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Homework 1

Due September 6th

Problem 1 (Equivalence relations) **:**
Let $f : X \to Y$ be a function and let $x_1 \sim x_2$ hold if $f(x_1) = f(x_2)$.

1. Show that $\sim$ is an equivalence relation on X.
2. Assume that f is surjective. Show that there exists a bijection $g : Y \to \{\overline{x} : x \in X\}$, where $\overline{x}$ denotes the $\sim$-class of x.

Problem 2 :

1. Which are the $x \in \mathbb{Z}$ such that there exists $y \in \mathbb{Z}$ with $x \equiv y^2 \mod 9$.
2. Which are the $x \in \mathbb{Z}$ such that there exists $y, z \in \mathbb{Z}$ with $x \equiv y^2 + z^2 \mod 9$.
3. Show that if $x, y, z \in \mathbb{Z}$ are such that $x^2 + y^2 \equiv 12 \cdot z^2 \mod 9$, then $x \equiv y \equiv z \equiv 0 \mod 3$.
4. Show that if there exists $x, y, z \in \mathbb{Z}_{>0}$ such that $x^2 + y^2 = 12 \cdot z^2$ then there exists $x', y', z' \in \mathbb{Z}_{>0}$ such that $(x')^2 + (y')^2 = 12 \cdot (z')^2$, $x' < x$, $y' < y$ and $z' < z$.
5. Conclude that if $x, y, z \in \mathbb{Z}$ are such that $x^2 + y^2 = 12 \cdot z^2$ then they are all equal to 0.

Problem 3 :
Let G be a non empty finite set and $\cdot$ a binary operation on G such that:

- The operation $\cdot$ is associative;
- For all x, y and $z \in G$ if $x \cdot y = x \cdot z$ then $y = z$ and if $y \cdot x = z \cdot x$ then $y = z$.

1. Show that there exists $e \in G$ such that for all $x \in G$, $e \cdot x = x$.

   (*Hint:* Show that for some $a \in G$, there exists e such that $e \cdot a = a$ and that any $x \in G$ can be written as $a \cdot y$ for some $y \in G$.)
2. Show that we also have $x \cdot e = x$ for all $x \in G$.

   (*Hint:* Show that there exists e' such that $x \cdot e' = x$ for all $x \in G$ and that $e' = e$).
3. Show that $(G, \cdot)$ is a group.