Université de Paris

From Boltzmann to Navier–Stokes with polynomial initial data

Pierre Gervais

ENS | PSL

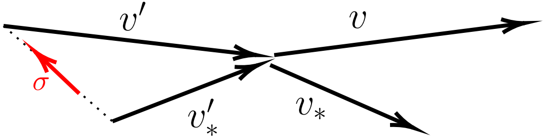
Introduction

The scaled Boltzmann equation

Boltzmann equation = evolution of particles density $F^\varepsilon(t,x,v) \geq 0$,
mean free path (Knudsen number) = ε and $x \in \Omega = \mathbb{R}^d, \mathbb{T}^d$, ($d = 2, 3$)

$$\varepsilon \partial_t F^\varepsilon + v \cdot \nabla_x F^\varepsilon = \frac{1}{\varepsilon} Q(F^\varepsilon, F^\varepsilon)$$

$$Q(F,G)(v) = \int_{\mathbb{R}^d_{v_*} \times \mathbb{S}^{d-1}_\sigma} |v - v_*| \Big(F(v')G(v'_*) - F(v)G(v_*) \Big) \mathrm{d}v_* \mathrm{d}\sigma$$



$$v' = \frac{v + v_*}{2} + \sigma \frac{|v - v_*|}{2}, \quad v'_* = \frac{v + v_*}{2} - \sigma \frac{|v - v_*|}{2}$$

Introduction

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$$\varepsilon \partial_t F^\varepsilon + v \cdot \nabla_x F^\varepsilon = \frac{1}{\varepsilon} Q(F^\varepsilon, F^\varepsilon)$$

Conserved macroscopic observables:

- Mass : $R^\varepsilon = \displaystyle\int F^\varepsilon \,\mathrm{d}v$
- Momentum : $R^\varepsilon U^\varepsilon = \displaystyle\int F^\varepsilon \, v \,\mathrm{d}v$
- Energy : $\frac{1}{2} R^\varepsilon |U|^2 + \frac{d}{2} R^\varepsilon T^\varepsilon = \displaystyle\int F^\varepsilon \, \frac{|v|}{2}^2 \mathrm{d}v$

Introduction

Relation with the incompressible Navier–Stokes–Fourier system

Gas at thermodynamic equilibrium (constant heat, mass density, at rest) :

$$M = (2\pi)^{-d/2} \exp\big(-|v|^2/2\big)$$

Statistical fluctuation of order ε:

$$F^\varepsilon = M + \varepsilon f^\varepsilon, \; F^\varepsilon_{|t=0} = M + \varepsilon f_{\text{in}}$$

Macroscopic fluctuations of order ε:

$$\begin{aligned} R^\varepsilon(t,x) &\approx 1 + \varepsilon\rho^\varepsilon(t,x), \\ U^\varepsilon(t,x) &\approx 0 + \varepsilon u^\varepsilon(t,x), \\ T^\varepsilon(t,x) &\approx 1 + \varepsilon\theta^\varepsilon(t,x). \end{aligned}$$

Relation with the incompressible Navier–Stokes–Fourier system

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Statistical fluctuation of order ε:

$$F^\varepsilon = M + \varepsilon f^\varepsilon, \; F^\varepsilon_{|t=0} = M + \varepsilon f_{\text{in}}$$

"Linearized" equation:

$$\begin{cases} \partial_t f^\varepsilon = \frac{1}{\varepsilon^2} \left(\mathcal{L} + \varepsilon v \cdot \nabla_x\right) f^\varepsilon + \frac{1}{\varepsilon} Q(f^\varepsilon, f^\varepsilon), \\ f^\varepsilon_{|t=0} = f_{\text{in}}, \end{cases} \qquad (B^\varepsilon)$$

where

$$\mathcal{L} := Q(M, \cdot) + Q(\cdot, M)$$

Introduction

Relation with the incompressible Navier–Stokes–Fourier system

Definition-Theorem (microscopic, macroscopic)

