Propositional logic: Formal proofs and completeness

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So far, the only way we were able to assert that some formula is a tautology or that some formula is a consequence of other formulas was purely semantical and was done using assignements. But when we do mathematics, we prove statements by formally deducing them from others using the rules of deduction. So if we want to give a complete account of (propositional) logic, we might want to also formalize this formal deduction process.
For example, if we know that φ and $\varphi \to \psi$ are consequences of Γ then we want to say that ψ is also a consequence of Γ. Similarly, if we show that ψ is a consequence of $\Gamma \cup \varphi$, then we know that $\varphi \to \psi$ is a consequence of Γ too. Note that if, in the previous sentences, consequence is meant is the semantic sense, then we can prove these two statements are indeed correct. But that is not what we are looking for. What we want is to define an alternative notion of consequence that relies solely on these deduction rules. Therefore we wish to define the set of pairs (Γ, φ) where $\Gamma \subseteq F$ and $\varphi \in F$ such that "φ is a formal consequence of Γ".

Definition 9.1:
Let $T \subseteq \mathfrak{P}(F) \times F$ be the smallest set such that for all Γ, Γ_1, Γ_2, $\Gamma_3 \subseteq F$ and φ, φ_1, φ_2, $\psi \in F$:

(i) *if $\varphi \in \Gamma$, $(\Gamma, \varphi) \in T$;*

(ii) *if $(\Gamma \cup \varphi, \psi) \in T$ then $(\Gamma, \varphi \to \psi) \in T$;*

(iii) *if $(\Gamma_1, \varphi \to \psi)$ and $(\Gamma_2, \varphi) \in T$ then $(\Gamma_1 \cup \Gamma_2, \psi) \in T$;*

(iv) *if (Γ_1, φ_1) and $(\Gamma_2, \varphi_2) \in T$ then $(\Gamma_1 \cup \Gamma_2, \varphi_1 \wedge \varphi_2) \in T$;*

(v) *if $(\Gamma, \varphi_1 \wedge \varphi_2) \in T$ then (Γ, φ_1) and $(\Gamma, \varphi_2) \in T$;*

(vi) *if $(\Gamma, \varphi) \in T$ then $(\Gamma, \varphi \wedge \psi)$ and $(\Gamma, \psi \wedge \varphi) \in T$ (for any choice of ψ);*

(vii) *if $(\Gamma_1 \cup \{\varphi_1\}, \psi)$, $(\Gamma_2 \cup \{\varphi_2\}, \psi)$ and $(\Gamma_3, \varphi_1 \vee \varphi_2) \in T$ then $(\Gamma_1 \cup \Gamma_2 \cup \Gamma_3, \psi) \in T$;*

(viii) *if $(\Gamma_1 \cup \{\varphi\} \vdash \psi)$ and $(\Gamma_2 \cup \{\psi\} \vdash \varphi) \in T$ then $(\Gamma_1 \cup \Gamma_2, \varphi \leftrightarrow \psi) \in T$;*

(ix) *if $(\Gamma, \varphi \leftrightarrow \psi) \in T$ then $(\Gamma \cup \varphi, \psi)$ and $(\Gamma \cup \psi, \varphi) \in T$;*

(x) *if (Γ_1, φ) and $(\Gamma_2, \neg\varphi) \in T$ then $(\Gamma_1 \cup \Gamma_2, \psi) \in T$ (for any choice of ψ);*

(xi) *$(\Gamma, \varphi \vee (\neg\varphi)) \in T$ (for any choice of φ).*

If $(\Gamma, \varphi) \in T$, we write $\Gamma \vdash \varphi$ and we say that φ is a syntactic consequence of Γ.

