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Solutions to homework 1

Problem 1 :
Let $\mathfrak{U}$ be some an ultrafilter on some infinite set I. Show that the following are equivalent:

1. The filter $\mathfrak{U}$ is not principal;
2. The filter $\mathfrak{U}$ contains the Fréchet filter on I.

Solution: First if $\mathfrak{U} = \langle i \rangle$ is principal, it does not contain the complement of $\{i\}$ which is cofinite so it cannot contain the Fréchet filter.
Let us now assume that $\mathfrak{U}$ does not contain the Fréchet filter. So $\mathfrak{U}$ does not contain some cofinite set. Since $\mathfrak{U}$ is an ultrafilter, it follows that $\mathfrak{U}$ contains some finite set X_0. We may assume that X_0 has minimal cardinality. Pick any $i_0 \in X_0$, if $\{i_0\} \notin \mathfrak{U}$ it follows that $X_0 \smallsetminus \{i_0\} \in \mathfrak{U}$, contradicting minimality of X_0. It follows that $\{i_0\} \in \mathfrak{U}$ and $\mathfrak{U}$ is principal.

Problem 2 :
Let $\mathcal{L}$ be a countable language, $(M_i)_{i\in\mathbb{Z}_{\geqslant 0}}$ a countable family of $\mathcal{L}$-structures, $\mathfrak{U}$ a non principal ultrafilter on $\mathbb{Z}_{\geqslant 0}$, and $\Sigma(\overline{x}) = \{\varphi_j : j \in \mathbb{Z}_{\geqslant 0}\}$ a set of $\mathcal{L}$-formulas (whose free variables are in the tuple $\overline{x}$). We assume that for all finite $\Sigma_0 \subseteq \Sigma$, $\prod_{i\to\mathfrak{U}} M_i \vDash \exists\overline{x} \bigwedge_{\varphi\in\Sigma_0} \varphi(\overline{x})$.

1. For all $n \in \mathbb{Z}_{\geqslant 0}$, let $S(n) := \{i \in \mathbb{Z}_{\geqslant 0} : M_i \vDash \exists\overline{x} \bigwedge_{j=0}^{n} \varphi_j(\overline{x})\}$. Show that the $S(n)$ form a decreasing chain of elements of $\mathfrak{U}$.

   Solution: Since $\prod_{i\to\mathfrak{U}} M_i \vDash \exists\overline{x} \bigwedge_{j=0}^{n} \varphi_j(\overline{x})$, by Łoś's theorem, it follows that $S(n) \in \mathfrak{U}$. Moreover, if $i \in S(n+1)$ then $M_i \vDash \exists\overline{x} \bigwedge_{j=0}^{n+1} \varphi_j(\overline{x})$ so, in particular, $M_i \vDash \exists\overline{x} \bigwedge_{j=0}^{n} \varphi_j(\overline{x})$ and $i \in S(n)$. So $S(n+1) \subseteq S(n)$.

2. Let $\overline{b}_i \in M_i^{\overline{x}}$ be defined as follows:
   - If $i \in S(0)$, let m be maximal such that $m \leqslant i$ and $i \in S(m)$. Then pick $\overline{b_i} \in M_i^{\overline{x}}$ such that $M_i \vDash \bigwedge_j \varphi_{j=0}^{m}(\overline{b}_i)$;
   - Otherwise, pick any $\overline{b}_i \in M_i$.

   Show that for all $\varphi \in \Sigma$, we have $\prod_{i\to\mathfrak{U}} M_i \vDash \varphi([(\overline{b}_i)_i]_{\mathfrak{U}})$.

   Solution: Pick some $j \in \mathbb{Z}_{\geqslant 0}$ and let $i \in \mathbb{Z}_{\geqslant 0}$ be such that $i \geqslant j$ and $i \in S(j)$. Then the maximal m such that $m \leqslant i$ and $i \in S(m)$ is larger than j. It follows that, by contraction, $M_i \vDash \varphi_j(\overline{b}_i)$. Since $\mathfrak{U}$ is a non principal ultrafilter, it contains the cofinite set $\{i \in \mathbb{Z}_{\geqslant 0} : i \geqslant j\}$. Therefore it also contains $\{i \in \mathbb{Z}_{\geqslant 0} : i \geqslant j\} \cap S(j)\} \subseteq \{i \in \mathbb{Z}_{\geqslant 0} : M_i \vDash \varphi_j(\overline{b}_i)\}$. It follows that that last set is in $\mathfrak{U}$ and, by Łoś's theorem, that $\prod_{i\to\mathfrak{U}} M_i \vDash \varphi_j([(\overline{b}_i)_i]_{\mathfrak{U}})$.

3. Let $M := \mathbb{Q}^{\mathfrak{U}}$ with the natural $\mathcal{L}_{\text{or}}$-structure. We identify $\mathbb{Q}$ with its image under the diagonal embedding. Let $\mathcal{O} := \{a \in M : \exists n \in \mathbb{Z}_{>0}\ -n < a < n\}$ and $\mathfrak{M} := \{a \in M : \forall n \in \mathbb{Z}_{>0}\ -\frac{1}{n} < a < \frac{1}{n}\}$. Show that $\mathcal{O}$ is a ring and that $\mathfrak{M}$ is a maximal ideal.

