_s \to X_s(\mathbb{C})$, see [Fox57]. Here $\mathcal{M}_s$ a manifold and $\mathcal{M}_s \to X_s(\mathbb{C})$ is a ramified covering space.

Proposition 5.2. *The image of μ in (13) is the group $P\Gamma_\theta$, and the induced homomorphism $\pi_1(X_0(\mathbb{C}), F_0)/\pi_1(\mathcal{M}_0, m_0) = \mathfrak{S}_5 \to P\Gamma/P\Gamma_\theta$ is an isomorphism. In other words, we obtain the following commutative diagram with exact rows:*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \pi_1(\mathcal{M}_0, m_0) & \longrightarrow & \pi_1(X_0(\mathbb{C}), F_0) & \xrightarrow{\tau} & \mathfrak{S}_5 & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \mu} & & \downarrow{\scriptstyle \rho} & & {\scriptstyle \gamma}\downarrow\wr & & \\ 0 & \longrightarrow & P\Gamma_\theta & \longrightarrow & P\Gamma & \longrightarrow & P\mathrm{Aut}(W) & \longrightarrow & 0. \end{array} \tag{14}$$

Proof. Consider the quotient $Q = G \setminus \mathcal{M}_0 = \mathrm{PGL}_2(\mathbb{C}) \setminus P_0$ and let $0 \in Q$ be the image of $m_0 \in \mathcal{M}_0$. In [DM86], Deligne and Mostow define a hermitian space bundle $B_Q \to Q$ over Q whose fiber over $0 \in Q$ is $\mathbb{C}H^2$. Consequently, writing $V_1 = \Lambda \otimes_{\mathcal{O}_K, \tau_1} \mathbb{C}$, this gives a monodromy representation $\pi_1(Q, 0) \to \mathrm{PU}(V_1, \mathfrak{h}^{\tau_1}) \cong \mathrm{PU}(2,1)$ whose image we denote by Γ_{DM}. Kondō has shown that in fact, $\Gamma_{\mathrm{DM}} = P\Gamma_\theta$ [Kon05, Theorem 7.1]. Since $\mathcal{M}_0 \to Q$ is a covering space (the action of G on $\mathcal{M}_0$ being free) we have an embedding $\pi_1(\mathcal{M}_0, m_0) \hookrightarrow \pi_1(Q, 0)$ whose composition with $\pi_1(Q, 0) \to \mathrm{PU}(2,1)$ is the map $\mu : \pi_1(\mathcal{M}_0, m_0) \to P\Gamma \subset \mathrm{PU}(2,1)$.

To prove that the image of μ is $P\Gamma_\theta$, it suffices to give a section of the map $\mathcal{M}_0 \to Q$. Indeed, this induces a retraction of $\pi_1(\mathcal{M}_0, m_0) \hookrightarrow \pi_1(Q, 0)$ so that the images of both groups in $\mathrm{PU}(2,1)$ are the same. To define such a section, observe that if $\Delta \subset \mathbb{P}^1(\mathbb{C})^5$ is the union of all hyperplanes $\{x_i = x_j\} \subset \mathbb{P}^1(\mathbb{C})^5$ for $i \neq j$, then

$$Q = \mathrm{PGL}_2(\mathbb{C}) \setminus P_0 = \mathrm{PGL}_2(\mathbb{C}) \setminus \left(\mathbb{P}^1(\mathbb{C})^5 - \Delta\right) \cong \{(x_4, x_5) \in \mathbb{A}^1(\mathbb{C})^2 : x_i \neq 0, 1 \text{ and } x_1 \neq x_2\}.$$

The section $Q \to \mathcal{M}_0$ may then be defined by sending (x_4, x_5) to the binary quintic $F(X,Y) = X(X-Y)Y(X-x_4Y)(X-x_5Y) \in X_0(\mathbb{C})$, marked by the labelling of its roots $\{0, 1, \infty, x_4, x_5\}$. It remains to prove that the homomorphism $\gamma : \mathfrak{S}_5 \to P\Gamma/P\Gamma_\theta$ appearing on the right in (14) is an isomorphism. We use Theorem 8.1, proven by Shimura in [Shi64], which says that $(\Lambda, \mathfrak{h}) \cong (\mathcal{O}_K^3, \mathrm{diag}(1, 1, \frac{1-\sqrt{5}}{2}))$. It follows that $P\Gamma/P\Gamma_\theta = P\mathrm{Aut}(W) \cong \mathrm{PO}_3(\mathbb{F}_5) \cong \mathfrak{S}_5$. Next, consider the manifold $\mathcal{M}_s$. Remark that $\mathfrak{S}_5$ embeds into $\mathrm{Aut}(G \setminus \mathcal{M}_s)$, and that the natural map $\mathcal{P} : G \setminus \mathcal{M}_s \to P\Gamma_\theta \setminus \mathbb{C}H^2$ (see (15)) is an isomorphism, see [DM86], [Kon05]. The composition $\mathfrak{S}_5 \subset \mathrm{Aut}(G \setminus \mathcal{M}_s) \cong \mathrm{Aut}(P\Gamma_\theta \setminus \mathbb{C}H^2)$ coincides with the composition $\mathfrak{S}_5 \to P\Gamma/P\Gamma_\theta \subset \mathrm{Aut}(P\Gamma_\theta \setminus \mathbb{C}H^2)$ by equivariance of $\mathcal{P}$ with respect to γ, hence γ is injective. □

Corollary 5.3. *The monodromy representation ρ in (11) is surjective.* □

5.3 Framed binary quintics

By a *framing* of a point $F \in X_0(\mathbb{C})$ we mean a projective equivalence class $[f]$, where $f : \mathbb{V}_F = H^1(C_F(\mathbb{C}), \mathbb{Z}) \to \Lambda$ is an $\mathcal{O}_K$-linear isometry: two such isometries are in the same class if and only if they differ by an element in μ_K. Let $\mathcal{F}_0$ be the collection of all framings of all points $x \in X_0(\mathbb{C})$. The set $\mathcal{F}_0$ is naturally a complex manifold. Note that Corollary 5.3 implies that $\mathcal{F}_0$ is connected, hence $\mathcal{F}_0 \to X_0(\mathbb{C})$ is a covering space with Galois group $P\Gamma$.

Lemma 5.4. *The spaces $P\Gamma_\theta \setminus \mathcal{F}_0$ and $\mathcal{M}_0$ are isomorphic as covering spaces of $X_0(\mathbb{C})$. In particular, there is a covering map $\mathcal{F}_0 \to \mathcal{M}_0$ with Galois group $P\Gamma_\theta$.*

Proof. The quotients $P\Gamma \to P\Gamma/P\Gamma_\theta$ and $P\Gamma \to \mathfrak{S}_5$ are isomorphic by Proposition 5.2. □

Lemma 5.5. *The variety $\Delta := X_s - X_0$ is an irreducible normal crossings divisor in X_s.*

Proof. The proof is similar to the proof of Proposition 6.7 in [Bea09]. □

Lemma 5.6. *The local monodromy transformations of $\mathcal{F}_0 \to X_0(\mathbb{C})$ around every $x \in \Delta(\mathbb{C})$ are of finite order. More precisely, if $x \in \Delta(\mathbb{C})$ lies on the intersection of k local components of $\Delta(\mathbb{C})$, then the local monodromy group around x is isomorphic to $(\mathbb{Z}/10)^k$.*

Proof. See [DM86, Proposition 9.2] or [CT99, Proposition 6.1] for the generic case, i.e. when a quintic $Z = \{F = 0\} \subset \mathbb{P}^1_{\mathbb{C}}$ aquires only one node. In this case, the local equation of the singularity is $x^2 = 0$, hence the curve C_F acquires a singularity of the form $y^5 + x^2 = 0$. If the quintic acquires two nodes, then C_F acquires two such singularities; the vanishing cohomology splits as an orthogonal direct sum hence the local monodromy transformations commute. □

Corollary 5.7. *There is a manifold $\mathcal{F}_s$, unique up to isomorphism, with a branched cover $\pi : \mathcal{F}_s \to X_s(\mathbb{C})$ extending $\mathcal{F}_0 \to X_0(\mathbb{C})$ such that for any $x \in \Delta(\mathbb{C})$ and any open neighborhood $x \in U \subset X_s(\mathbb{C})$ with $U \cong D^6$ and $U \cap X_0(\mathbb{C}) \cong (D^*)^k \times D^{6-k}$, any component of $\pi^{-1}(U) \subset \mathcal{F}_s$ is isomorphic to $D^k \times D^{6-k}$, mapping to U by $(z_1, \ldots, z_6) \mapsto (z_1^{r_1}, \ldots, z_k^{r_k}; z_{k+1}, \ldots, z_6)$.*

Proof. See [Bea09, Lemma 7.2]. See also [Fox57] and [DM86, Section 8]. □

The group $G = \mathrm{GL}_2(\mathbb{C})/\mu_K$ acts on $\mathcal{F}_0$ over its action on X_0. Explicitly, $g \in G$ and if $[\phi], \phi : \mathbb{V}_t \to \Lambda$ is a framing of $t \in X_0(\mathbb{C})$, then $[\phi \circ g^*], \phi \circ g^* : \mathbb{V}_{gt} \to \Lambda$ is a framing of $gt \in X_0(\mathbb{C})$. This is a left action. Note that $P\Gamma$ also acts on $\mathcal{F}_0$ from the left, and that the actions of $P\Gamma$ and G on $\mathcal{F}_0$ commute. By the naturality of the Fox completion, the action of G on $\mathcal{F}_0$ extends to an action of G on $\mathcal{F}_s$.