Let us now rewrite the previous rules for formal deduction in a more legible way. The convention is that the bottom deduction (the conclusion) is a consequence of the deductions above the lines (the premises), and the symbol in parenthesis next to each deduction rule is its name. For example $(\to_E)$ stands for "elimination of $\to$" and $(\vee_{IR})$ stand for "introduction of $\vee$ on the right".

$$\text{(Ax)}\ \dfrac{}{\Gamma \cup \{\varphi\} \vdash \varphi}$$

$$(\to_I)\ \dfrac{\Gamma \cup \{\varphi\} \vdash \psi}{\Gamma \vdash \varphi \to \psi} \qquad (\to_E)\ \dfrac{\Gamma_1 \vdash \varphi \to \psi \qquad \Gamma_2 \vdash \varphi}{\Gamma_1 \cup \Gamma_2 \vdash \psi}$$

$$(\wedge_I)\ \dfrac{\Gamma_1 \vdash \varphi_1 \qquad \Gamma_2 \vdash \varphi_2}{\Gamma_1 \cup \Gamma_2 \vdash \varphi_1 \wedge \varphi_2} \qquad (\wedge_{EL})\ \dfrac{\Gamma \vdash \varphi_1 \wedge \varphi_2}{\Gamma \vdash \varphi_1} \qquad (\wedge_{ER})\ \dfrac{\Gamma \vdash \varphi_1 \wedge \varphi_2}{\Gamma \vdash \varphi_2}$$

$$(\vee_{IL})\ \dfrac{\Gamma \vdash \varphi}{\Gamma \vdash \varphi \vee \psi} \qquad (\vee_{IR})\ \dfrac{\Gamma \vdash \varphi}{\Gamma \vdash \psi \vee \varphi}$$

$$(\vee_E)\ \dfrac{\Gamma_1 \cup \{\varphi_1\} \vdash \psi \qquad \Gamma_2 \cup \{\varphi_2\} \vdash \psi \qquad \Gamma_3 \vdash \varphi_1 \vee \varphi_2}{\Gamma_1 \cup \Gamma_2 \cup \Gamma_3 \vdash \psi}$$

$$(\leftrightarrow_I)\ \dfrac{\Gamma_1 \cup \{\varphi\} \vdash \psi \qquad \Gamma_2 \cup \{\psi\} \vdash \varphi}{\Gamma_1 \cup \Gamma_2 \vdash \varphi_1 \leftrightarrow \varphi_2} \qquad (\leftrightarrow_{EL})\ \dfrac{\Gamma \vdash \varphi \leftrightarrow \psi}{\Gamma \vdash \varphi \to \psi} \qquad (\leftrightarrow_{ER})\ \dfrac{\Gamma \vdash \varphi \leftrightarrow \psi}{\Gamma \vdash \psi \to \varphi}$$

$$(\neg_E)\ \dfrac{\Gamma_1 \vdash \varphi \qquad \Gamma_2 \vdash \neg\varphi}{\Gamma_1 \cup \Gamma_2 \vdash \psi} \qquad \text{(ExMid)}\ \dfrac{}{\Gamma \vdash \varphi \vee \neg\varphi}$$

We will allow ourselves some abusive notations. We will write $\vdash \varphi$ instead of $\varnothing \vdash \varphi$ and $\varphi \vdash \psi$ instead of $\{\varphi\} \vdash \psi$.
We often call a witness that some deduction $\Gamma \vdash \varphi$ holds a derivation of this deduction. Let me now give some examples of valid derivations:

$$(\wedge_I)\ \dfrac{(\wedge_{ER})\ \dfrac{\text{(Ax)}\ \dfrac{}{\varphi \wedge \psi \vdash \varphi \wedge \psi}}{\varphi \wedge \psi \vdash \psi} \qquad (\wedge_{EL})\ \dfrac{\text{(Ax)}\ \dfrac{}{\varphi \wedge \psi \vdash \varphi \wedge \psi}}{\varphi \wedge \psi \vdash \varphi}}{\varphi \wedge \psi \vdash \psi \wedge \varphi}$$

This commutativity of $\wedge$

$$(\vee_E)\ \dfrac{(\vee_{IR})\ \dfrac{\text{(Ax)}\ \dfrac{}{\varphi \vdash \varphi}}{\varphi \vdash \psi \vee \varphi} \qquad (\vee_{IL})\ \dfrac{\text{(Ax)}\ \dfrac{}{\psi \vdash \psi}}{\psi \vdash \psi \vee \varphi} \qquad \text{(Ax)}\ \dfrac{}{\varphi \vee \psi \vdash \varphi \vee \psi}}{\varphi \vee \psi \vdash \psi \vee \varphi}$$

This commutativity of $\vee$.