- We say $f(x,v)$ is **macroscopic** if it satisfies the **equivalent conditions**
  - $\mathcal{L}f = 0$
  - $f(x,v) = \Big(\rho(x) + u(x)\cdot v + \frac{1}{2}(|v|^2 - d)\theta(x)\Big)M(v)$

  and **well-prepared** if $\nabla_x \cdot u(x) = 0,\ \rho(x) + \theta(x) = 0.$
- We say f is **microscopic** if

$$\int f(v)\varphi(v)M(v)\mathrm{d}v = 0,\ \varphi(v) = 1, v, |v|^2$$

Introduction

Relation with the incompressible Navier–Stokes–Fourier system

Theorem (1991-2004)

If $F^\varepsilon = M + \varepsilon f^\varepsilon$ is a "renormalized" solution to the Boltzmann equation, then f^ε converges in a weak sense to

$$f^0(t,x,v) = \left(\rho(t,x) + u(t,x)\cdot v + \frac{1}{2}\left(|v|^2 - d\right)\theta(t,x)\right)M(v),$$

where (ρ, u, θ) are Leray solutions to the Navier-Stokes-Fourier

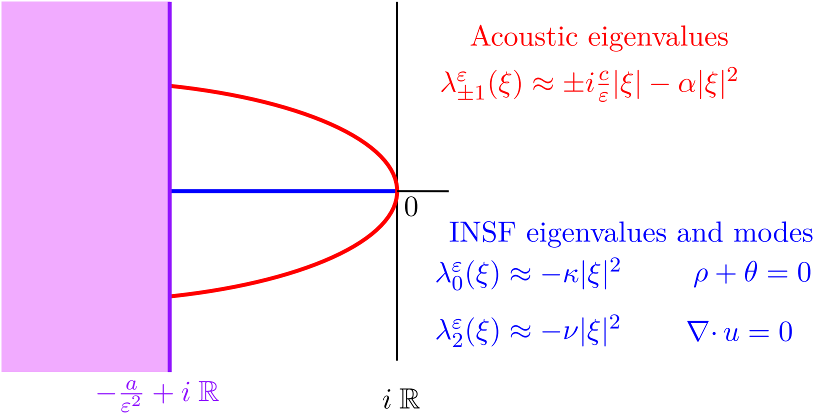
$$\begin{cases} \partial_t u + u\cdot\nabla_x u = \mu\Delta_x u - \nabla_x p, \\ \partial_t \theta + u\cdot\nabla_x\theta = \kappa\Delta_x\theta, \\ \nabla_x\cdot u = 0, \quad \rho + \theta = 0, \end{cases} \tag{INSF}$$

and $\mu, \kappa > 0$ depend only on Q and M.

Construction of solutions and convergence

Initial data with Gaussian decay (Bardos–Ukai/Gallagher–Tristani)

- Functional space : $\mathbf{G} = L^\infty_v H^s_x \left(M^{-1/2}\langle v\rangle^\beta \mathrm{d}v\right)$
- Spectral study of $\mathcal{L} + v \cdot \nabla_x$ from R. Ellis, M. Pinsky, S. Ukai (c.f. figure)
- "Grad's decomposition" of $\mathcal{L}$



Construction of solutions and convergence

Initial data with Gaussian decay (Bardos–Ukai/Gallagher–Tristani)

Duhamel formulation, initial data $f^\varepsilon_{|t=0} = f_{\mathrm{in}}$

$$\partial_t f^\varepsilon = \frac{1}{\varepsilon^2}\big(\mathcal{L} + \varepsilon v\cdot\nabla_x\big) f^\varepsilon + \frac{1}{\varepsilon} Q(f^\varepsilon, f^\varepsilon), \qquad (B^\varepsilon)$$

$$\downarrow$$

$$f^\varepsilon(t) = U^\varepsilon(t) f_{\mathrm{in}} + \Psi^\varepsilon(t)(f^\varepsilon, f^\varepsilon),$$

Where we denote

$$U^\varepsilon(t) := \exp\left(\frac{1}{\varepsilon^2}\left(\mathcal{L} + \varepsilon v\cdot\nabla_x\right)\right),$$

$$\Psi^\varepsilon(t)(f^\varepsilon, f^\varepsilon) := \frac{1}{\varepsilon}\int_0^t U^\varepsilon(t-t') Q\left(f^\varepsilon(t')\mathrm{d}t', f^\varepsilon(t')\right)$$