Lemma 5.8. *The group $G = \mathrm{GL}_2(\mathbb{C})/\mu_K$ acts freely on $\mathcal{F}_s$.*

Proof. The universality of the Fox completion gives an action of G on $\mathcal{M}_s$ such that, by Lemma 5.4, $P\Gamma_\theta \setminus \mathcal{F}_s$ and $\mathcal{M}_s$ are G-equivariantly isomorphic as branched covering spaces of $X_s(\mathbb{C})$. In particular, it suffices to show that G acts freely on $\mathcal{M}_s$. Note that $\mathcal{M}_0$ admits a natural $\mathbb{G}_m$-covering map $\mathcal{M}_0 \to P_0$ where $P_0 \subset \mathbb{P}^1(\mathbb{C})^5$ is the space of distinct ordered 5-tuples in $\mathbb{P}^1(\mathbb{C})$. Consequently, there is a $\mathbb{G}_m$-quotient map $\mathcal{M}_s \to P_s$, where P_s is the space of stable ordered 5-tuples, and this map is equivariant for the homomorphism $\mathrm{GL}_2(\mathbb{C}) \to \mathrm{PGL}_2(\mathbb{C})$. But it is clear that $\mathrm{PGL}_2(\mathbb{C})$ acts freely on P_s. Therefore, if $g \in \mathrm{GL}_2(\mathbb{C})$ and $x \in \mathcal{M}_s$ are such that $gx = x$, then $g = \lambda \in \mathbb{C}^*$. Let $F \in X_s(\mathbb{C})$ be the image of $x \in \mathcal{M}_s$; then $gF(x,y) = F(g^{-1}(x,y)) = F(\lambda^{-1}x, \lambda^{-1}y) = \lambda^{-5}F(x,y)$. The equality $gF = F$ implies that $\lambda^5 = 1 \in \mathbb{C}$; consequently, $\lambda \in \mu_K$. We conclude that $[g] = [\mathrm{id}] \in G$ and the lemma follows. □

5.4 *Complex uniformization*

Consider the hermitian space $V_1 = \Lambda \otimes_{\mathcal{O}_K,\tau_1} \mathbb{C}$; define $\mathbb{C}H^2$ to be the space of negative lines in V_1. Using Proposition 4.5 we see that the abelian scheme $J \to X_0$ induces a G-equivariant morphism $\mathcal{P} : \mathcal{F}_0 \to \mathbb{C}H^2$. Explicitly, if $(F, [f]) \in \mathcal{F}_0$ is the framing $[f : H^1(C_F(\mathbb{C}), \mathbb{Z}) \xrightarrow{\sim} \Lambda]$ of the binary quintic $F \in X_0(\mathbb{C})$, and A_F is the Jacobian of the curve C_F, then

$$\mathcal{P} : \mathcal{F}_0 \to \mathbb{C}H^2, \qquad \mathcal{P}(F, [f]) = f\left(H^{0,-1}(A_F)_{\tau_1}\right) = f\left(H^{1,0}(C_f)_{\zeta^3}\right) \in \mathbb{C}H^2. \tag{15}$$

By general considerations, $\mathcal{P}$ is holomorphic. It descends to a morphism of analytic spaces

$$\mathscr{M}_{\mathbb{C}} = G \setminus X_0(\mathbb{C}) \to P\Gamma \setminus \mathbb{C}H^2. \tag{16}$$

By Riemann extension, (15) extends to a G-equivariant holomorphic map $\overline{\mathcal{P}} : \mathcal{F}_s \to \mathbb{C}H^2$.

Theorem 5.9 (Deligne-Mostow)**.** *The map $\overline{\mathcal{P}}$ defines an isomorphism of complex manifolds $\overline{\mathscr{M}}^f_{\mathbb{C}} := G \setminus \mathcal{F}_s \cong \mathbb{C}H^2$. Taking $P\Gamma$-quotients gives an isomorphism of complex analytic spaces*

$$\overline{\mathscr{M}}_{\mathbb{C}} = G \setminus X_s(\mathbb{C}) \cong P\Gamma \setminus \mathbb{C}H^2. \tag{17}$$

Proof. In [DM86], Deligne and Mostow define $\widetilde{Q} \to Q$ to be the covering space corresponding to the monodromy representation $\pi_1(Q, 0) \to \mathrm{PU}(2,1)$; since the image of this homomomorphism is $P\Gamma_\theta$ (see the proof of Proposition 5.2), it follows that $G \setminus \mathcal{F}_0 \cong \widetilde{Q}$ as covering spaces of Q. Consequently, if $\widetilde{Q}_{\mathrm{st}}$ is the Fox completion of the spread $\widetilde{Q} \to Q \to Q_{\mathrm{st}} := G \setminus \mathcal{M}_s = \mathrm{PGL}_2(\mathbb{C}) \setminus P_s$, then there is an isomorphism $G \setminus \mathcal{F}_s \cong \widetilde{Q}_{\mathrm{st}}$ of branched covering spaces of Q_{st}. Consequently, we obtain commutative diagrams, where the lower right morphism uses (14):

$$\begin{array}{ccccc}
G \setminus \mathcal{F}_s & \xrightarrow{\sim} & \widetilde{Q}_{\mathrm{st}} & \longrightarrow & \mathbb{C}H^2 \\
\downarrow & & \downarrow & & \downarrow \\
G \setminus \mathcal{M}_s & \xrightarrow{\sim} & Q_{\mathrm{st}} & \longrightarrow & P\Gamma_\theta \setminus \mathbb{C}H^2 \\
\downarrow & & \downarrow & & \downarrow \\
G \setminus X_s(\mathbb{C}) & \xrightarrow{\sim} & Q_{\mathrm{st}}/\mathfrak{S}_5 & \longrightarrow & P\Gamma \setminus \mathbb{C}H^2.
\end{array}$$

Now the map $\widetilde{Q}_{\rm st} \to \mathbb{C}H^2$ is an isomorphism by [DM86, (3.11)]. Therefore, we are done if the composition $G \setminus \mathcal{F}_0 \to \widetilde{Q} \to \mathbb{C}H^2$ is nothing but the period map $\mathcal{P} : G \setminus \mathcal{F}_0 \to \mathbb{C}H^2$ that we defined in Equation (15). This follows from [DM86, (2.23) and (12.9)]. □

Proposition 5.10. *The isomorphism (17) induces an isomorphism of complex analytic spaces*

$$\mathscr{M}_{\mathbb{C}} = G \setminus X_0 \cong P\Gamma \setminus \left(\mathbb{C}H^2 - \mathscr{H}\right). \tag{18}$$

Proof. We have $\overline{\mathcal{P}}(\mathcal{F}_0) \subset \mathbb{C}H^2 - \mathscr{H}$ by Proposition 4.7 because the Jacobian of a smooth curve cannot contain an abelian subvariety whose induced polarization is principal. Therefore $\overline{\mathcal{P}}^{-1}(\mathscr{H}) \subset \mathcal{F}_s - \mathcal{F}_0$. Since $\mathcal{F}_s$ is irreducible (it is smooth (Corollary 5.7) and connected (Corollary 5.3)), $\overline{\mathcal{P}}^{-1}(\mathscr{H})$ is a divisor; but $\mathcal{F}_s - \mathcal{F}_0$ is also a divisor (Corollary 5.7) hence $\overline{\mathcal{P}}^{-1}(\mathscr{H}) = \mathcal{F}_s - \mathcal{F}_0$ and we are done. (Alternatively, let $H_{0,5}$ be the moduli space of degree 5 covers of $\mathbb{P}^1$ ramified along five distinct marked points [HM98, §2.G]. The period map $H_{0,5}(\mathbb{C}) \to P\Gamma \setminus \mathbb{C}H^2$, mapping a curve $C \to \mathbb{P}^1$ to its Jacobian $J(C)$ with $\mathbb{Z}[\zeta_5]$-action, extends to the stable compactification $\overline{H}_{0,5}(\mathbb{C}) \supset H_{0,5}(\mathbb{C})$ because the curves in the limit are of compact type. Since the divisor $\mathscr{H} \subset \mathbb{C}H^2$ parametrizes abelian varieties that are products of lower dimensional ones (Proposition 4.7), the image of the boundary is exactly $P\Gamma \setminus \mathscr{H}$.) □

6 The Period Map for Real Binary Quintics

Write $\mathcal{F}_0(\mathbb{R}) = \mathcal{F}_0(\mathbb{R})$ for the preimage of $X_0(\mathbb{R})$ in the space $\mathcal{F}_0$. Then $\mathcal{F}_0 = \cup_{\alpha \in P\mathscr{A}} \mathcal{F}_0^{\alpha}$, where $\mathscr{A}$ is the set of anti-unitary involutions $\alpha : \Lambda \to \Lambda$ and $P\mathscr{A} = \mu_K \setminus \mathscr{A}$ as in Section 3, $\kappa : X_0(\mathbb{C}) \to X_0(\mathbb{C})$ the anti-holomorphic involution that sends a binary quintic $F(x,y) = \sum_{i+j=5} a_{ij}x^iy^j \in \mathbb{C}[x,y]$ to the binary quintic $\overline{F(x,y)} = \sum_{i+j=5} \overline{a_{ij}}x^iy^j \in \mathbb{C}[x,y]$, and $\mathcal{F}_0^{\alpha}$ the fixed point set of the natural anti-holomorphic involution $\alpha : \mathcal{F}_0 \to \mathcal{F}_0$ lying over κ which is induced by α as follows. Consider a framing $(F,[f]) \in \mathcal{F}_0$, $f : \mathbb{V}_F \to \Lambda$, and let $[\alpha] \in P\mathscr{A}$. Let $C_F \to \mathbb{P}^1_{\mathbb{C}}$ be the quintic cover defined by a smooth binary quintic $F \in X_0(\mathbb{C})$. Complex conjugation $\sigma : \{F = 0\} = Z \to \kappa Z = \{\kappa \cdot F = 0\}$ extends to an anti-holomorphic map $\sigma : C_F \to C_{\kappa \cdot F}$ which in turn extends to an anti-holomorphic map $\sigma : J(C_F) \to J(C_{\kappa\cdot F})$. Define $F_{\infty,F} = \sigma^* : \mathbb{V}_{\kappa \cdot F} \to \mathbb{V}_F$; then the composite $\mathbb{V}_{\kappa\cdot F} \xrightarrow{F_{\infty,F}} \mathbb{V}_F \xrightarrow{f} \Lambda \xrightarrow{\alpha} \Lambda$ defines a framing of $\kappa \cdot F \in X_0(\mathbb{C})$. For the set of fixed points $\mathcal{F}_0^{\alpha}$, we have $\mathcal{F}_0^{\alpha} = \{(F,[f]) \in \mathcal{F}_0 : \kappa \cdot F = F \text{ and } [f \circ F_{\infty,\kappa\cdot F} \circ f^{-1}] = [\alpha].\}$. If $[\alpha] \neq [\beta] \in P\mathscr{A}$, then $\mathcal{F}_0^{\alpha} \cap \mathcal{F}_0^{\beta} = \emptyset$, for if $(F,[f])$ is in the intersection, then $[\alpha] = [f \circ F_{\infty,\kappa\cdot F} \circ f^{-1}] = [\beta]$.

Lemma 6.1. *The anti-holomorphic involution $[\alpha] : \mathcal{F}_0 \to \mathcal{F}_0$ defined by $[\alpha] \in P\mathscr{A}$ makes the period map $\mathcal{P} : \mathcal{F}_0 \to \mathbb{C}H^2$ Galois-equivariant.*

Proof. Indeed, if $\sigma : \mathbb{C} \to \mathbb{C}$ is complex conjugation, then the induced map $F_{\infty,t} \otimes \sigma : \mathbb{V}_{\kappa t} \otimes_{\mathbb{Z}} \mathbb{C} \to \mathbb{V}_t \otimes_{\mathbb{Z}} \mathbb{C}$ is anti-linear and preserves the Hodge decomposition as well as the eigenspaces. □

Consequently, we obtain a *real period map* $\mathcal{P}_{\mathbb{R}} : \coprod_{\alpha \in P\mathscr{A}} \mathcal{F}_0^{\alpha} = \mathcal{F}_0(\mathbb{R}) \to \widetilde{Y} = \coprod_{\alpha \in P\mathscr{A}} \mathbb{R}H^2_{\alpha}$. It is constant on $G(\mathbb{R})$-orbits since the same is true for $\mathcal{P} : \mathcal{F}_0 \to \mathbb{C}H^2$.

