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Homework 9

Due November 19th

Problem 1 :

1. Recall that a finite Boolean algebra is atomic. Hence, if $\bigcap_{a\in\mathcal{A}} a^{\mathbf{c}} \neq 0$, there exists a atome $b \in \mathcal{A}$ such that $b \leqslant \bigcap_{a\in\mathcal{A}} a^{\mathbf{c}}$. In particular $b \leqslant b^{\mathbf{c}}$, but that is absurd as $b \neq 0$.
2. Recall that an ultrafilter is principal if and only if it contains a atome. Let $F \in \mathcal{S}(A)$ be non principal, then F does not contain any atom. Becaus F is an ultrafilter, it must contain all $a^{\mathbf{c}}$ for $a \in \mathcal{A}$ and hence it must contain $\bigcap_{a\in\mathcal{A}} a^{\mathbf{c}} = 0$, a contradiction.
3. We also showed in class that $F = U_a$ is principal if and only if a is an atom. Let us show that $V_a = \{F\}$ (and hence $\{F\}$ is open). First, we obviously have $F \in V_a$ as $a \in U_a = F$. Now let $U \in V_a$. Then, by defintion, $a \in U$ and hence U is principal. Let b be such that $U = U_b$. As $a \in U_b$, we have $b \leqslant a$, and as both are atoms, $a = b$. So $U = U_a = F$.
4. We have $X = \bigcup_{F\in X}\{F\}$ and each of the $\{F\}$ is open (as every ultrafilter over A is principal), so X itself is open.
5. By Stone duality, we know that A is isomorphic to $\mathcal{B}(\mathcal{S}(A))$. But we have just show that every subset of $\mathcal{S}(A)$ is both open and closed, so $\mathcal{B}(\mathcal{S}(A)) = \mathcal{P}(\mathcal{S}(A))$. Moreover, the map $\mathcal{A} \to \mathcal{S}(A)$ that sends a to U_a is surjective by the above questions. It is obviously bijective as $b \in U_a$ if and only if $b = a$ and hence $\mathcal{S}(A)$ is in bijection with $\mathcal{A}$. One can easily check that this bijection extends to a morphism of Boolean algebras btetween $\mathcal{P}(\mathcal{S}(A))$ and $\mathcal{P}(\mathcal{A})$. Composition those two iso;orphism, we get an isomorphism between A and $\mathcal{P}(\mathcal{A})$.

Problem 2 :

1. First, as in $\mathcal{P}(E)$, and for the same reason, the atoms in $\mathcal{F}(E)$ are exactly the singletons. Let F be a non principal ultrafilter on $\mathcal{F}(E)$, then F does not contain any singleton and hence F contains all the conplements of singletons. Let $X \subseteq E$ be cofinite, then $X = \bigcup_{x\in X^{\mathbf{c}}} {}^{\mathbf{c}}\{x\}$ and thus $X \in F$ So F contains the Fréchet filter (which is non principal as it does not contain any atom, i.e. singletons). Moreover, let $X \in \mathcal{F}(E)$, the either X is cofinite and $X \in F$ or X is finite and $X^{\mathbf{c}} \in F$. It follows that $\mathfrak{F}$ is an ultrafilter and hence $F = \mathfrak{F}$.
2. Let $F \in \mathcal{S}(E)$ be an ultrafilter. The either F is non principal and hence $F = \mathfrak{F}$ by the previous question, or F is principal and $F = U_a$ for some atome a. But the atoms of $\mathcal{F}(E)$ are the singletons and hence $F = U_{\{e\}}$ for some $e \in E$.
3. First of all, by definition of $\mathfrak{F}$, $\mathfrak{F} \in V_X$ if and only $X \in \mathfrak{F}$ if and only if X is cofinite.

   Now let us prove that if X is finite then $V_X = \{U_x : x \in X\}$. Indeed, for all $x \in X$, $X \in U_{\{x\}}$ and hence $\{U_x : x \in X\} \subseteq V_X$. Now let $F \in V_X$, then $X \in F$ and hence $F \neq \mathfrak{F}$. It follows that $F = U_{\{x\}}$ for some $x \in E$. But we have $X \in U_{\{x\}}$ and hence $x \in X$, so $V_X \subseteq \{U_x : x \in X\}$. In particular, if X is finite, then V_X is finite. We have just proved, the contropsition of c) implies a).

The implication b) imples c) is trivial, so there only remains to show that a) implies b). Let us assume that X is cofnite. Then $X^{\mathbf{c}}$ is finite and hence $V_{X^{\mathbf{c}}}$ is finite. Thus $V_X = V_{X^{\mathbf{c}}}^{\mathbf{c}}$ is cofinite.

4. Let $V \subseteq \mathcal{S}(\mathcal{F}(E))$ be open. Then $V = \bigcup_i V_{X_i}$ for some $X_i \in \mathcal{F}(E)$. If all of the X_i are finite then V does not contain $\mathfrak{F}$. If X_{i_0} is cofinite, then V contains $\mathfrak{F}$ and $V_{i_0} \subseteq V$ and V must also be cofinite.

   Conversely, assume V does not contains $\mathfrak{F}$. Let $X = \{x \in E : U_{\{x\}} \in V\}$. As $\{x\}$ is an atom, $V_{\{x\}} = \{U_{\{x\}}\}$ and $V = \bigcup_{x \in X} V_{\{x\}}$ is open. If V is cofinite and contains $\mathfrak{F}$, then let $V^{\mathbf{c}}$ is finite and does not contain $\mathfrak{F}$. Let $X = \{x \in E : U_{\{x\}} \notin V\}$. As above, $V^{\mathbf{c}} = \bigcup_{x \in X} V_{\{x\}}$, but this union is finite and $V_{\{x\}}$ is closed, so $V^{\mathbf{c}}$ is closed and V is open.