Construction of solutions and convergence

Initial data with Gaussian decay (Bardos–Ukai/Gallagher–Tristani)

$$f^\varepsilon(t) = U^\varepsilon(t) f_{\text{in}} + \Psi^\varepsilon(t)(f^\varepsilon, f^\varepsilon)$$

- Bardos-Ukai (1991):
  - uniform bounds on U^ε and Ψ^ε
  - convergence of U^ε and Ψ^ε
  - $\rightarrow$ global solutions for $\|f_{\text{in}}\|_{\mathbf{G}} \ll 1$, then strong limit
- Gallagher-Tristani (2019)
  - Well-prepared part of $f_{\text{in}} \rightarrow$ strong f^0 solution of (INSF) on $[0, T]$
  - Write equation on $f^\varepsilon - f^0 -$ ac. waves, fixed point, then limit

Construction of solutions and convergence

Initial data with Gaussian decay (Bardos–Ukai/Gallagher–Tristani)

Reminder

- Mass density : $\displaystyle\int F^\varepsilon \, \mathrm{d}v$
- Energy : $\frac{1}{2}R^\varepsilon |U|^2 + \frac{d}{2}R^\varepsilon T^\varepsilon = \displaystyle\int F^\varepsilon \frac{|v|^2}{2} \mathrm{d}v$

Question: Can we only assume $f_{\text{in}} \in [\dots]_x L^1_v \left(\langle v \rangle^2 \mathrm{d}v \right)$?

Construction of solutions and convergence

Initial data with polynomial decay

Theorem (G. 2021)

Let $s > \frac{d}{2}$, $k > 3$, $f_{\text{in}} \in L^1_v H^s_x\left(\langle v\rangle^k\right)$, there exists $T \in (0,\infty]$ s.t.

- for $\varepsilon \ll 1$, the equation (B^ε) has a a **unique solution**

$$f^\varepsilon \in \mathcal{C}_b\left([0,T); L^1_v H^s_x\left(\langle v\rangle^{k+1}\right)\right) \\ \cap L^1\left([0,T); L^1_v H^s_x\left(\langle v\rangle^{k+1}\right)\right)$$

- $f^\varepsilon = f^0 + u^\varepsilon_{\text{ac}} + u^\varepsilon_1 + u^\varepsilon_\infty$, where f^0 is **the strong solution** to (INSF) generated by the well-prepared part of f_{in},

$$u^\varepsilon_1(t) = O(e^{-\lambda t/\varepsilon^2}),\ u^\varepsilon_\infty(t) = o(1),\ u^\varepsilon_{\text{ac}} \rightharpoonup 0,$$

- macroscopic part of f_{in} **well-prepared** $\Rightarrow u^\varepsilon_w = 0$
- f_{in} **purely macroscopic** (micro. part = 0) $\Rightarrow u^\varepsilon_1 = 0$

Construction of solutions and convergence

Initial data with polynomial decay

Functional space: $\mathbf{P} := L^p_v H^s_x \left(\langle v\rangle^{\beta} \mathrm{d}v\right)$

- C. Mouhot (2005): Enlargement Theory
- M.P. Gualdani, S. Mischler, C. Mouhot (2017): strong solution for (B^ε) when $\varepsilon = 1$ and $\|f_{\text{in}}\|_{\mathbf{P}} \ll 1$
- M. Briant, S. Merino, C. Mouhot (2019): weak hydrodynamic limit
  - write $f^\varepsilon = g^\varepsilon + h^\varepsilon \in \mathbf{G} + \mathbf{P} \to$ coupled system
  - uniform estimates on h^ε and g^ε

Proof of the theorem

Strategy

Grad's decomposition: $\mathcal{L} = -\nu(v) + K$

$$\nu_0\langle v\rangle \leq \nu(v) \leq \nu_1\langle v\rangle, \; K \to \text{moment gain}$$

GMM decomposition: $\mathcal{L} = \mathcal{B} + \mathcal{A}$

$$\mathcal{B} = -\nu + \text{perturbation}, \; \mathcal{A} \; : \; \mathbf{P} \xrightarrow{\text{bounded}} \mathbf{G}$$