Proposition 6.2. *The real period map $\mathcal{P}_{\mathbb{R}}$ descends to a $P\Gamma$-equivariant diffeomorphism $G(\mathbb{R}) \setminus \mathcal{F}_0(\mathbb{R}) \cong \coprod_{\alpha \in P\mathscr{A}} \mathbb{R}H^2_{\alpha} - \mathscr{H}$. Thus there is an isomorphism of real-analytic orbifolds*

$$\mathcal{P}_{\mathbb{R}} : \mathscr{M}_{\mathbb{R}} = G(\mathbb{R}) \setminus X_0(\mathbb{R}) \cong \coprod_{\alpha \in C\mathscr{A}} \left[P\Gamma_{\alpha} \setminus \left(\mathbb{R}H^2_{\alpha} - \mathscr{H}\right)\right]. \tag{19}$$

Proof. This follows from [ACT10, *proof of Theorem 3.3*]. It is crucial that the actions of G and $P\Gamma$ on $\mathcal{F}_0$ commute and are free, which is the case, see Corollary 5.8. $\square$

The goal is to prove the real analogue of the isomorphism $\overline{\mathscr{M}}_{\mathbb{C}} = G(\mathbb{C}) \setminus X_s(\mathbb{C}) \cong P\Gamma \setminus \mathbb{C}H^2$ in Theorem 5.9. We need two more lemma's.

Lemma 6.3. *The period map $\overline{\mathcal{P}} : \mathcal{F}_s \to \mathbb{C}H^2$ sends the points $f \in \widetilde{\Delta} := \mathcal{F}_s - \mathcal{F}_0$ lying above binary quintics with k nodes to the locus of $\mathbb{C}H^2$ where exactly k of the hyperplanes of $\mathscr{H}$ meet. If $f \in \widetilde{\Delta}$ is such a point, and $\boldsymbol{r} = (r_1, \dots, r_k)$ a vector of short roots such that $\overline{\mathcal{P}}(f) = x \in \cap_i H_{r_i}$, then $\overline{\mathcal{P}} : \mathcal{F}_s \to \mathbb{C}H^2$ induces a group isomorphism $P\Gamma_f \cong G(\boldsymbol{r})$.* $\square$

The naturality of the Fox completion implies that each anti-unitary involution $\alpha : \mathcal{F}_0 \to \mathcal{F}_0$ extends to an anti-unitary involution $\alpha : \mathcal{F}_s \to \mathcal{F}_s$.

Lemma 6.4. *For every $\alpha \in P\mathscr{A}$, the restriction of $\overline{\mathcal{P}} : \mathcal{F}_s \to \mathbb{C}H^2$ to $\mathcal{F}_s^\alpha$ defines a diffeomorphism $G(\mathbb{R}) \setminus \mathcal{F}_s^\alpha \cong \mathbb{R}H^2_\alpha$.*

Proof. See [ACT10, Lemma 11.3]. It is crucial that G acts freely on $\mathcal{F}_s$ (Corollary 5.8). $\square$

We arrive at the main theorem of this section. Define $\mathcal{F}_s(\mathbb{R}) = \cup_{\alpha \in P\mathscr{A}} \mathcal{F}_s^\alpha = \pi^{-1}(X_s(\mathbb{R}))$. This is not a manifold because of the ramification of $\pi : \mathcal{F}_s \to X_s(\mathbb{C})$, but a union of embedded submanifolds.

Theorem 6.5. *The real stable period map $\overline{\mathcal{P}}_{\mathbb{R}} : \coprod_{\alpha \in P\mathscr{A}} \mathcal{F}_s^\alpha \to \widetilde{Y} = \coprod_{\alpha \in P\mathscr{A}} \mathbb{R}H^2_\alpha$ extends $\mathcal{P}_{\mathbb{R}}$ and induces the following commutative diagram:*

$$\begin{array}{ccc}
\coprod_{\alpha \in P\mathscr{A}} \mathcal{F}_s^\alpha & \xrightarrow{\overline{\mathcal{P}}_{\mathbb{R}}:} & \widetilde{Y} = \coprod_{\alpha \in P\mathscr{A}} \mathbb{R}H^2_\alpha \\
\downarrow & & \downarrow \\
\mathcal{F}_s(\mathbb{R}) & \xrightarrow{\overline{\mathcal{P}}_{\mathbb{R}}} & Y \\
\downarrow & & \| \\
G(\mathbb{R}) \setminus \mathcal{F}_s(\mathbb{R}) & \xrightarrow{\mathscr{P}_{\mathbb{R}}} & Y \\
\downarrow & & \downarrow \\
G(\mathbb{R}) \setminus X_s(\mathbb{R}) & \xrightarrow{\mathscr{P}_{\mathbb{R}}} & P\Gamma \setminus Y,
\end{array}$$

and $\mathscr{P}_{\mathbb{R}} : G(\mathbb{R}) \setminus \mathcal{F}_s(\mathbb{R}) \to Y$ and $\mathscr{P}_{\mathbb{R}} : G(\mathbb{R}) \setminus \mathcal{F}_s(\mathbb{R})/P\Gamma \to P\Gamma \setminus Y$ are homeomorphisms.

Proof. We first show that $\overline{\mathcal{P}}_{\mathbb{R}} : \coprod_{\alpha \in P\mathscr{A}} \mathcal{F}_s^\alpha \to \widetilde{Y} \to Y$ factors through $\mathcal{F}_s(\mathbb{R})$. Now (f, α) and $(g, \beta) \in \coprod_{\alpha \in P\mathscr{A}} \mathcal{F}_s^\alpha$ have the same image in $\mathcal{F}_s(\mathbb{R})$ if and only if $f = g \in \mathcal{F}_s^\alpha \cap \mathcal{F}_s^\beta$, in which case $x := \mathcal{P}_s(f) = \mathcal{P}_s(g) \in \mathbb{R}H^2_\alpha \cap H^2_\beta$, so all we need to show is that $(x, \alpha) \sim (x, \beta) \in \widetilde{Y}$. But note that $\alpha\beta \in P\Gamma_f \cong (\mathbb{Z}/10)^k$, and $\overline{\mathcal{P}}$ induces an isomorphism $P\Gamma_f \cong G(\mathbf{r})$ by Lemma 6.3. Hence $\alpha\beta \in G(\mathbf{r})$ so that indeed, $(x, \alpha) \sim (x, \beta)$. Let us prove the $G(\mathbb{R})$-equivariance of $\overline{\mathcal{P}}_{\mathbb{R}}$: if $f \in \mathcal{F}_s^\alpha, g \in \mathcal{F}_s^\beta$ with $a \cdot f = g \in \mathcal{F}_s(\mathbb{R})$ for some $a \in G(\mathbb{R})$, then $x := \overline{\mathcal{P}}(f) = \overline{\mathcal{P}}(g) \in \mathbb{C}H^2$, so we need to show that $\alpha\beta \in G(\mathbf{r})$. The actions of $G(\mathbb{C})$ and $P\Gamma$ on $\mathbb{C}H^2$ commute, and the same holds for the actions of $G(\mathbb{R})$ and $P\Gamma'$ on $\mathcal{F}_s^{\mathbb{R}}$, where $P\Gamma' = \Gamma'/\mu_K$, Γ' being the group of all unitary and anti-unitary automorphisms of Λ. It follows that $\alpha(g) = \alpha(a \cdot f) = a \cdot \alpha(f) = a \cdot f = g$, hence $g \in \mathcal{F}_s^\alpha \cap \mathcal{F}_s^\beta$. This implies in turn that $\alpha\beta(g) = g$, hence $\alpha\beta \in P\Gamma_g \cong G(\mathbf{r})$, so that indeed, $(x, \alpha) \sim (x, \beta)$. To prove that $\mathscr{P}_{\mathbb{R}}$ is injective, let again $(f, \alpha), (g, \beta) \in \coprod_{\alpha \in P\mathscr{A}} \mathcal{F}_s^\alpha$ and suppose that they have the same image in Y. This implies that $x := \overline{\mathcal{P}}(f) = \overline{\mathcal{P}}(g) \in \mathbb{R}H^2_\alpha \cap H^2_\beta$ and $\beta = \phi \circ \alpha$ for some $\phi \in G(\mathbf{r})$. Now $\phi \in G(\mathbf{r}) \cong P\Gamma_f$ (by Lemma 6.3) hence

$$\beta(f) = \phi(\alpha(f)) = \phi(f) = f. \tag{20}$$

Therefore $f, g \in \mathcal{F}_s^\beta$; since $\overline{\mathcal{P}}(f) = \overline{\mathcal{P}}(g)$, it follows from Lemma 6.4 that there exists $a \in G(\mathbb{R})$ such that $a \cdot f = g$, which proves injectivity of $\mathscr{P}_\mathbb{R}$. Surjectivity of $\mathscr{P}_\mathbb{R} : G(\mathbb{R}) \setminus \mathcal{F}_s(\mathbb{R}) \to Y$ is straightforward, using surjectivity of $\overline{\mathcal{P}}_\mathbb{R}$ (Lemma 6.4). Finally, we claim that $\mathscr{P}_\mathbb{R}$ is open. Let $U \subset G(\mathbb{R}) \setminus \mathcal{F}_s^\mathbb{R}$; we can write $U = \mathscr{P}_\mathbb{R}^{-1} \mathscr{P}_\mathbb{R}(U)$. Let V be the preimage of U in $\coprod_{\alpha \in P\mathscr{A}} \mathcal{F}_s^\alpha$. Then $V = \overline{\mathcal{P}}_\mathbb{R}^{-1}\left(p^{-1}\left(\mathscr{P}_\mathbb{R}(U)\right)\right)$ and hence $\overline{\mathcal{P}}_\mathbb{R}(V) = p^{-1}\left(\mathscr{P}_\mathbb{R}(U)\right)$. But the map $\overline{\mathcal{P}}_\mathbb{R}$ is open, being the coproduct of the maps $\mathcal{F}_s^\alpha \to \mathbb{R}H^2_\alpha$, each of which is open since its differential is surjective at each point. Thus $\mathscr{P}_\mathbb{R}(U)$ is open in Y. □

Corollary 6.6. *There is a lattice $P\Gamma_\mathbb{R} \subset \mathrm{PO}(2,1)$ and a homeomorphism*

$$\mathscr{M}_\mathbb{R} = G(\mathbb{R}) \setminus X_s(\mathbb{R}) \cong P\Gamma_\mathbb{R} \setminus \mathbb{R}H^2. \tag{21}$$

Proof. $G(\mathbb{R}) \setminus X_s(\mathbb{R}) \cong P\Gamma \setminus Y$ (Theorem 6.5) and $\Gamma \setminus Y \cong P\Gamma_\mathbb{R} \setminus \mathbb{R}H^2$ (Theorem 3.10). □

Remark 6.7. The proof of Theorem 6.5 also shows that $\mathcal{M}_s(\mathbb{R}) \cong P\Gamma \setminus Y$ if $\mathcal{M}_s$ is the moduli stack of either cubic surfaces or binary sextics. Notice that this strategy to uniformize the real moduli space is somewhat different than the one in [ACT10], [ACT06], [ACT07], since we first glue together the different components of the real locus of the Shimura variety and only afterwards prove that our moduli space is homeomorphic to the resulting hyperbolic gluing.