$$
(\vee_E)\,\dfrac{(\neg_E)\,\dfrac{\begin{matrix}\vdots\\ \Gamma_1 \cup \{\neg\varphi\} \vdash \psi\end{matrix} \qquad \begin{matrix}\vdots\\ \Gamma_1 \cup \{\neg\varphi\} \vdash \neg\psi\end{matrix}}{\Gamma_1 \cup \Gamma_2 \cup \{\neg\varphi\} \vdash \varphi} \qquad (\mathrm{Ax})\,\dfrac{}{\varphi \vdash \varphi} \qquad (\mathrm{ExMid})\,\dfrac{}{\vdash \varphi \vee \neg\varphi}}{\Gamma_1 \cup \Gamma_2 \vdash \varphi}
$$

The previous derivation shows that proof by contradiction is valid in our formal setting.

Note that because of the way the relation $\Gamma \vdash \varphi$ was defined, we can prove results by induction on how we obtained $\Gamma \vdash \varphi$. What that means is that to prove some statement on deductions it suffices to prove that for each rule, if the proposition holds for the premises then it holds for the conclusion.
Note also that now we have two notions of truth, a syntactic one and a semantic one: $\Gamma \vdash \varphi$ and $\Gamma \vDash \varphi$. We wish to prove that they coincide. Let us first prove the easy direction.

Proposition 9.2 (Soundness)**:**
Let $\Gamma \subseteq F$ and $\varphi \in F$, then if $\Gamma \vdash \varphi$ then $\Gamma \vDash \varphi$.

The meaning of this proposition is that our deduction rules are not completely crazy and that they respect the semantic, i.e. the way the logical connectives are supposed to work.

Proof. We proceed by induction on the deduction $\Gamma \vdash \varphi$. Let $\delta \in \{0,1\}^P$ be an assignement.

(i) If $\varphi \in \Gamma$ and δ satisfies Γ, then $\varphi_\delta = 1$ and hence $\Gamma \vDash \varphi$.

(ii) Assume that δ satisfies Γ. If $\varphi_\delta = 1$, then δ satifies $\Gamma \cup \{\varphi\}$. By induction, we have $\Gamma \cup \{\varphi\} \vDash \psi$ and hence $\psi_\delta = 1$. It follows that $(\varphi \to \psi)_\delta = 1$. If $\varphi_\delta = 0$ then whatever the value of ψ_δ, $(\varphi \to \psi)_\delta = 1$.

(iii) Assume that δ satisfies $\Gamma_1 \cup \Gamma_2$, then, in particular δ satifies Γ_1 and δ satifies Γ_2. By induction, we get that $(\varphi \to \psi)_\delta = 1$ and $\varphi_\delta = 1$. Then by the semantic of $\to$, $\psi_\delta = 1$.

(iv) Assume that δ satisfies $\Gamma_1 \cup \Gamma_2$, then, by induction, $(\varphi_1)_\delta = 1$ and $(\varphi_2)_\delta = 1$ so $(\varphi_1 \wedge \varphi_2)_\delta = 1$.

(v) Assume that δ satisfies Γ, then, by induction, $(\varphi_1 \wedge \varphi_2)_\delta = 1$. It follows that $(\varphi_1)_\delta = 1$ and $(\varphi_2)_\delta = 1$.

(vi) Assume that δ satisfies Γ, then, by induction, $\varphi_\delta = 1$, it follows that, by the semantic of $\vee$, that whatever the valued of φ_δ, $(\varphi \vee \psi)_\delta = (\psi \vee \varphi)_\delta = 1$.

(vii) Assume that δ satisfies $\Gamma_1 \cup \Gamma_2 \cup \Gamma_3$, then, by induction, $(\varphi_1 \to \psi)_\delta = 1$, $(\varphi_2 \to \psi)_\delta = 1$ and $(\varphi_1 \vee \varphi_2)_\delta = 1$. If $(\varphi_1)_\delta = 1$, then $\psi_\delta = 1$. If $(\varphi_1)_\delta = 0$, then we must have $(\varphi_2)_\delta = 1$ and so $\psi_\delta = 1$. Hence, in both cases, $\psi_\delta = 1$.

(viii) Assume that δ satisfies $\Gamma_1 \cup \Gamma_2$ and $\varphi_\delta = 1$. Then, by induction, $\psi_\delta = 1$. On the other hand, if $\varphi_\delta = 0$, then for $\Gamma \cup \{\psi\} \vDash \varphi$ to hold, we must have $\psi_\delta = 0$ too. Hence $\varphi_\delta = \psi_\delta$.