- Split $f^\varepsilon = h^\varepsilon + g^\varepsilon$ in the way of Briant-Merino-Mouhot
  - h^ε satisfies nice equation
  - Build g^ε close to f^0 =solution to (INSF) on $[0,T)$ in the way of Gallagher-Tristani

$$\mathcal{A}f(v) := \int \Theta\left(M'_* f' + M' f'_* - M f_*\right)|v - v_*|\mathrm{d}v_*\mathrm{d}\sigma,$$

$$\Theta \in \mathcal{C}_c^\infty$$

Proof of the theorem

Splitting of the equation

- Use the GMM splitting $\mathcal{L} = \mathcal{B} + \mathcal{A}$

$$\mathcal{B}h \approx -(1+|v|)h, \quad \mathcal{A} : \mathbf{P} \xrightarrow{\text{bded.}} \mathbf{G}$$

- Write $f^\varepsilon = h^\varepsilon + g^\varepsilon \in \mathbf{P} + \mathbf{G}$

$$\partial_t f^\varepsilon = \frac{1}{\varepsilon^2}(\mathcal{L} + \varepsilon v \cdot \nabla_x) f^\varepsilon + \frac{1}{\varepsilon} Q(f^\varepsilon, f^\varepsilon),$$

$$\Uparrow$$

$$\mathbf{P} \ : \ \partial_t h^\varepsilon = \frac{1}{\varepsilon^2}\left(\mathcal{B} + \varepsilon v \cdot \nabla_x\right) h^\varepsilon + \frac{1}{\varepsilon} Q(h^\varepsilon, h^\varepsilon + 2g^\varepsilon),$$

$$\mathbf{G} \ : \ \partial_t g^\varepsilon = \frac{1}{\varepsilon^2}\left(\mathcal{L} + \varepsilon v \cdot \nabla_x\right) g^\varepsilon \underbrace{+ \frac{1}{\varepsilon^2} \mathcal{A} h^\varepsilon}_{\in \mathbf{G}} + \frac{1}{\varepsilon} Q(g^\varepsilon, g^\varepsilon),$$

$$(h^\varepsilon, g^\varepsilon)_{|t=0} = (f_{\text{in,mic}}, f_{\text{in,mac}}) \in \mathbf{P} \times \mathbf{G}$$

Proof of the theorem

control of the polynomial part

$$\partial_t h^\varepsilon = \frac{1}{\varepsilon^2}\left(\mathcal{B} + \varepsilon v \cdot \nabla_x\right) h^\varepsilon + \frac{1}{\varepsilon} Q(h^\varepsilon, h^\varepsilon + 2g^\varepsilon),$$

- Energy estimate:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\|h^\varepsilon(t)\|_{\mathbf{P}} \leq &-\frac{\Lambda}{\varepsilon^2}\|\langle v\rangle h^\varepsilon(t)\|_{\mathbf{P}} \\ &+ \frac{M}{\varepsilon}\|\langle v\rangle h^\varepsilon(t)\|_{\mathbf{P}}\|h^\varepsilon(t)\|_{\mathbf{P}} + (\dots)\end{aligned}$$

- Grönwall for some $0 < \lambda < \Lambda$:

$$\begin{aligned}\sup_{0\leq t<T}&\left(e^{\lambda t/\varepsilon^2}\|h^\varepsilon(t)\|_{\mathbf{P}} + \frac{\Lambda-\lambda}{\varepsilon^2}\int_0^t e^{\lambda t'/\varepsilon^2}\|\langle v\rangle h^\varepsilon(t')\|_{\mathbf{P}}\,\mathrm{d}t'\right) \\ &=: \|\!|h^\varepsilon|\!\|_{\mathbf{P}^\varepsilon} \leq C\varepsilon \|\!|h^\varepsilon|\!\|_{\mathbf{P}^\varepsilon}\left(\|\!|h^\varepsilon|\!\|_{\mathbf{P}^\varepsilon} + \|g^\varepsilon\|_{L^\infty_t\mathbf{G}}\right) + \|f_{\mathrm{in,mic}}\|_{\mathbf{P}}\end{aligned}$$