7 The Moduli Space of Real Binary Quintics as Hyperbolic Triangle

The goal of this section is to prove Theorem 7.9, which states that the moduli space $\overline{\mathscr{M}}_\mathbb{R} = G(\mathbb{R}) \setminus X_s(\mathbb{R})$ of stable binary quintics equipped with the metric of Corollary 6.6 is isometric to the triangle $\Delta_{3,5,10} \subset H^2$ of angles $\pi/3, \pi/5$ and $\pi/10$ in the real hyperbolic plane $\mathbb{R}H^2$.

7.1 Classification of the stabilizers groups of $\overline{\mathscr{M}}_\mathbb{R}$

The aim of this section is then to describe all possible elements $x = [\alpha_1, \dots, \alpha_5] \in (P_s/\mathfrak{S}_5)(\mathbb{R})$ whose stabilizer $H_x := \mathrm{PGL}_2(\mathbb{R})_x$ is non-trivial, and also to calculate H_x in these cases:

Proposition 7.1. *All stabilizer groups $H_x \subset \mathrm{PGL}_2(\mathbb{R})$ for points $x \in (P_s/\mathfrak{S}_5)(\mathbb{R})$ are among $\mathbb{Z}/2, D_3, D_5$. For $n \in \{3, 5\}$, there is a unique $\mathrm{PGL}_2(\mathbb{R})$-orbit in $(P_s/\mathfrak{S}_5)(\mathbb{R})$ of points x with stabilizer D_n.*

Proof. We have an injection $(P_s/\mathfrak{S}_5)(\mathbb{R}) \hookrightarrow P_s/\mathfrak{S}_5$ which is equivariant for the embedding $\mathrm{PGL}_2(\mathbb{R}) \hookrightarrow \mathrm{PGL}_2(\mathbb{C})$. In particular, $H_x \subset \mathrm{PGL}_2(\mathbb{C})_x$ for every $x \in (P_s/\mathfrak{S}_5)(\mathbb{R})$. The groups $\mathrm{PGL}_2(\mathbb{C})_x$ for equivalence classes of distinct points $x \in P_0/\mathfrak{S}_5$ are calculated in [WX19, Theorem 22], and such a group is isomorphic to $\mathbb{Z}/2, D_3, \mathbb{Z}/4$ or D_5. None of these have subgroups isomorphic to $D_2 = \mathbb{Z}/2 \rtimes \mathbb{Z}/2$ or $D_4 = \mathbb{Z}/2 \rtimes \mathbb{Z}/4$. Define an involution

$$\nu := (z \mapsto 1/z) \in \mathrm{PGL}_2(\mathbb{R}). \tag{22}$$

Lemma 7.2. *Consider an element $\tau \in \mathrm{PGL}_2(\mathbb{R})$ and three distinct elements $x, y, z \in \mathbb{P}^1(\mathbb{C})$ such that $S = \{x, y, z\}$ is stabilized by complex conjugation, and such that $\tau(x) = x$, $\tau(y) = z$ and $\tau(z) = y$. Then there is a transformation $g \in \mathrm{PGL}_2(\mathbb{R})$ that maps S to either $\{-1, 0, \infty\}$ or $\{-1, i, -i\}$, and that satisfies $g\tau g^{-1} = \nu = (z \mapsto 1/z) \in \mathrm{PGL}_2(\mathbb{R})$. In particular, $\tau^2 = \mathrm{id}$.*

Proof. This follows readily from the fact that two transformations $g, h \in \mathrm{PGL}_2(\mathbb{C})$ that satisfy $g(x_i) = h(x_i)$ for three different points $x_1, x_2, x_3 \in \mathbb{P}^1(\mathbb{C})$ are necessarily equal. □

Lemma 7.3. *There is no $x \in (P_s/\mathfrak{S}_5)(\mathbb{R})$ stabilized by an element $\phi \in \mathrm{PGL}_2(\mathbb{R})$ of order 4.*

Proof. By [Bea10, Theorem 4.2], all subgroups $G \subset \mathrm{PGL}_2(\mathbb{R})$ that are isomorphic to $\mathbb{Z}/4$ are conjugate to each other. Since the transformation $I : z \mapsto (z-1)/(z+1)$ is of order 4, it gives a representative $G_I = \langle I \rangle$ of this conjugacy class. Hence, assuming there exists x and ϕ as in the lemma, possibly after replacing x by gx for some $g \in \mathrm{PGL}_2(\mathbb{R})$, we may and do assume that $\phi = I$. On the other hand, it is easily shown that I cannot fix any $x \in (P_s/\mathfrak{S}_5)(\mathbb{R})$. □

Define

$$\rho \in \mathrm{PGL}_2(\mathbb{R}), \quad \rho(z) = \frac{-1}{z+1}.$$

Lemma 7.4. *Let $x = (x_1, \dots, x_5) \in (P_s/\mathfrak{S}_5)(\mathbb{R})$. Suppose $\phi(x) = x$ for an element $\phi \in \mathrm{PGL}_2(\mathbb{R})$ of order 3. Then there is a transformation $g \in \mathrm{PGL}_2(\mathbb{R})$ mapping x to $z = (-1, \infty, 0, \omega, \omega^2)$ with ω a primitive third root of unity, and the stabilizer of x to the subgroup of $\mathrm{PGL}_2(\mathbb{R})$ generated by ρ and ν. In particular, the stabilizer H_x is isomorphic to D_3.*

Proof. It follows from Lemma 7.2 that there must be three elements x_1, x_2, x_3 which form an orbit under ϕ. Since complex conjugation preserves this orbit, one element in it is real; since g is defined over $\mathbb{R}$, they are all real. Let $g \in \mathrm{PGL}_2(\mathbb{R})$ such that $g(x_1) = -1$, $g(x_2) = \infty$ and $g(x_3) = 0$. Define $\kappa = g\phi g^{-1}$. Then $\kappa^3 = \mathrm{id}$, and κ preserves $\{-1, \infty, 0\}$ and sends -1 to ∞ and ∞ to 0. Consequently, $\kappa(0) = -1$, and it follows that $\kappa = \rho$. Hence x is equivalent to an element of the form $z = (-1, \infty, 0, \alpha, \beta)$. Moreover, $\beta = \bar{\alpha}$ and $\alpha^2 + \alpha + 1 = 0$. □

Recall that $\zeta_5 = e^{2i\pi/5} \in \mathbb{P}^1(\mathbb{C})$ and define

$$\lambda = \zeta_5 + \zeta_5^{-1} \in \mathbb{R}, \qquad \gamma(z) = \frac{(\lambda+1)z - 1}{z+1} \in \mathrm{PGL}_2(\mathbb{R})$$

Lemma 7.5. *Let $x = (x_1, \dots, x_5) \in (P_s/\mathfrak{S}_5)(\mathbb{R})$. Suppose x is stabilized by a subgroup of $\mathrm{PGL}_2(\mathbb{R})$ of order 5. Then there is a transformation $g \in \mathrm{PGL}_2(\mathbb{R})$ mapping x to $z = (0, -1, \infty, \lambda+1, \lambda)$ and identifying the stabilizer of x with the subgroup of $\mathrm{PGL}_2(\mathbb{R})$ generated by γ and ν. In particular, the stabilizer H_x of x is isomorphic to D_5.*

Proof. Let $\phi \in H_x$ be an element of order 5. Using Lemma 7.2 one shows that x must be smooth, i.e. all x_i are distinct, and $x_i = \phi^{i-1}(x_1)$. Since there is one real x_i and ϕ is defined over $\mathbb{R}$, all x_i are real. Now note that $z = (0, -1, \infty, \lambda+1, \lambda)$ is the orbit of 0 under $\gamma : z \mapsto ((\lambda+1)z - 1)/(z+1)$. The reflection $\nu : z \mapsto 1/z$ preserves z as well: if $\zeta = \zeta_5$ then $\lambda = \zeta + \zeta^{-1}$ hence $\lambda + 1 = -(\zeta^2 + \zeta^{-2}) = -\lambda^2 + 2$, so that $\lambda(\lambda+1) = 1$. So we have $H_z \cong D_5$. But then, by [WX19, Theorem 22], the point z with its stabilizer H_z must be equivalent under $\mathrm{PGL}_2(\mathbb{C})$ to the point $(1, \zeta, \zeta^2, \zeta^3, \zeta^4)$ with its stabilizer $\langle x \mapsto \zeta x, x \mapsto 1/x \rangle$. Consequently, there exists an element $g \in \mathrm{PGL}_2(\mathbb{C})$ such that $g(x_1) = 0$, $g(x_2) = -1$, $g(x_3) = \infty$, $g(x_4) = \lambda + 1$ and $g(x_5) = \lambda$, and such that $gH_xg^{-1} = H_z$. Since all x_i and $z_i \in z$ are real, we see that $\bar{g}(x_i) = z_i$ for every i, hence g and $\bar{g}$ coincide on more than 2 points, hence $g = \bar{g} \in \mathrm{PGL}_2(\mathbb{R})$. □

This finishes the proof of Proposition 7.1. □

7.2 *Points of* $(P_s/\mathfrak{S}_5)(\mathbb{R})$ *with stabilizer* $\mathbb{Z}/2 \hookrightarrow \mathrm{PGL}_2(\mathbb{R})$

The goal of this section is to prove that there are no cone points in the orbifold $\mathrm{PGL}_2(\mathbb{R}) \setminus (P_s/\mathfrak{S}_5)(\mathbb{R})$, i.e. orbifold points whose stabilizer group is $\mathbb{Z}/n$ for some n acting on the orbifold chart by rotations. By Proposition 7.1, this fact will follow from the following:

Proposition 7.6. *Let* $x = (x_1, \dots, x_5) \in (P_s/\mathfrak{S}_5)(\mathbb{R})$ *be such that its stabilizer group* $H_x = \langle \tau \rangle$ *has order* 2*. Then there is a* H_x*-stable open neighborhood* $U \subset (P_s/\mathfrak{S}_5)(\mathbb{R})$ *of* x *such that* $H_x \setminus U \to \overline{\mathscr{M}}_{\mathbb{R}}$ *is injective, and a homeomorphism* $\phi : (U, x) \to (B, 0)$ *for* $0 \in B \subset \mathbb{R}^2$ *an open ball, such that* ϕ *identifies* H_x *with* $\mathbb{Z}/2$ *acting on* B *by reflections in a line through* 0.