(ix) Assume that δ satisfies Γ, then, by induction $\varphi_\delta = \psi_\delta$. One can then check that in that case $(\varphi \to \psi)_\delta = (\psi \to \varphi)_\delta = 1$.

(x) Let us show that Γ is not satisfiable. We proceed by contradiction. Assume that δ satisfies Γ, then by induction, $\varphi_\delta = (\neg\varphi)_\delta = 1$ which is absurd. It follows that because Γ is not satifiable, then $\Gamma \vDash \varphi$ for any formula φ.

(xi) We know that $X \vee \neg X$ is a tautology, so by substitution, so is $\varphi \vee \neg\varphi$ for all formula φ. In particular $\Gamma \vDash \varphi \vee \neg\varphi$.

That concludes the proof. ∎

Before we prove the converse, we need some lemmas.

Lemma 9.3:
If $\Gamma \vdash \varphi$ the there exists a finite $\Gamma_0 \subseteq \Gamma$ such that $\Gamma_0 \vdash \varphi$.

Proof. This proved by induction on the deduction $\Gamma \vdash \varphi$. We are not going to prove the eleven cases but the main idea is that the union of finite sets being finite, if the premises are proved using only finitely many formulas from Γ, then so is the conclusion, by implying an instance of the deduction rule where the hypotheses of the deductions have been replaced by their finite subsets.
The only two exceptions to this global pattern are the axiom rule and the excluded middle rule but they can easily be replaced by $\varphi \vdash \varphi$ and $\vdash \varphi \vee \neg\varphi$ where the hypothesis, being either empty or a singleton, are indeed finite. ∎

Lemma 9.4:
Let $\Gamma \subseteq \Gamma' \subseteq F$ and $\varphi \in F$. If $\Gamma \vdash \varphi$, then $\Gamma' \vdash \varphi$.

Proof. Let us assume that Γ' is not empty (otherwise, $\Gamma = \Gamma' = \varnothing$ and the lemma is quite easy to prove). Let $\psi \in \Gamma'$. Then we have the following derivation:

$$\dfrac{\dfrac{\begin{matrix}\vdots \\ \Gamma \vdash \varphi\end{matrix} \qquad (\mathrm{Ax})\,\dfrac{}{\Gamma' \vdash \psi}}{\Gamma' \vdash \varphi \wedge \psi}\,(\wedge_I)}{\Gamma' \vdash \varphi}\,(\wedge_{EL})$$

Hence $\Gamma' \vdash \varphi$ also holds. ∎

Definition 9.5 (Consistence)**:**
Let $\Gamma \subseteq F$. We say that Γ is inconsistent if there is a formula φ such that $\Gamma \vdash \varphi$ and $\Gamma \vdash \neg\varphi$.

Lemma 9.6:
Let $\Gamma \subseteq F$. If the set Γ is inconsistent, then it is not satisfiable.

Proof. This is an easy consequence of soundness. Let φ be such that $\Gamma \vdash \varphi$ and $\Gamma \vdash \neg\varphi$. Then $\Gamma \vDash \varphi$ and $\Gamma \vDash \neg\varphi$. If Γ was satisfiable, then let $\delta \in \{0,1\}^P$ satisfy Γ. But then $\varphi_\delta = 1 = (\neg\varphi)_\delta$ which is absurd. ∎

Remark that if Γ is inconsistent, applying rule $(\neg_E)$, we obtain that $\Gamma \vdash \psi$ for all formulas ψ. And that is in fact an equivalence: Γ is inconsistent if and only if $\Gamma \vdash \varphi$ for all formulas φ.

Lemma 9.7:
Let $\Gamma \subseteq F$ and $\varphi \in F$. If $\Gamma \cup \{\neg\varphi\}$ is inconsistent, then $\Gamma \vdash \varphi$.

