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Solutions to homework 8

Due November 8th

Problem 1 (nilpotent elements and radical ideals) **:**
Let R be a unitary commutative ring. An element $x \in R$ is said to be nilpotent if there exists $n \in \mathbb{Z}_{>0}$ such that $x^n = 0$.

1. What are the nilpotent elements in $\mathbb{Z}/36\mathbb{Z}$?

   Solution: We have $36 = 2^2 \cdot 3^2$. Let $x \in \mathbb{Z}$, $\overline{x}$ is nilpotent in $\mathbb{Z}/36\mathbb{Z}$ if for n big enough $36|x^n$. So x^n is divisble by 2 and 3 and because 2 and 3 are prime, x itself is divisible by 6. Conversely, if x is divisible by 6, x^2 is divisible by 36 and $\overline{x}$ is nilpotent in $\mathbb{Z} \mod 36\mathbb{Z}$. So the nilpotent elements in $\mathbb{Z}/36\mathbb{Z}$ are $\overline{0}$, $\overline{6}$, $\overline{12}$, $\overline{18}$, $\overline{24}$ and $\overline{30}$.

2. Show that $\{x \in R : x \text{ nilpotent}\}$ is an ideal. It is called the nilradical of R.

   Solution: Let x, $y \in R$ be nilpotent and let n, $m \in \mathbb{Z}_{>0}$ be such that $x^n = 0 = y^m$. Then $(x+y)^{m+n} = \sum_{i+j=m+n} \binom{i+j}{i} x^i y^j$. But if $i + j = m + n$ and both i, $j \geqslant 0$, then either $i \geqslant m$ or $j \geqslant n$. It follows that $x^i y^j = 0$ and $(x+y)^{m+n} = 0$, so $x + y$ is nilpotent. Now pick $a \in R$. $(ax)^n = a^n x^n = 0$. It follows that the set of nilpotent elements in R is an ideal.

3. Assume that x is nilpotent, show that $1 - x$ is a unit.

   Solution: Let $n \in \mathbb{Z}_{>0}$ be such that $x^n = 0$. We have $1 = 1 - x^n = (1-x)(\sum_{i=0}^{n-1} x^i)$ so $1 - x$ is a unit in R.

4. Assume that x is nilpotent, show that for all $u \in R^\star$, $u + x$ is a unit.

   Solution: Because the set of nilpotent elements is an ideal, $-u^{-1}x$ is nilpotent. By the previous question $1 + u^{-1}x$ is a unit and hence so is $u(1 + u^{-1}x) = u + x$.

5. (Harder) Let $S \subseteq R \smallsetminus \{0\}$ be closed under multiplication. Show that there exists a prime ideal $\mathfrak{p} \subseteq R$ such that $\mathfrak{p} \cap S = \varnothing$.

   Solution: Adding it, we may assume that S contains 1. The ring R_S is not the trivial ring (because $0 \notin S$) so there exists a maximal (and hence prime) ideal $\mathfrak{q} \subseteq R_S$. Let $\mathfrak{p} = \mathfrak{q} \cap R$ (where R is identified with a subring of R_S in the usual way). Then $\mathfrak{p}$ is a prime ideal (you can check it by hand or apply question 9 to $\mathfrak{q}$ and the inclusion morphism $R \to R_S$). Moreover, if $x \in \mathfrak{p} \cap S$, then $x \in (R_S)^\star \cap \mathfrak{q}$ and $\mathfrak{q} = R_S$ contradicting the fact that it is a proper ideal. So $\mathfrak{p} \cap S = \varnothing$.

6. Let $x \in R$ not be nilpotent. Show that there exists a prime ideal $\mathfrak{p} \subseteq R$ such that $x \notin \mathfrak{p}$.

   Hint: Use the previous question!

   Solution: Let $S = \{x^n : n \in \mathbb{Z}_{\leqslant 0}\}$, then S is closed under multiplication and does not contain 0. So, by the previous question, there exist a prime ideal $\mathfrak{p} \subseteq R$ such that $S \cap \mathfrak{p} = \varnothing$. In particular, $x \notin \mathfrak{p}$.

7. Let N be the nilradical of R, show that

$$N = \bigcap_{\mathfrak{p} \subseteq R \text{ prime}} \mathfrak{p}.$$

Solution: Let $\mathfrak{p} \subseteq R$ be a prime ideal. Let us show by induction on $n > 0$ that if $x^n \in \mathfrak{p}$ then $x \in \mathfrak{p}$. If $n = 1$, this is obvious. Now assume it is true for n and $x^{n+1} = xx^n \in \mathfrak{p}$. Then, because $\mathfrak{p}$ is prime, we either have $x \in \mathfrak{p}$ or $x^n \in \mathfrak{p}$ in which case, by induction, $x \in \mathfrak{p}$.