Proof. Using Lemma 7.2, it is a routine exercise to check that the only possibilities for the element $x = (x_1, \dots, x_5) \in (P_s/\mathfrak{S}_5)(\mathbb{R})$ are as follows: 1. $x = (-1, 0, \infty, \beta, \beta^{-1})$ with $\beta \in \mathbb{A}^1(\mathbb{R}) \setminus \{-1, 0, 1\}$ or $\beta \in \mathbb{S}^1 \setminus \{-1, 1\} \subset \mathbb{A}^1(\mathbb{C}), H_x = \langle \nu \rangle$. 2. $x = (-1, i, -i, \beta, \beta^{-1})$ with $\beta \in \mathbb{A}^1(\mathbb{R}) \setminus \{-1, 1\}$ or $\beta \in \mathbb{S}^1 \setminus \{-1, 1, i, -i\} \subset \mathbb{A}^1(\mathbb{C}), H_x = \langle \nu \rangle$. 3. $x = (-1, -1, \beta, 0, \infty)$ with $\beta \in \mathbb{A}^1(\mathbb{R}) \setminus \{-1, 0, 1\}, H_x = \langle \nu \rangle$. 4. $x = (-1, -1, \beta, i, -i)$ with $\beta \in \mathbb{A}^1(\mathbb{R}) \setminus \{-1, 1\}, H_x = \langle \nu \rangle$. 5. $x = (0, 0, \infty, \infty, -1), H_x = \langle \nu \rangle$. 6. $x = (-1, i, i, -i, -i), H_x = \langle \nu \rangle$. □

7.3 *Comparing the orbifold structures*

The aim of this section is to prove the following:

Proposition 7.7. *Let* $\overline{\mathscr{M}}_{\mathbb{R}}$ *be the moduli space of real stable binary quintics. The hyperbolic orbifold structure of* $\overline{\mathscr{M}}_{\mathbb{R}}$*, induced by the homeomorphism of Corollary 6.6, has no cone points and three corner reflectors, whose orders are* $\pi/3, \pi/5$ *and* $\pi/10$.

Proof. There are two topological orbifold structures on $\overline{\mathscr{M}}_{\mathbb{R}}$: one via its definition as $[G(\mathbb{R}) \setminus X_s(\mathbb{R})]$ and one via the isomorphism $\overline{\mathscr{M}}_{\mathbb{R}} \cong P\Gamma \setminus Y$ of Theorem 6.5 and the orbifold structure on $P\Gamma \setminus Y$ given by Theorem 3.10. We claim that these topological orbifold structures differ only at the moduli point $(\infty, i, i, -i, -i)$. Indeed, this can be deduced from Proposition 3.20. The notation of that proposition was as follows: for $f \in Y \cong G(\mathbb{R}) \setminus \mathcal{F}_s(\mathbb{R})$ (see Theorem 6.5) the group $A_f \subset P\Gamma$ is the stabilizer of $f \in K$; if $x \in \mathcal{F}_s(\mathbb{R})$ represents f and if $F = [x] \in X_s(\mathbb{R})$ has $k = 2a+b$ nodes, then the image $y \in \mathbb{C}H^2$ lies on k orthogonal hyperplanes H_r, with a pairs of complex conjugate hyperplanes and b real hyperplanes. If F has no nodes ($k = 0$), then $G(\mathbf{r})$ is trivial by Proposition 3.20.1 and $G_F = A_f = \Gamma_f$. If F has only real nodes, then $B_f = G(\mathbf{r})$ hence $G_F = A_f/G(\mathbf{r}) = A_f/B_f = \Gamma_f$. Now suppose that $a = 1$ and $b = 0$: the equation F defines a pair of complex conjugate nodes. In other words, the zero set of F defines a 5-tuple $x = (\alpha_1, \dots, \alpha_5) \in \mathbb{P}^1(\mathbb{C})$, well-defined up to the $\mathrm{PGL}_2(\mathbb{R}) \times \mathfrak{S}_5$ action on $\mathbb{P}^1$, where $\alpha_1 \in \mathbb{P}^1(\mathbb{R})$ and $\alpha_3 = \bar{\alpha}_2 = \alpha_5 = \bar{\alpha}_4 \in \mathbb{P}^1(\mathbb{C}) \setminus \mathbb{P}^1(\mathbb{R})$. So we may write $x = (\rho, \alpha, \bar{\alpha}, \alpha, \bar{\alpha})$ with $\rho \in \mathbb{P}^1(\mathbb{R})$ and $\alpha \in \mathbb{P}^1(\mathbb{C}) \setminus \mathbb{P}^1(\mathbb{R})$. Then there is a unique $T \in \mathrm{PGL}_2(\mathbb{R})$ such that $T(\rho) = \infty$ and $T(\alpha) = i$. But this gives $T(x) = (\infty, i, -i, i, -i)$ hence F is unique up to isomorphism. As for the stabilizer $G_F = A_f/G(\mathbf{r})$, we have $G(\mathbf{r}) \cong (\mathbb{Z}/10)^2$. Since there are no real nodes, B_f is trivial. By Proposition 3.20.3, K_f is the union of 10 copies of $\mathbb{B}^2(\mathbb{R})$ meeting along a common point $\mathbb{B}^0(\mathbb{R})$. In fact, in the local coordinates (t_1, t_2) around f, the $\alpha_j : \mathbb{B}^2(\mathbb{C}) \to \mathbb{B}^2(\mathbb{C})$ are defined by $(t_1, t_2) \mapsto (\bar{t}_2 \zeta^j, \bar{t}_1 \zeta^j)$, for $j \in \mathbb{Z}/10$, and so the fixed points sets are given by the equations $\mathbb{R}H^2_j = \{t_2 = \bar{t}_1 \zeta^j\} \subset \mathbb{B}^2(\mathbb{C})$, $j \in \mathbb{Z}/10$. Notice that the subgroup $G \subset G(\mathbf{r})$ that stabilizes $\mathbb{R}H^2_j$ is the cyclic group of order 10 generated by the transformations $(t_1, t_2) \mapsto (\zeta t_1, \zeta^{-1} t_2)$. On the other hand, there is only one non-trivial transformation $T \in \mathrm{PGL}_2(\mathbb{R})$ that fixes ∞ and sends the subset $\{i, -i\} \subset \mathbb{P}^1(\mathbb{C})$ to itself, and T is of order 2. Hence $G_F = \mathbb{Z}/2$ so that we have an exact sequence $0 \to \mathbb{Z}/10 \to \Gamma_f \to \mathbb{Z}/2 \to 0$ and this splits since G_F is a subgroup of Γ_f. We are done by Propositions 7.1 and 7.6. □

7.4 The real moduli space of 5 unordered points on $\mathbb{P}^1$ is a hyperbolic triangle

In the sequal, we shall consider the topological space $\overline{\mathscr{M}}_{\mathbb{R}} = G(\mathbb{R}) \setminus X_s(\mathbb{R})$ as a hyperbolic orbifold via the metric induced by the homeomorphism $G(\mathbb{R}) \setminus X_s(\mathbb{R}) \cong P\Gamma \setminus K$ of Theorem 6.5, the metric and hyperbolic orbifold structure on $P\Gamma \setminus K$ of Theorem 3.10.1. The goal of Section 7.4 will be to show that $\overline{\mathscr{M}}_{\mathbb{R}}$, as a hyperbolic orbifold, is isomorphic to the triangle $\Delta_{3,5,10}$ in the real hyperbolic plane H^2 with angles $\pi/3, \pi/5$ and $\pi/10$, by which we mean that there exists an isometry $\phi : \overline{\mathscr{M}}_{\mathbb{R}} \xrightarrow{\sim} \Delta_{3,5,10}$ that preserves the orbifold structures.

The results in the above Sections 7.1, 7.2 and 7.3 give the orbifold singularities of $\overline{\mathscr{M}}_{\mathbb{R}}$ together with their stabilizer groups. In order to complete determine the hyperbolic orbifold structure of $\overline{\mathscr{M}}_{\mathbb{R}}$, however, we shall also need to know the underlying topological space of $\overline{\mathscr{M}}_{\mathbb{R}}$. This will be our first goal. The first observation is that $\overline{\mathscr{M}}_{\mathbb{R}}$ is compact. Indeed, it is classical that the topological space $\overline{\mathscr{M}}_{\mathbb{C}} = G(\mathbb{C}) \setminus X_s(\mathbb{C})$ parametrizing complex stable binary quintics is compact - for example because is homeomorphic to $\overline{M}_{0,5}(\mathbb{C})/\mathfrak{S}_5$ and the stack of stable 5-pointed curves $\overline{M}_{0,5}$ is proper [Knu83] or because it is homeomorphic to a compact ball quotient [Shi64] - and the map $\overline{\mathscr{M}}_{\mathbb{R}} \to \overline{\mathscr{M}}_{\mathbb{C}}$ is proper. The second observation is that $\overline{\mathscr{M}}_{\mathbb{R}}$ is connected: $X_s(\mathbb{R})$ is obtained from the manifold $X(\mathbb{R}) = \{F \in \mathbb{R}[x,y] : F \text{ is homogeneous of degree } 5\}$ by removing a subspace of codimension two. We can prove more:

Proposition 7.8. *The moduli space $\overline{\mathscr{M}}_{\mathbb{R}}$ of real stable binary quintics is simply connected.*