Proof. Because $\Gamma \cup \{\neg\varphi\}$, $\Gamma \cup \{\neg\varphi\} \vdash \varphi$ and we have the following derivation:

$$\dfrac{\dfrac{\vdots}{\Gamma \cup \{\neg\varphi\} \vdash \varphi} \qquad (\mathrm{Ax})\,\dfrac{}{\Gamma \cup \{\varphi\} \vdash \varphi} \qquad (\mathrm{ExMid})\,\dfrac{}{\Gamma \vdash \varphi \vee \neg\varphi}}{\Gamma \vdash \varphi}\,(\vee_E)$$

So $\Gamma \vdash \varphi$ holds. ∎

Lemma 9.8:
Let $\Gamma \subseteq F$ and $\varphi \in F$. If Γ is consistent then $\Gamma \cup \{\varphi\}$ or $\Gamma \cup \{\neg\varphi\}$ is consistent.

Proof. Let us assume that both $\Gamma \cup \{\varphi\}$ and $\Gamma \cup \{\neg\varphi\}$ are inconsistent. Then by Lemma **(9.7)**, $\Gamma \vdash \varphi$. Moreover, we have the following derivation:

$$\dfrac{(\to_I)\,\dfrac{\dfrac{\vdots}{\Gamma \cup \{\varphi\} \vdash \neg\varphi}}{\Gamma \vdash \varphi \to \neg\varphi} \qquad \dfrac{\vdots}{\Gamma \vdash \varphi}}{\Gamma \vdash \neg\varphi}\,(\to_E)$$

So both $\Gamma \vdash \varphi$ and $\Gamma \vdash \neg\varphi$ hold, i.e. Γ is inconsistent. ∎

Definition 9.9 (Syntactic completeness)**:**
Let $\Gamma \subseteq F$. We say that Γ is syntactically complete if for all $\varphi \in F$, $\varphi \in \Gamma$ or $\neg\varphi \in \Gamma$.

Lemma 9.10:
Let $\Delta \subseteq F$ be consistent and syntactically complete. If $\Delta \vdash \varphi$ then $\varphi \in \Delta$.

Proof. If $\varphi \notin \Delta$, because Δ is syntactically complete, we have $\neg\varphi \in \Delta$ and hence using (Ax), $\Delta \vdash \neg\varphi$. But that contradicts the consistence of Δ. Hence $\varphi \in \Delta$. ∎

Lemma 9.11:
Let $\Delta \subseteq F$ be consistent and syntactically complete. Then:

(i) $\neg\varphi \in \Delta$ *if and only if* $\varphi \notin \Delta$;

(ii) $\varphi \wedge \psi \in \Delta$ *if and only if* $\varphi \in \Delta$ *and* $\psi \in \Delta$;

(iii) $\varphi \vee \psi \in \Delta$ *if and only if* $\varphi \in \Delta$ *or* $\psi \in \Delta$;

(iv) $\varphi \to \psi \in \Delta$ *if and only if* $\varphi \notin \Delta$ *or* $\psi \in \Delta$;

(v) $\varphi \leftrightarrow \psi \in \Delta$ *if and only if* $\varphi \in \Delta$ *is equivalent to* $\psi \in \Delta$;

Proof.

(i)
- Assume that $\neg\varphi \in \Delta$, if $\varphi \in \Delta$ too then, by the (Ax) rule $\Delta \vdash \varphi$ and $\Delta \vdash \neg\varphi$ which contradicts its consistency.

- Assume that $\varphi \notin \Delta$, then because Δ is syntactically complete, we must have $\neg\varphi \in \Delta$.

(ii)

- Assume that $\varphi \wedge \psi \in \Delta$, then because of the $(\wedge_{ER})$ and $(\wedge_{EL})$ rules $\Delta \vdash \varphi$ and $\Delta \vdash \psi$. Now apply Lemma **(9.10)**.
- Assume that φ and $\psi \in \Delta$, then because of the $(\wedge_I)$ rule, $\Delta \vdash \varphi\psi$. By Lemma **(9.10)**, we have $\varphi \wedge \psi \in \Delta$.

(iii)

- Assume that $\varphi \vee \psi \in \Delta$ and $\varphi \notin \Delta$. By syntactic completeness $\neg\varphi \in \Delta$ and we have the following derivation:

$$\dfrac{(\neg_E)\dfrac{\text{Ax}\dfrac{}{\Delta \vdash \neg\varphi} \quad \text{Ax}\dfrac{}{\varphi \vdash \varphi}}{\Delta \cup \{\varphi\} \vdash \psi} \quad \text{Ax}\dfrac{}{\psi \vdash \psi} \quad \text{Ax}\dfrac{}{\Delta \vdash \varphi \vee \psi}}{\Delta \vdash \psi}(\vee_E)$$

  So $\Delta \vdash \psi$ and hence $\psi \in \Delta$.

