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# Homework 9

Due November 15th

The questions indicated as (Harder) are optional and will not be taken in account in the grade.

**Problem 1 :**
Let $D \in \mathbb{Z}$ and let $\alpha \in \mathbb{C}$ be such that $\alpha^2 = D$.

1. Show that $\mathbb{Z}[\alpha] = \{a + b\alpha : a, b \in \mathbb{Z}\}$ is a subring of $\mathbb{C}$.
2. If $D = 1 \mod 4$, show that $\mathbb{Z}[\frac{1+\alpha}{2}] = \{a + b\frac{1+\alpha}{2} : a, b \in \mathbb{Z}\}$ is a subring of $\mathbb{C}$ that contains $\mathbb{Z}[\alpha]$.
3. Let $\beta = \frac{1+\alpha}{2}$ and $\overline{\beta} = \frac{1-\alpha}{2}$ if $D = 1 \mod 4$ and $\beta = \alpha$, $\overline{\beta} = -\alpha$ otherwise. We define $N(a + \beta b) = (a + \beta b) \cdot (a + \overline{\beta} b)$. Show that for all $x \in \mathbb{Z}[\beta]$, $N(x) \in \mathbb{Z}$ and if $D < 0$, then $N(x) = |x|^2 \geqslant 0$ where $|x|$ is the complex norm.
4. Show that for all $x$, $y \in \mathbb{Z}[\beta]$, $N(xy) = N(x)N(y)$.
5. Show that $x \in \mathbb{Z}[\beta]$ is a unit if and only if $N(x) \in \{1, -1\}$.
6. Let us now assume that $D = -3$, show that for all $x \in \mathbb{C}$ there exists $a \in \mathbb{Z}[\beta]$ such that $|x - a| < 1$.
7. Let us still assume that $D = -3$, show that $\mathbb{Z}[\beta]$ is Euclidian (with respect to $N$).
8. (Harder) Assume $D < 0$ and $D = 1 \mod 4$, show that for all $x \in \mathbb{C}$ there exists $a \in \mathbb{Z}[\beta]$ such that $|x - a| \leqslant \frac{1+|D|}{4\sqrt{|D|}}$. Conclude that, if $D \in \{-3, -7, -11\}$, $\mathbb{Z}[\beta]$ is Euclidian with respect to $N$.

**Problem 2 :**
Let $R$ be an integral domain. We say that $R$ is Bezout domain if for all $a, b \in R$, there exists $c \in R$ such that $(a, b) = (c)$.

1. Show that the following are equivalent:
   a) $R$ is a Bezout domain;
   b) Every finitely generated ideal of $R$ is principal;
   c) For every $a, b \in R$, a greatest common divisor $d$ of $a$ and $b$ exists and there exists $u, v \in R$ such that $d = ua + bv$.
2. Assume that $R$ is a UFD and a Bezout domain. Let $I \subset R$ be a proper ideal and let $a \in I$ be such that the factorization of $a$ into irreducibles has a minimal number of factors, among elements of $I \smallsetminus \{0\}$. Show that $I = (a)$.
3. Show that $R$ is a PID if and only if it is both a UFD and a Bezout domain.