Proof. We shall prove that the closures of the three connected components of $\mathscr{M}_{\mathbb{R}}$ in $\overline{\mathscr{M}}_{\mathbb{R}}$ are homeomorphic to the closed disc D in $\mathbb{R}^2$, and that they are glued together to form $\overline{\mathscr{M}}_{\mathbb{R}}$ in the way that the components $\mathscr{M}_0$ and $\mathscr{M}_2$ are glued to $\mathscr{M}_1$ in Figure 1. Let $M_{0,5}(\mathbb{R})$ (resp. $\overline{M}_{0,5}(\mathbb{R})$) be the moduli space of real smooth (resp. stable) genus zero curves with five real marked points [Knu83]. Recall that $P_0 \subset \mathbb{P}^1(\mathbb{C})$ is the subset of distinct five-tuples, and $P_s \subset \mathbb{P}^1(\mathbb{C})$ the subset of five-tuples where no three coordinates coincides. We have $M_{0,5}(\mathbb{R}) = \mathrm{PGL}_2(\mathbb{R}) \setminus P_0(\mathbb{R})$ and $\overline{M}_{0,5}(\mathbb{R}) = \mathrm{PGL}_2(\mathbb{R}) \setminus P_s(\mathbb{R})$. Let $\sigma_1 : \mathbb{P}^1(\mathbb{C})^5 \to \mathbb{P}^1(\mathbb{C})^5$ be the anti-holomorphic involution $(x_1, x_2, x_3, x_4, x_5) \mapsto (\bar{x}_1, \bar{x}_2, \bar{x}_3, \bar{x}_5, \bar{x}_4)$, and let $\sigma_2 : \mathbb{P}^1(\mathbb{C})^5 \to \mathbb{P}^1(\mathbb{C})^5$ be the anti-holomorphic involution $(x_1, x_2, x_3, x_4, x_5) \mapsto (\bar{x}_1, \bar{x}_3, \bar{x}_2, \bar{x}_5, \bar{x}_4)$. Then define

$$P_0^1(\mathbb{R}) = P_0^{\sigma_1}, \quad P_s^1(\mathbb{R}) = P_1^{\sigma_1}, \quad P_0^2(\mathbb{R}) = P_0^{\sigma_2}, \quad P_s^2(\mathbb{R}) = P_1^{\sigma_2}.$$

The action of $\mathfrak{S}_3 \times \mathfrak{S}_2$ on $\mathbb{P}^1(\mathbb{C})^5$ commutes with σ_1 and hence $S_3 \times S_2$ preserves the sets $P_0^1(\mathbb{R})$ and $P_s^1(\mathbb{R})$. Similarly, $\mathfrak{S}_2 \times \mathfrak{S}_2$ preserves $P_0^2(\mathbb{R})$ and $P_s^2(\mathbb{R})$. On the other hand, $\mathrm{PGL}_2(\mathbb{R})$ acts on all the $P_0^i(\mathbb{R})'s$ and $P_s^i(\mathbb{R})'s$, and this commutes with the $\mathfrak{S}_3 \times \mathfrak{S}_2$ and $\mathfrak{S}_2 \times \mathfrak{S}_2$ actions.

Recall that $\mathscr{M}_i \subset \mathscr{M}_{\mathbb{R}}$ was the connected component of unordered five-tuples $(x_1, \dots, x_5) \in \mathrm{PGL}_2(\mathbb{R}) \setminus (P_s/\mathfrak{S}_5)(\mathbb{R})$ that have i pairs of complex conjugate points $x_i \neq \bar{x}_i \in x$. One readily observes that $\mathscr{M}_0 = \mathrm{PGL}_2(\mathbb{R}) \setminus P_0(\mathbb{R})/\mathfrak{S}_5, \mathscr{M}_1 = \mathrm{PGL}_2(\mathbb{R}) \setminus P_0^1(\mathbb{R})/\mathfrak{S}_3 \times \mathfrak{S}_2, \mathscr{M}_2 = \mathrm{PGL}_2(\mathbb{R}) \setminus P_0^2(\mathbb{R})/\mathfrak{S}_2 \times \mathfrak{S}_2$, and we define $\overline{\mathscr{M}}_0 = \mathrm{PGL}_2(\mathbb{R}) \setminus P_s(\mathbb{R})/\mathfrak{S}_5$,

$$\overline{\mathscr{M}}_1 = \mathrm{PGL}_2(\mathbb{R}) \setminus P_s^1(\mathbb{R})/\mathfrak{S}_3 \times \mathfrak{S}_2, \quad \overline{\mathscr{M}}_2 = \mathrm{PGL}_2(\mathbb{R}) \setminus P_s^2(\mathbb{R})/\mathfrak{S}_2 \times \mathfrak{S}_2.$$

Note that the natural maps $\overline{\mathscr{M}}_i \to \overline{\mathscr{M}}_{\mathbb{R}}$ are closed embeddings of topological spaces; we identify $\overline{\mathscr{M}}_i$ with its image in $\overline{\mathscr{M}}_{\mathbb{R}}$. Moreover, we have $\overline{\mathscr{M}}_{\mathbb{R}} = \overline{\mathscr{M}}_0 \cup \overline{\mathscr{M}}_1 \cup \overline{\mathscr{M}}_2$. The first step is to show that each $\overline{\mathscr{M}}_i$ is homeomorphic to the closed disc D in $\mathbb{R}^2$. We start with $\overline{\mathscr{M}}_0$. Let $\Gamma \subset \mathbb{P}^1(\mathbb{R}) \times \mathbb{P}^1(\mathbb{R})$ be the union of the lines

$$l_1 = \mathbb{P}^1(\mathbb{R}) \times \{0\}, l_2 = \{0\} \times \mathbb{P}^1(\mathbb{R}), k_1 = \mathbb{P}^1(\mathbb{R}) \times \{1\}, k_2 = \{1\} \times \mathbb{P}^1(\mathbb{R}),$$

$$m_1 = \mathbb{P}^1(\mathbb{R}) \times \{\infty\}, m_2 = \{\infty\} \times \mathbb{P}^1(\mathbb{R}), \delta = \Delta = \{(x,x) : x \in \mathbb{P}^1(\mathbb{R})\}.$$

Then $\mathrm{PGL}_2(\mathbb{R}) \setminus P_0(\mathbb{R}) \cong (\mathbb{P}^1(\mathbb{R}) \times \mathbb{P}^1(\mathbb{R})) - \Gamma$. Hence $\mathrm{PGL}_2(\mathbb{R}) \setminus P_0(\mathbb{R})$ has 12 components $C \subset \mathrm{PGL}_2(\mathbb{R}) \setminus P_0(\mathbb{R})$. If $N = \mathrm{Stab}_{\mathfrak{S}_5}(C)$ is the stabilizer in $\mathfrak{S}_5$ of one of them, then $|N| = 120/12 = 10$ hence $N = D_5$. On the other hand, $\overline{M}_{0,5}(\mathbb{R}) = \mathrm{PGL}_2(\mathbb{R}) \setminus P_s(\mathbb{R})$ is homeomorphic to the blow-up of $\mathbb{P}^1(\mathbb{R}) \times \mathbb{P}^1(\mathbb{R})$ in three points $0, 1, \infty$. Let $\bar{C}$ be the closure of C in $\overline{M}_{0,5}(\mathbb{R})$. Then $\bar{C}$ is simply connected as well as equivariant under $N \subset \mathfrak{S}_5$. Since the only orbifold point of $\overline{\mathscr{M}}_{\mathbb{R}}$ with stabilizer D_5 is $\alpha_5 := (0, -1, \infty, \lambda + 1, \lambda)$ with $\lambda = \zeta_5 + \zeta_5^{-1} \in \mathbb{P}^1(\mathbb{R})$ by Lemma 7.5, and $\alpha_5 \in \overline{M}_{0,5}(\mathbb{R})/\mathfrak{S}_5$, there is a D_5-equivariant homeomorphism between $\bar{C}$ and $D \subset \mathbb{R}^2$, mapping α_5 to $0 \in D$, with D_5 acting on $(D, 0)$ in the natural way. Hence

$$\overline{\mathscr{M}}_0 = \overline{M}_{0,5}(\mathbb{R})/\mathfrak{S}_5 = \bar{C}/N = D/D_5$$

is simply connected, and homeomorphic to the closed disc $D \subset \mathbb{R}^2$.

Next, we treat the case $\overline{\mathscr{M}}_1$. Let $\overline{M}^1_{0,5}(\mathbb{R}) = \mathrm{PGL}_2(\mathbb{R}) \setminus P^1_s(\mathbb{R})$ and $M^1_{0,5}(\mathbb{R}) = \mathrm{PGL}_2(\mathbb{R}) \setminus P^1_0(\mathbb{R})$. Now $M^1_{0,5}(\mathbb{R}) = \mathrm{PGL}_2(\mathbb{R}) \setminus P^1_0(\mathbb{R}) \cong$ and $\overline{M}^1_{0,5}$ is the real blow-up $B = \mathrm{Bl}_{0,1,\infty}\mathbb{P}^1(\mathbb{C})$ of $\mathbb{P}^1(\mathbb{C})$ in the three points $0, 1, \infty$, with $M^1_{0,5}(\mathbb{R}) \hookrightarrow \overline{M}^1_{0,5}(\mathbb{R})$ corresponding to the natural inclusion $\mathbb{P}^1(\mathbb{C}) \setminus \mathbb{P}^1(\mathbb{R}) \hookrightarrow B$. Let C be any of the two connected components of $\mathbb{P}^1(\mathbb{C}) \setminus \mathbb{P}^1(\mathbb{R})$. Then C is a manifold and an open subset of B, whose closure $\bar{C}$ in B is a the simply connected manifold with corners diffeomorphic to the hexagon. Moreover, action of the subgroup $\mathfrak{S}_2$ of $\mathfrak{S}_3 \times \mathfrak{S}_2$ on $\mathbb{P}^1(\mathbb{C}) \setminus \mathbb{P}^1(\mathbb{R})$ interchanges the connected components, hence the stabilizer $N = \mathrm{Stab}_{\mathfrak{S}_3 \times \mathfrak{S}_2}(C)$ is $\mathfrak{S}_3$, and $\bar{C}$ is stabilized by N. By Lemma 7.4, the only orbifold point of $\overline{\mathscr{M}}_{\mathbb{R}}$ with stabilizer group D_3 is given by the point $\alpha_3 := (-1, \infty, 0, \omega, \omega^2)$ with ω a primitive third root of unity. Hence $(\bar{C}, \alpha_3)$ is homeomorphic to the closed disc $(D, 0) \subset (\mathbb{R}^2, 0)$ with $N = \mathfrak{S}_3$ acting on it in the natural way. In particular,

$$\overline{\mathscr{M}}_1 = \overline{M}^1_{0,5}/\mathfrak{S}_3 \times \mathfrak{S}_2 = \bar{C}/N = D/D_3$$