- If $\varphi\Delta$, by the $(\vee_{IL})$ rule $\Delta \vdash \varphi \vee \psi$ and if $\psi\Delta$, by the $(\vee_{IR})$ rule $\Delta \vdash \psi \vee \varphi$. In both cases we do get $\psi \vee \varphi \in \Delta$.

(iv)

- Assume that $\varphi \to \varphi \in \Delta$ and $\varphi \in \Delta$, then by the $(\to_E)$ rule, $\Delta \vdash \psi$ and hence $\psi \in \Delta$.
- If $\varphi \notin \Delta$, then by syntactic completeness $\neg\varphi \in \Delta$ and we have the following derivation:

$$\to_I \dfrac{(\neg_E)\dfrac{\text{Ax}\dfrac{}{\varphi \vdash \varphi} \quad \text{Ax}\dfrac{}{\Delta \vdash \neg\varphi}}{\Delta \cup \{\varphi\} \vdash \psi}}{\Delta \vdash \varphi \to \psi}$$

  On the other hand, if $\psi \in \Delta$, then $\Delta \cup \{\varphi\} \vdash \psi$ and hence by the $(\to_I)$ rule $\Delta \vdash \varphi \to \psi$. In both cases, we do have $\varphi \to \psi \in \Delta$.

(v)

- Assume that $\varphi \leftrightarrow \psi\Delta$, then by the $(\leftrightarrow_{EL})$ and $(\leftrightarrow_{ER})$ rules, we get that $\Delta \vdash \varphi \to \psi$ and $\Delta \vdash \psi \to \varphi$. Let us now assume that $\varphi \in \Delta$, then by the $(\to_E)$ rule we get that $\Delta \vdash \psi$. Similarly, if $\psi \in \Delta$, then $\varphi \in \Delta$.
- Assume that both φ and $\psi \in \Delta$, then $\Delta \cup \{\varphi\} \vdash \psi$ and $\Delta \cup \{\varphi\} \vdash \psi$ and hence by the $(\leftrightarrow_I)$ rule, $\Delta \vdash \varphi \leftrightarrow \psi$.

  On the other hand, if both $\neg\varphi$ and $\neg\psi \in \Delta$, then we have the following derivation:

$$(\leftrightarrow_I)\dfrac{(\neg_E)\dfrac{\text{Ax}\dfrac{}{\varphi \vdash \varphi} \quad \text{Ax}\dfrac{}{\Delta \vdash \neg\varphi}}{\Delta \cup \{\varphi\} \vdash \psi} \quad (\neg_E)\dfrac{\text{Ax}\dfrac{}{\psi \vdash \psi} \quad \text{Ax}\dfrac{}{\Delta \vdash \neg\psi}}{\Delta \cup \{\psi\} \vdash \varphi}}{\Delta \vdash \varphi \leftrightarrow \psi}$$

This concludes the proof. ■

Corollary 9.12:
Let $\Delta \subseteq F$ be consistent and syntactically complete, then Δ is satisfiable.

Proof. Let us define the assigmenent δ as follows: $\delta(X) = 1$ if $X \in \Delta$ and $\delta(X) = 0$ otherwise. We now prove by induction on φ that $\varphi \in \Delta$ is equivalent to $\varphi_\delta = 1$.