If follows that if n is nilpotent, $x^n = 0 \in \mathfrak{p}$ for some n, so $x \in \mathfrak{p}$. We have shown that $N \subseteq \mathfrak{p}$ for all prime ideal $\mathfrak{p} \subseteq R$. Therefore, $N \subseteq \bigcap_{\mathfrak{p} \subseteq R \text{ prime}} \mathfrak{p}$. Conversely, let x not be nilpotent. By the previous question, there exists a prime $\mathfrak{p} \subseteq R$ such that $x \notin \mathfrak{p}$, so $x \notin \bigcap_{\mathfrak{p} \subseteq R \text{ prime}} \mathfrak{p}$. So $N = \bigcap_{\mathfrak{p} \subseteq R \text{ prime}} \mathfrak{p}$.

8. Let $I \subseteq R$ be an ideal. We define $\sqrt{I} := \{x \in R : x^n \in \text{ for some } n \in \mathbb{Z}_{>0}\}$. Show that $\sqrt{I} \subseteq R$ is an ideal.

Solution: Let x, $y \in R$ and n, $m \in \mathbb{Z}_{>0}$ such that x^n, $y^m \in I$. Then $(x+y)^{m+n} = \sum_{i+j=m+n} \binom{i+j}{i} x^i y^j$. But if $i + j = m + n$ and both i, $j \geqslant 0$, then either $i \geqslant m$ or $j \geqslant n$. It follows that $x^i y^j \in I$ and $(x+y)^{m+n} \in I$, so $x + y \in \sqrt{I}$. Now pick $a \in R$. $(ax)^n = a^n x^n \in I$. It follows that $\sqrt{I}$ is an ideal.

9. Let $f : R \to S$ be a unitary ring homomorphism, $\mathfrak{p} \subseteq S$ be a prime ideal. Show that $f^{-1}(\mathfrak{p}) \subseteq R$ is a prime ideal.

Solution: We know that $f^{-1}(\mathfrak{p}) \subseteq R$ is an ideal. Because f is unitary, $f(1) = 1 \notin \mathfrak{p}$ so $f^{-1}(\mathfrak{p})$ is a proper ideal. Let us show that it is prime. Let x, $y \in R$ be such that $xy \in f^{-1}(\mathfrak{p})$, i.e. $f(xy) = f(x)f(y) \in \mathfrak{p}$. Because $\mathfrak{p}$ is prime, it follows that either $f(x) \in \mathfrak{p}$ (and $x \in f^{-1}(\mathfrak{p})$) of $f(y) \in \mathfrak{p}$ (and $y \in f^{-1}(\mathfrak{p})$).

10. Let $I \subseteq R$ be an ideal and $\pi : R \to R/I$ be the canonical projection. Let $N_I \subseteq R/I$ be its nilradical. Show that $\sqrt{I} = \pi^{-1}(N_I)$.

Solution: We have $x \in \sqrt{I}$ if and only if $x^n \in I$ for some I if and only if $\pi(x)^n = \pi(x^n) = x^n + I = I$ if and only if $\pi(x) \in N_I$ if and only if $x \in \pi^{-1}(N_I)$.

11. Let $I \subseteq R$ be an ideal. Show that

$$\sqrt{I} = \bigcap_{\substack{I \subseteq \mathfrak{p} \subseteq R \\ \mathfrak{p} \text{ prime}}} \mathfrak{p}.$$

Solution: Let $\mathfrak{p} \subseteq R$ be a prime ideal containing and $x \in R$ be such that $x^n \in I \subseteq \mathfrak{p}$. By the computation in question 7, we have $x \in \mathfrak{p}$. There only remains to show that if $x \notin \sqrt{I}$, there exists a prime ideal containing I that does not contain x. By the previous question, $\pi(x) \notin N_I$. So, by question 7, there exists a prime $\mathfrak{q} \subseteq R/I$ such that $\pi(x) \notin \mathfrak{q}$. Let $\mathfrak{p} = \pi^{-1}(\mathfrak{q})$. Then, by question 9, $\mathfrak{p}$ is a prime ideal. Also, $I = \pi^{-1}(0) \subseteq \pi^{-1}(\mathfrak{q}) = \mathfrak{p}$ and $x \notin \mathfrak{p}$.