is homeomorphic to the closed disc $D \subset \mathbb{R}^2$. Finally, consider $\overline{\mathscr{M}}_2$. Let $\overline{M}^2_{0,5}(\mathbb{R}) = \mathrm{PGL}_2(\mathbb{R}) \setminus P^2_s(\mathbb{R})$ and $M^2_{0,5}(\mathbb{R}) = \mathrm{PGL}_2(\mathbb{R}) \setminus P^2_0(\mathbb{R})$. Now $M^2_{0,5}(\mathbb{R}) = \mathrm{PGL}_2(\mathbb{R}) \setminus P^2_0(\mathbb{R}) \cong \mathbb{P}^1(\mathbb{C}) \setminus (\mathbb{P}^1(\mathbb{R}) \cup \{\infty, i, -i\})$ and $\overline{M}^2_{0,5}$ is the real blow-up $B' = \mathrm{Bl}_\infty \mathbb{P}^1(\mathbb{C})$ of the sphere $\mathbb{P}^1(\mathbb{C})$ in the point ∞, with $M^2_{0,5}(\mathbb{R}) \hookrightarrow \overline{M}^2_{0,5}(\mathbb{R})$ corresponding to the natural inclusion of $\mathbb{P}^1(\mathbb{C}) \setminus (\mathbb{P}^1(\mathbb{R}) \cup \{i, -i\})$ in B'. Let C be any of the two connected components of $\mathbb{P}^1(\mathbb{C}) \setminus \mathbb{P}^1(\mathbb{R})$. Then C is a manifold, homeomorphic to the punctured unit disc, and forms an open subset of B whose closure $\bar{C}$ in B is the simply connected manifold-with-corners that has 2 two corners. The subgroup N of $\mathfrak{S}_2 \times \mathfrak{S}_2$ that stabilizes C is isomorphic to $\mathbb{Z}/2$, and $\bar{C}$ is equivariant under N. The induced action of $\mathbb{Z}/2$ on D' is reflection in a line through the origin; in particular,

$$\overline{\mathscr{M}}_2 = \overline{M}^2_{0,5}/\mathfrak{S}_2 \times \mathfrak{S}_2 = \bar{C}/N = D'/\mathbb{Z}/2$$

is homeomorphic to the closed disc $D \subset \mathbb{R}^2$. Finally, it is clear that $\overline{\mathscr{M}}_{\mathbb{R}} = \overline{\mathscr{M}}_0 \cup \overline{\mathscr{M}}_1 \cup \overline{\mathscr{M}}_2$ is topologically the gluing of three closed discs in the prescribed way, i.e. as in Figure 1. □

Theorem 7.9. *The moduli space $\overline{\mathscr{M}}_{\mathbb{R}} = G(\mathbb{R}) \setminus X_s(\mathbb{R})$ of stable binary quintics equipped with the metric given by Theorem 6.5 is isometric to the triangle $\Delta_{3,5,10} \subset \mathbb{R}H^2$ of angles $\pi/3, \pi/5$ and $\pi/10$ in the real hyperbolic plane $\mathbb{R}H^2$.*

Proof. To any closed 2-dimensional orbifold O one can associate a set of natural numbers $S_O = \{n_1, \ldots, n_k; m_1, \ldots, m_l\}$ by letting k be the number of cone points of X_O, l the number of corner reflectors, n_i the order of the i-th cone point and $2m_j$ the order of the j-th corner reflector. A closed 2-dimensional orbifold O is then determined, up to orbifold-structure preserving homeomorphism, by its underlying space X_O and the set S_O. By Proposition 7.8, $\overline{\mathscr{M}}_{\mathbb{R}}$ is simply connected. By Proposition 7.7, $\overline{\mathscr{M}}_{\mathbb{R}}$ has no cone points and three corner reflectors whose orders are $\pi/3, \pi/5$ and $\pi/10$. This implies $\overline{\mathscr{M}}_{\mathbb{R}}$ and $\Delta_{3,5,10}$ are isomorphic as topological orbifolds. Consequently, the orbifold fundamental group of $\overline{\mathscr{M}}_{\mathbb{R}}$ is abstractly isomorphic to the group $\Gamma_{3,5,10} := \langle a, b, c | a^2 = b^2 = c^2 = (ab)^3 = (ac)^5 = (bc)^{10} = 1\rangle$. Now let $\phi : \Gamma_{3,5,10} \hookrightarrow \mathrm{PSL}_2(\mathbb{R})$ be *any* embedding such that $X := \phi(\Gamma_{3,5,10}) \setminus \mathbb{R}H^2$ is a hyperbolic orbifold; we claim that there is a fundamental domain Δ of X isometric to $\Delta_{3,5,10}$. Consider the generator $a \in \Gamma_{3,5,10}$. Since $\phi(a)^2 = 1$, there exists a geodesic $L_1 \subset \mathbb{R}H^2$ such that $\phi(a) \in \mathrm{PSL}_2(\mathbb{R}) = \mathrm{Isom}(\mathbb{R}H^2)$ is the reflection across L_1. Next, consider the generator $b \in \Gamma_{3,5,10}$. Again, there exists a geodesic $L_2 \subset \mathbb{R}H^2$ such that $\phi(b)$ is the reflection across L_2. One easily shows that $L_2 \cap L_1 \neq \emptyset$. Let $x \in L_1 \cap L_2$. Then $\phi(a)\phi(b)$ is an element of order three that fixes x, hence $\phi(a)\phi(b)$ is a rotation around x. Therefore, one of the angles between L_1 and L_2 must be $\pi/3$. Finally, we know that $\phi(c)$ is an element of order 2 in $\mathrm{PGL}_2(\mathbb{R})$, hence a reflection across a line L_3. By the previous arguments, $L_3 \cap L_2 \neq \emptyset$ and $L_3 \cap L_1 \neq \emptyset$. It also follows that $x \in L_3 \cap L_2 \cap L_1 = \emptyset$. Consequently, the three geodesics $L_i \subset \mathbb{R}H^2$ enclose a hyperbolic triangle; the orders of $\phi(a)\phi(b)$, $\phi(a)\phi(c)$ and $\phi(b)\phi(c)$ imply that the three interior angles of the triangle are $\pi/3$, $\pi/5$ and $\pi/10$. □

8 Monodromy Groups of Complex and Real Binary Quintics

In this section, we describe the monodromy group $P\Gamma$ attached to the universal $\mathscr{C} \to X_0(\mathbb{C})$ family of cyclic quintic covers $C \to \mathbb{P}^1_{\mathbb{C}}$ ramified along a smooth binary quintic, as well as the groups $P\Gamma_\alpha$ appearing in Proposition 6.2, i.e. the groups that make $P\Gamma_\alpha \setminus (\mathbb{R}H^2 - \mathscr{H})$ isomorphic to a connected component $\mathscr{M}_i$ of the moduli space of smooth binary quintics $\mathscr{M}_{\mathbb{R}}$.

We start with an explicit description of $P\Gamma$. Let $\mathscr{C} \to X_0(\mathbb{C})$ be the universal family of cyclic quintic covers of $\mathbb{P}^1_{\mathbb{C}}$ ramified along a smooth binary quintic. Let $K = \mathbb{Q}(\zeta_5)$ and let Λ be the $\mathcal{O}_K$-module $H^1(C(\mathbb{C}), \mathbb{Z})$ for the cyclic quintic $C \to \mathbb{P}^1_{\mathbb{C}}$ ramified along the smooth binary quintic defined by $F_0 \in X_0(\mathbb{C})$. Let $\mathfrak{h}$ be the hermitian form of Equation (10), define $\Gamma = \mathrm{Aut}_{\mathcal{O}_K}(\Lambda)$, $P\Gamma = \Gamma/\mu_K$ and consider the monodromy representation $\rho : \pi_1(X_0(\mathbb{C}), F_0) \to P\Gamma$ defined in Equation (11). Recall that ρ is surjective by Corollary 5.3. Moreover, we have:

Theorem 8.1 (Shimura)**.** *There is an isomorphism* $(\Lambda, \mathfrak{h}) \cong \left(\mathcal{O}_K^3, \mathrm{diag}(1, 1, \frac{1-\sqrt{5}}{2})\right)$.

Proof. See [Shi64, Section 6] as well as item (5) in the table on page 1. □

So let us simply write $\Lambda = \mathcal{O}_K^3$ and $\mathfrak{h} = \mathrm{diag}(1, 1, \frac{1-\sqrt{5}}{2})$ in the remaining part of Section 8. Write $\alpha = \zeta_5 + \zeta_5^{-1} = \frac{\sqrt{5}-1}{2}$. Recall that $\theta = \zeta_5 - \zeta_5^{-1}$ and observe that $|\theta|^2 = \frac{\sqrt{5}+5}{2}$. Define three quadratic forms q_0, q_1 and q_2 on $\mathbb{Z}[\alpha]^3$ as follows:

$$\begin{aligned} q_0(x_0, x_1, x_2) &= x_0^2 + x_1^2 - \alpha x_2^2 \\ q_1(x_0, x_1, x_2) &= |\theta|^2 x_0^2 + x_1^2 - \alpha x_2^2 \\ q_2(x_0, x_1, x_2) &= |\theta|^2 x_0^2 + |\theta|^2 x_1^2 - \alpha x_2^2 \end{aligned} \tag{23}$$

We consider $\mathbb{Z}[\alpha]$ as a subring of $\mathbb{R}$ via the standard embedding.

Theorem 8.2. *Consider the quadratic forms* $q_j : \mathbb{Z}[\alpha]^3 \to \mathbb{Z}[\alpha]$. *There is a union geodesic subspaces* $\mathscr{H}_j \subset \mathbb{R}H^2$ *for each* $j \in \{0, 1, 2\}$ *and an isomorphism of real hyperbolic orbifolds*

$$\mathscr{M}_{\mathbb{R}} \cong \coprod_{j=0}^{2} \mathrm{PO}(q_j, \mathbb{Z}[\alpha]) \setminus \left(\mathbb{R}H^2 - \mathscr{H}_j\right). \tag{24}$$

Proof. Recall that $\theta = \zeta_5 - \zeta_5^{-1}$; we consider the $\mathbb{F}_5$-vector space W equipped with the quadratic form $q = \mathfrak{h} \bmod \theta$. We consider three explicit anti-involutions α_j on the $\mathcal{O}_K$-lattice Λ:

$$\begin{aligned} \alpha_0 &: (x_0, x_1, x_2) \mapsto (\ \ \bar{x}_0, \ \ \ \bar{x}_1, \bar{x}_2) \\ \alpha_1 &: (x_0, x_1, x_2) \mapsto (-\bar{x}_0, \ \ \ \bar{x}_1, \bar{x}_2) \\ \alpha_2 &: (x_0, x_1, x_2) \mapsto (-\bar{x}_0, -\bar{x}_1, \bar{x}_2). \end{aligned} \tag{25}$$