- If $\varphi = X$, the by definition of δ, $X_\delta = \delta(X) = 1$ if and only if $X \in \Delta$.
- If $\varphi = \neg\psi$, then $\neg\psi \in \Delta$ if and only if $\psi \notin \Delta$ if and only if, by induction, $\psi_\delta = 0$ if and only if $(\neg\psi)_\delta = 1$.
- If $\varphi = \varphi_1 \wedge \varphi_2$ then $\varphi_1 \wedge \varphi_2 \in \Delta$ if and only if φ_1 and $\varphi_2 \in \Delta$ if and only if, by induction, $(\varphi_1)_\delta = (\varphi_2)_\delta = 1$ if and only if $(\varphi_1 \wedge \varphi_2)_\delta = 1$.
- If $\varphi = \varphi_1 \vee \varphi_2$ then $\varphi_1 \vee \varphi_2 \in \Delta$ if and only if φ_1 or $\varphi_2 \in \Delta$ if and only if, by induction, $(\varphi_1)_\delta = 1$ or $(\varphi_2)_\delta = 1$ if and only if $(\varphi_1 \vee \varphi_2)_\delta = 1$.
- If $\varphi = \varphi_1 \to \varphi_2$ then $\varphi_1 \to \varphi_2 \in \Delta$ if and only if $\varphi_1 \notin \Delta$ or $\varphi_2 \in \Delta$ if and only if, by induction, $(\varphi_1)_\delta = 0$ or $(\varphi_2)_\delta = 1$ if and only if $(\varphi_1 \to \varphi_2)_\delta = 1$.
- If $\varphi = \varphi_1 \leftrightarrow \varphi_2$ then $\varphi_1 \leftrightarrow \varphi_2 \in \Delta$ if and only if $\varphi_1 \in \Delta$ equivalent to $\varphi_2 \in \Delta$ if and only if, by induction, $(\varphi_1)_\delta = (\varphi_2)_\delta$ if and only if $(\varphi_1 \leftrightarrow \varphi_2)_\delta = 1$.

■

Theorem 9.13:

If $\Gamma \subseteq F$ is consistent then Γ is statisfiable.

We will only prove this theorem when P is countable. If it is more than countable then a slightly more complicated argument involving Zorn's lemma works.

Proof. First, if we assume P to be countable, then by induction on n, F_n is countable for all n and hence so is F. So let us write $F = \{\varphi_i : i \in \mathbb{N}\}$. We define by induction on i the set $\Gamma_i \subseteq F$ as follows. Let $\Gamma_0 := \Gamma$. If $\Gamma_i \cup \{\varphi\}$ is consistent, let $\Gamma_{i+1} := \varphi_i$ and $\Gamma_{i+1} := \neg\varphi_i$ otherwise. By Lemma **(9.8)** and induction, Γ_i is consistent. Let us show that $\Delta := \bigcup_i \Gamma_i$ is consistent. Assume it is not, then there exists a formula $\varphi \in F$ such that $\Delta \vdash \varphi$ and $\Delta \vdash \neg\varphi$. But then, by Lemma **(9.3)**, there exists a finite $\Delta_0 \subseteq \Delta$ such that $\Delta_0 \vdash \varphi$ and $\Delta_0 \vdash \neg\varphi$. But $\Delta = \bigcup_i \Gamma_i$ and the Γ_i are increasing, so there exists some i_0 such that $\Delta_0 \subseteq \Gamma_{i_0}$ and hence $\Gamma_{i_0} \vdash \varphi$ and $\Gamma_{i_0} \vdash \neg\varphi$, but that contradicts the consistency of Γ_{i_0}.
So Δ is consistent and, by contruction, syntactically complete. So, by Corollary **(9.12)** Δ is satisfiable. As $\Gamma \subseteq \Delta$, Γ is also satisifable. ■

Corollary 9.14 (Completeness)**:**
Let $\Gamma \subseteq F$ and $\varphi \in F$ then $\Gamma \vDash \varphi$ if and only if $\Gamma \vdash \varphi$.

So our two notions of truth do coincide.

Proof. In Proposition **(9.2)**, we proved that $\Gamma \vdash \varphi$ implies $\Gamma \vDash \varphi$. Let us now assume that $\Gamma \vDash \varphi$. Then $\Gamma \cup \{\neg\varphi\}$ is not satisfiable and by Theorem **(9.13)**, $\Gamma \cup \{\neg\varphi\}$ is inconsistent. We conclude by Lemma **(9.7)**. ■

Remark 9.15:
Note that we just reproved the compactness theorem. Let $\Gamma \subseteq F$ and $\varphi \in F$, then if $\Gamma \vDash \varphi$ then $\Gamma \vdash \varphi$ and by Lemma **(9.3)**, for some finite $\Gamma_0 \subseteq \Gamma$, $\Gamma_0 \vdash \varphi$ and hence $\Gamma_0 \vDash \varphi$.