Proof of the theorem

Study of the Gaussian part

- Duhamel formulation:

$$g^\varepsilon(t) = U^\varepsilon(t) f_{\mathrm{in,mac}} + \Psi^\varepsilon(t)(g^\varepsilon, g^\varepsilon) + \frac{1}{\varepsilon^2} U^\varepsilon * \mathcal{A} h^\varepsilon(t),$$

$$\frac{1}{\varepsilon^2} U^\varepsilon * \mathcal{A} h^\varepsilon(t) := \frac{1}{\varepsilon^2} \int_0^t U^\varepsilon(t-t') \mathcal{A} h^\varepsilon(t') \mathrm{d}t',$$

- Usual Duhamel form of (B^ε) but $\|h^\varepsilon(t)\| \lesssim e^{-\lambda t/\varepsilon^2}$
$\rightarrow$ convolution bounded but not small

$$U^\varepsilon(t) := \exp\left(\frac{1}{\varepsilon^2}\left(\mathcal{L} + \varepsilon v \cdot \nabla_x\right)\right),$$

$$\Psi^\varepsilon(t)(f^\varepsilon, f^\varepsilon) := \frac{1}{\varepsilon} \int_0^t U^\varepsilon(t-t') Q\left(f^\varepsilon(t'), f^\varepsilon(t')\right)$$

Proof of the theorem

Study of the Gaussian part

Lemma (G. 21)

Uniformly in t and ε,

$$\frac{1}{\varepsilon^2} U^\varepsilon * \mathcal{A} h^\varepsilon(t) = U^\varepsilon(t) f_{\text{in,mic}} + O(\varepsilon) + O\left(e^{-\lambda t/\varepsilon^2}\right)$$
$$= o(1) + O\left(e^{-\lambda t/\varepsilon^2}\right)$$

Proof: Denote $V^\varepsilon(t) := \exp\left(\frac{t}{\varepsilon^2}\left(\mathcal{B} + \varepsilon v \cdot \nabla_x\right)\right)$

$$\text{Duhamel} \to \begin{cases} U^\varepsilon = V^\varepsilon + \frac{1}{\varepsilon^2} U^\varepsilon \mathcal{A} * V^\varepsilon, \\ h^\varepsilon = V^\varepsilon f_{\text{in,mic}} + \frac{1}{\varepsilon} V^\varepsilon * Q(h^\varepsilon, h^\varepsilon + 2g^\varepsilon) \end{cases}$$

$$\to \frac{1}{\varepsilon^2} U^\varepsilon * \mathcal{A} h^\varepsilon(t) = U^\varepsilon(t) f_{\text{in,mic}} + \text{(bi)linear in } \frac{h^\varepsilon}{\varepsilon}$$

$$\xrightarrow[\text{a priori bound on } h^\varepsilon]{\text{spectral study}} o(1) + O\left(e^{-\lambda t/\varepsilon^2}\right)$$

Proof of the theorem

Study of the Gaussian part

- New unknown $\overline{g}^\varepsilon := g^\varepsilon - f^0 - O\left(e^{-\lambda t/\varepsilon^2}\right) -$ aco. waves

$$g^\varepsilon = U^\varepsilon f_{\text{in,mac}} + \Psi^\varepsilon(g^\varepsilon, g^\varepsilon) + \frac{1}{\varepsilon^2} U^\varepsilon * \mathcal{A} h^\varepsilon,$$

$$\downarrow$$

$$\overline{g}^\varepsilon = o(1) + \underbrace{\{\text{Linear}\}}_{\text{contraction}}(\overline{g}^\varepsilon) + \underbrace{\{\text{Bilinear}\}}_{\text{bounded}}(\overline{g}^\varepsilon, \overline{g}^\varepsilon),$$

- {Linear} and {Bilinear} depend on f^0
  $\to$ use norm equivalent to $\|\cdot\|_{L^\infty \mathbf{G}}$ $\to$ {Linear} is a contraction
- ... **and on h^ε** $\to$ generalize some estimates/convergence on U^ε and Ψ^ε to $\mathbf{P}$.
  - Factorization techniques using $\mathcal{L} = \mathcal{B} + \mathcal{A}$
  - Estimates/convergence in $\mathbf{G}$ $\to$ Estimates/convergence in $\mathbf{P}$

Thank you for your attention!