For isometries $\alpha : W \to W$, the dimension and determinant of the fixed space $(W^{\alpha}, q|_{W^{\alpha}})$ are conjugacy-invariant. Using this, one easily shows that an anti-unitary involution of Λ is Γ-conjugate to exactly one of the $\pm\alpha_j$, hence $C\mathscr{A}$ has cardinality 3 and is represented by $\alpha_0, \alpha_1, \alpha_2$ of (25). By Proposition 6.2, we obtain $\mathscr{M}_{\mathbb{R}} \cong \coprod_{j=0}^{2} P\Gamma_{\alpha_j} \setminus (\mathbb{R}H^2_{\alpha_j} - \mathscr{H})$ where each hyperbolic quotient $P\Gamma_{\alpha_j} \setminus (\mathbb{R}H^2_{\alpha_j} - \mathscr{H})$ is connected. Next, consider the fixed lattices

$$\begin{aligned} \Lambda_0 &:= \Lambda^{\alpha_0} = \mathbb{Z}[\alpha] \oplus \mathbb{Z}[\alpha] \oplus \mathbb{Z}[\alpha] \\ \Lambda_1 &:= \Lambda^{\alpha_1} = \theta\mathbb{Z}[\alpha] \oplus \mathbb{Z}[\alpha] \oplus \mathbb{Z}[\alpha] \\ \Lambda_2 &:= \Lambda^{\alpha_2} = \theta\mathbb{Z}[\alpha] \oplus \theta\mathbb{Z}[\alpha] \oplus \mathbb{Z}[\alpha]. \end{aligned} \tag{26}$$

One easily shows that $P\Gamma_{\alpha_j} = N_{P\Gamma}(\alpha_j)$ for the normalizer $N_{P\Gamma}(\alpha_j)$ of α_j in $P\Gamma$. Moreover, if h_j denotes the restriction of $\mathfrak{h}$ to Λ^{α_j}, then there is a natural embedding

$$\iota : N_{P\Gamma}(\alpha_j) \hookrightarrow \mathrm{PO}(\Lambda_j, h_j, \mathbb{Z}[\alpha]). \tag{27}$$

We claim that ι is actually an isomorphism. Indeed, this follows from the fact that the natural homomorphism $\pi : N_{\Gamma}(\alpha_j) \to O(\Lambda_j, h_j)$ is surjective, where $N_{\Gamma}(\alpha_j) = \{g \in \Gamma : g \circ \alpha_j = \alpha_j \circ g\}$ is the normalizer of α_j in Γ. The surjectivity of π follows in turn from the equality

$$\Lambda = \mathcal{O}_K \cdot \Lambda_j + \mathcal{O}_K \cdot \theta\Lambda_j^{\vee} \subset K^3$$

which can be deduced from (26). Since $\mathrm{PO}(\Lambda_j, h_j, \mathbb{Z}[\alpha]) = \mathrm{PO}(q_j, \mathbb{Z}[\alpha])$, we are done. ☐

References

[ACT02] Daniel Allcock, James Carlson, and Domingo Toledo. “Orthogonal complex hyperbolic arrangements”. In: *Symposium in Honor of C. H. Clemens (Salt Lake City, UT, 2000)*. Vol. 312. Contemp. Math. Amer. Math. Soc., Providence, RI, 2002, pp. 1–8. DOI: 10.1090/conm/312/05388.

[ACT06] Daniel Allcock, James Carlson, and Domingo Toledo. “Non-Arithmetic Uniformization of Some Real Moduli Spaces”. In: *Geometriae Dedicata* 122 (2006).

[ACT07] Daniel Allcock, James Carlson, and Domingo Toledo. “Hyperbolic Geometry and the Moduli Space of Real Binary Sextics”. In: June 2007, pp. 1–22. ISBN: 978-3-7643-8283-4. DOI: 10.1007/978-3-7643-8284-1_1.

[ACT10] Daniel Allcock, James Carlson, and Domingo Toledo. "Hyperbolic geometry and moduli of real cubic surfaces". en. In: *Annales scientifiques de l'École Normale Supérieure* Ser. 4, 43.1 (2010), pp. 69–115. DOI: 10.24033/asens.2116.

[AV04] Alessandro Arsie and Angelo Vistoli. "Stacks of cyclic covers of projective spaces". In: *Compositio Mathematica* 140.3 (2004), 647–666. DOI: 10.1112/S0010437X03000253.

[Bea09] Arnaud Beauville. "Moduli of cubic surfaces and Hodge theory (after Allcock, Carlson, Toledo)". In: *Géométries à courbure négative ou nulle, groupes discrets et rigidités*. Vol. 18. Sémin. Congr. Soc. Math. France, Paris, 2009, pp. 445–466.

[Bea10] Arnaud Beauville. "Finite subgroups of $\mathrm{PGL}_2(K)$". In: *Vector bundles and complex geometry*. Vol. 522. Contemp. Math. Amer. Math. Soc., Providence, RI, 2010, pp. 23–29. DOI: 10.1090/conm/522/10289.

[Bea86] Arnaud Beauville. "Le groupe de monodromie des familles universelles d'hypersurfaces et d'intersections complètes". In: *Complex analysis and algebraic geometry (Göttingen, 1985)*. Vol. 1194. Lecture Notes in Math. Springer, Berlin, 1986, pp. 8–18. DOI: 10.1007/BFb0076991.

[CT99] James Carlson and Domingo Toledo. "Discriminant complements and kernels of monodromy representations". In: *Duke Math. J.* 97.3 (1999), pp. 621–648. ISSN: 0012-7094. DOI: 10.1215/S0012-7094-99-09723-5.

[Deb96] Olivier Debarre. "Polarisations sur les variétés abéliennes produits". In: *C. R. Acad. Sci. Paris, t. 323* I (1996), pp. 631–635.

[DM86] Pierre Deligne and George Mostow. "Monodromy of hypergeometric functions and non-lattice integral monodromy". en. In: *Publications Mathématiques de l'IHÉS* 63 (1986). URL: http://www.numdam.org/item/PMIHES_1986__63__5_0/.

[Fox57] Ralph Fox. "Covering spaces with singularities:" in: *Algebraic Geometry and Topology: A Symposium in Honor of Solomon Lefschetz*. Ed. by Ralph Hartzler Fox. Princeton University Press, 1957, pp. 243–257. DOI: doi:10.1515/9781400879915-019. URL: https://doi.org/10.1515/9781400879915-019.

[GH81] Benedict Gross and Joe Harris. "Real algebraic curves". en. In: *Annales scientifiques de l'École Normale Supérieure* Ser. 4, 14.2 (1981), pp. 157–182. DOI: 10.24033/asens.1401. URL: http://www.numdam.org/item/ASENS_1981_4_14_2_157_0.

[GPS87] Mikhael Gromov and Ilya Iosifovich Piatetski-Shapiro. "Non-arithmetic groups in Lobachevsky spaces". en. In: *Publications Mathématiques de l'IHÉS* 66 (1987), pp. 93–103. URL: http://www.numdam.org/item/PMIHES_1987__66__93_0/.

[Gro07] Misha Gromov. "Metric Structures for Riemannian and Non-Riemannian Spaces". In: Boston, MA: Birkhäuser Boston, 2007. ISBN: 978-0-8176-4583-0. DOI: 10.1007/978-0-8176-4583-0_1. URL: https://doi.org/10.1007/978-0-8176-4583-0_1.

[HM98] Joe Harris and Ian Morrison. "Moduli of curves". In: 1998.

[HR16] Gert Heckman and Sander Rieken. *Hyperbolic Geometry and Moduli of Real Curves of Genus Three*. 2016. arXiv: 1612.05785 [math.AG].

[Knu83] Finn Knudsen. "The projectivity of the moduli space of stable curves, II: The stacks $M_{g,n}$". In: *Mathematica Scandinavica* 52 (1983), 161–199. DOI: 10.7146/math.scand.a-12001. URL: https://www.mscand.dk/article/view/12001.

[Kon05] Shigeyuki Kondō. "The Moduli Space of 5 Points on $\mathbb{P}^1$ and K3 Surfaces". In: *Progress in Mathematics* 260 (Aug. 2005). DOI: 10.1007/978-3-7643-8284-1_7.

[Lee13] John Lee. *Introduction to smooth manifolds*. Second. Vol. 218. Graduate Texts in Mathematics. Springer, New York, 2013, pp. xvi+708. ISBN: 978-1-4419-9981-8.

[Lia76] Joseph Liang. "On the integral basis of the maximal real subfield of a cyclotomic field." In: *Journal für die reine und angewandte Mathematik* 0286_0287 (1976), pp. 223–226. URL: http://eudml.org/doc/151784.

[Mil20] James Milne. *Complex Multiplication*. 2020. URL: https://www.jmilne.org/math/CourseNotes/CM.pdf.

[MO11] Ben Moonen and Frans Oort. "The Torelli locus and special subvarieties". In: 2011.

[MS72] David Mumford and Kalevi Suominen. "Introduction to the theory of moduli". In: *Algebraic geometry, Oslo 1970 (Proc. Fifth Nordic Summer-School in Math.)* 1972.

[Neu99] Jürgen Neukirch. *Algebraic number theory*. Vol. Grundlehren der mathematischen Wissenschaften. Springer, 1999.

[Nik80] Viacheslav Nikulin. "Integral Symmetric Bilinear FOrms and Some of their Applications". In: *Mathematics of the USSR-Izvestiya* 14.1 (1980), pp. 103–167. DOI: 10.1070/im1980v014n01abeh001060.

[Shi63] Goro Shimura. "On Analytic Families of Polarized Abelian Varieties and Automorphic Functions". In: *Annals of Mathematics* 78 (1963), p. 149.

[Shi64] Goro Shimura. "On purely transcendental fields automorphic functions of several variable". In: *Osaka Journal of Mathematics* 1.1 (1964), pp. 1 –14. DOI: ojm/1200690995. URL: https://doi.org/.

[Shi75] Goro Shimura. "On the real points of an arithmetic quotient of a bounded symmetric domain". In: *Math. Ann.* 215 (1975), pp. 135–164. ISSN: 0025-5831. DOI: 10.1007/BF01432692. URL: https://doi.org/10.1007/BF01432692.

[SS89] Mika Seppälä and Robert Silhol. "Moduli Spaces for Real Algebraic Curves and Real Abelian Varieties". In: *Mathematische Zeitschrift* 201.2 (1989), pp. 151–166. URL: http://eudml.org/doc/174043.

[Stacks] The Stacks Project Authors. *Stacks Project*. https://stacks.math.columbia.edu. 2018.

[Tak77] Kisao Takeuchi. "Arithmetic triangle groups". In: *J. Math. Soc. Japan* 29.1 (1977), pp. 91–106. ISSN: 0025-5645. DOI: 10.2969/jmsj/02910091. URL: https://doi.org/10.2969/jmsj/02910091.

[Thu80] William Thurston. *The Geometry and Topology of Three-Manifolds*. Vol. 1. Lecture notes from Princeton University 1978-80. Princeton University Press, 1980. URL: http://www.msri.org/publications/books/gt3m.

[Wam99] Paul Van Wamelen. "Examples of genus two CM curves defined over the rationals". In: *Math. Comp* 68 (1999), pp. 307–320.

[WX19] Yue Wu and Bin Xu. *Moduli Spaces of Unordered $n \geq 5$ Points on the Riemann Sphere and Their Singularities*. 2019. arXiv: 1704.07583 [math.AG].