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Homework 7

Due October 23rd

Problem 1 (Action of the dihedral group on the diagonals) **:**
Let $n \geqslant 4$ be an even positive integer. Let us number the vertices of the n-gon by $\mathbb{Z}/n\mathbb{Z}$ (clockwise for example). And let $X = \{\{i,j\} : i \text{ and } j \text{ number opposite vertices}\}$. So X is the set of diagonals of the n-gon. Let $r \in D_{2n}$ be the rotation sending the vertex 0 to the vertex 1 and s be the symmetry that fixes the vertex 0.

1. Let $\sigma \in D_{2n}$ and $\{i,j\} \in X$, show that $\{\sigma(i), \sigma(j)\} \in X$ where the action of D_{2n} on the vertices is the usual one.
2. Show that $\sigma \star \{i,j\} = \{\sigma(i), \sigma(j)\}$ is an action of D_{2n} on X.
3. Show that this action has a unique orbit.
4. Show that $\mathrm{Stab}_{D_{2n}}(\{i,j\}) = \{1, r^{n/2}, r^{2i}s, r^{2i+n/2}s\}$.
5. Let $n = 4$, show that $\mathrm{Stab}_{D_8}(X) = \{1, r^2, s, r^2 s\}$.
6. Let $n > 4$, show that $\mathrm{Stab}_{D_{2n}}(X) = \{1, r^{n/2}\}$.

Problem 2 :
Let K be a field. We define $K[[X]] = \{\sum_{i\in\mathbb{Z}_{\geqslant 0}} a_i X^i : a_i \in K\}$ the set of formal power series with coefficients in K. The main difference with polynomials is that we now allow infinitely many coefficients to be non zero. We define addition as follows $\sum_i a_i X^i + \sum_i b_i X^i = \sum_i (a_i + b_i) X^i$ and multiplication as follows $(\sum_i a_i X^i) \cdot (\sum_i b_i X^i) = \sum_k (\sum_{i=0}^k a_i b_{k-i}) X^k$.

1. Show that $(K[[X]], +, \cdot)$ is a commutative ring.
2. Show that $S = \sum_i s_i X^i$ is a unit if and only if $s_0 \neq 0$.
3. Show that $K[[X]]/(X)$ is isomorphic to K.
4. Show that every non zero ideal in $K[[X]]$ is of the form (X^n) for some $n \in \mathbb{Z}_{\geqslant 0}$.