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Homework 4

Due October 2nd

The questions indicated as Harder will not be taken in account when grading.

Problem 1 :
Let $G \leqslant \mathbb{R}$.

1. Assume that for all $b \in \mathbb{R}_{>0}$, there exists $g \in G$ such that $0 < g < b$. Show that for all $x, y \in \mathbb{R}$ such that $x < y$, there is $g \in G$ such that $x < g < y$.
2. (Harder) If $a := \inf\{g \in G : g > 0\} \neq 0$, show that $G = a\mathbb{Z}$.

Problem 2 :

1. Show that $x \mapsto e^{2i\pi x}$ is group homomorphism from $\mathbb{R}$ to $\mathbb{C}^\star$.
2. Let $\mathbb{T} = \{x \in \mathbb{C} : |x| = 1\}$ (here $|x|$ denotes the absolute value). Show that $\mathbb{R}/\mathbb{Z} \cong \mathbb{T}$.
3. Show that in $\mathbb{Q}/\mathbb{Z} \leqslant \mathbb{R}/\mathbb{Z}$ all elements have finite order but the order can be arbitrarily large.
4. Show that $\mathbb{Q}/\mathbb{Z} \cong \mu_\infty = \{x \in \mathbb{C} : x^n = 1 \text{ for some } n \in \mathbb{Z}_{>0}\} \leqslant \mathbb{T}$.

Note that the group μ_∞ was called Z in the previous homework.

Problem 3 :
Let G be a group and $H \leqslant G$ such that $[G : H] = n < \infty$

1. Assume that $H \trianglelefteq G$, show that for all $g \in G$, $g^n \in H$.
2. (Harder) Find a counterexample when H is not normal.

Problem 4 :
Let $n \in \mathbb{Z}$, $n \geqslant 3$ and $d|n$. Let r denote one of the rotations in D_{2n} and $H = \langle r^d \rangle$.

1. Show that $H \trianglelefteq D_{2n}$.
2. If $d = 1$, show that $D_{2n}/H \cong \mathbb{Z}/2\mathbb{Z}$.
3. If $d = 2$ (in particular, n has to be even), $D_{2n}/H \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$.
4. If $d > 2$, $D_{2n}/H \cong D_{2d}$.