Midterm 2

October 30th

- To do a later question in a problem, you can always assume a previous question even if you have not answered it.
- I am aware that this is long. I don't expect you to do everything.
- There are 2 class material questions (in Problem 1) and 2 independent problems. You don't have to do them in any particular order. Question 3.2 is harder, you should probably leave it for the end.
- Using a pen and writing clearly makes it easier for me to grade.

Problem 1 :
Let R be a ring and $I, J \subseteq R$ be two ideals. Using only the definitions and nothing we have proved in class:

1. Show that $I \cap J$ is an ideal of R.

2. Assume R to be a commutative ring and I, J comaximal. Show that $I \cdot J = I \cap J$.

Problem 2 :
Let R be an integral domain.

1. Let $\varphi : R \to S$ be a ring homomorphism. We define $\psi : R[X] \to S[X]$ by $\psi(\sum_{i=0}^{n} a_i X^i) = \sum_{i=0}^{n} \varphi(a_i) X^i$. Show that ψ is a ring homomorphism.

2. For all $P = \sum_{i=0}^{n} a_i X^i \in R[X] \setminus \{0\}$, we define $v(P) = \min\{i : a_i \neq 0\}$. Show that, for all $P, Q \in R[X] \setminus \{0\}$, $v(P \cdot Q) = v(P) + v(Q)$.

3. Let $P, Q \in R[X]$ be such that $P \cdot Q = aX^n$ for some $a \in R \setminus \{0\}$ and $n \in \mathbb{Z}_{\geqslant 0}$. Show that there exists $r, s \in R$ and $i, j \in \mathbb{Z}_{\geqslant 0}$ such that $P = rX^i$ and $Q = sX^j$.

Problem 3 :
Let G be a finite group.

1. For all $x \in G$ of order n, show that the action of $\langle x \rangle$ on G by multiplication on the left — i.e. $x^i \star g = x^i \cdot g$ — has $|G|/|x|$ orbits and they are all of size $|x|$.

2. Let $f : G \to \{\mathbb{Z}/2\mathbb{Z}\}$ be such that $f(x) = \overline{1}$ if and only if $|x|$ is even and $|G|/|x|$ is odd. Show that f is a group homomorphism.

   Hint: Think about the signature of a permutation.

3. Assume $|G| = 2n$ where n is odd. Show that there exists a normal subgroup of G of index 2.