   Solution: Since $\mathbb{Q} \preccurlyeq M$, it is quite clear that M is a field. To prove that $\mathcal{O}$ is a ring, we have to show that it is closed under $+$ and $\cdot$. If $-n < x < n$ and $-n < y < n$, we have $-2n < x + y < 2n$ and $-n^2 < xy < n^2$.

Let us now show that $\mathfrak{M}$ is an ideal. Let x, $y \in \mathfrak{M}$. For all $n \in \mathbb{Z}_{>0}$, we have $-(2n)^{-1} < x < (2n)^{-1}$ and $-(2n)^{-1} < y < (2n)^{-1}$. So $-n^{-1} < x + y < n^{-1}$. If $x \in \mathcal{O}$ and $y \in \mathfrak{M}$, then for some $n \in \mathbb{Z}_{>0}$, we have $-n < x < n$ and, for all $m \in \mathbb{Z}_{>0}$, $-(mn)^{-1} < y < (mn)^{-1}$. It follows that $-m^{-1} < xy < m^{-1}$. Moreover, if $x \in \mathcal{O} \smallsetminus \mathfrak{M}$, there exists $n \in \mathbb{Z}_{>0}$ such that $x < -n^{-1}$ or $n^{-1} < x$. In both cases, we have $-n < x^{-1} < n$ and $x \in \mathcal{O}^{\star}$. It follows that any ideal containing $\mathfrak{M}$ contains a unit and is therefore $\mathcal{O}$ itself.

4. Show that $\mathcal{O}/\mathfrak{M}$ is isomorphic to $\mathbb{R}$.

   Solution: Note first $\mathbb{Q} \leqslant \mathcal{O}$ and $\mathbb{Q} \cap \mathfrak{M} = \{0\}$, it follows that $\mathbb{Q}$ can be identified with a subring of $k := \mathcal{O}/\mathfrak{M}$. Moreover, $\mathfrak{M}$ is convex, so $\mathcal{O}/\mathfrak{M}$ can be ordered by $\overline{x} \leqslant \overline{y}$ if sor some choice of x and y, $x \leqslant y$. It is then easy to check that, since $\mathcal{O}$ is an ordered ring, k is an ordered field. In particular, $+$ and $\cdot$ are continuous on k.

   Now, pick $\overline{x} < \overline{y} \in k$. There exists $n \in \mathbb{Z}_{>0}$ such that $y - x > n^{-1}$. Moreover, we can find $m \in \mathbb{Z}_{>0}$ such that $-m < x < y < m$. If e partition $[-m, m]$ into $2nm$ intervals of length n^{-1}. The bounds of these intervals are rational and x and y cannot be in the same interval. It follows that there exist $q \in \mathbb{Q}$ such that $x < q < y$ and hence $\overline{x} < q < \overline{y}$. We have just proved that $\mathbb{Q}$ is dense in k.

   Let us now prove that k is complete. Let $X \subset k$ be bounded above and non empty. Let $X_{\mathbb{Q}} := \{x \in \mathbb{Q} : x < y \text{ for some } y \in X\}$ and $Y_{\mathbb{Q}} := \mathbb{Q} \smallsetminus X_{\mathbb{Q}}$. Let $\Sigma(x) := \{q < x : q \in X_{\mathbb{Q}}\} \cup \{x < q : q \in Y_{\mathbb{Q}}\}$. Then $\Sigma(x)$ is finitely satisfiable in $\mathbb{Q}$ so there exists $a \in M$ realizing Σ. Since $X_{\mathbb{Q}}$ and $Y_{\mathbb{Q}}$ are non empty, $a \in \mathcal{O}$ and since $\mathbb{Q}$ is dense in k, $\overline{a}$ is the sup[remum of X. So k is a complete topological field containing $\mathbb{Q}$ as a dense subfield.

   So we define $f : \mathbb{R} \to k$ by $f(x) = \sup\{q \in \mathbb{Q} : q < x\}$. This is injective and surjective by density of $\mathbb{Q}$ in both $\mathbb{R}$ and k. This si also a continuous map since, for all $\varepsilon \in \mathbb{Q}_{>0}$, if $x \leqslant y$ and $y - x < \varepsilon$, then $X := \{q \in \mathbb{Q} : q < x\} \leqslant \{q \in \mathbb{Q} : q < x\} =: Y$ and thus $f(x) = \sup X \leqslant \sup Y = f(y)$. Moroever, $Y \subseteq X + \varepsilon$ and hence $f(y) \leqslant f(x) + \varepsilon$. Finally, the maps $f(x + y)$ and $f(x) + f(y)$ (resp. $f(x \cdot y)$ and $f(x) \cdot f(y)$) are continuous and coincide on $\mathbb{Q} \times \mathbb{Q}$ which is dense in $\mathbb{R} \times \mathbb{R}$ so they are equal.

Problem 3 :
Prove whether each of the following classes is elementary, finitely axiomatisable or none of those.

1. Infinite sets in some language $\mathcal{L}$ with one sort.

   Solution: Let $\varphi_n := \exists x_1 \ldots \exists x_n \bigwedge_{i \neq j} \neg x_i = x_j$ and $T_\infty := \{\varphi_n : n \in \mathbb{Z}_{>0}\}$. Models of T_∞ are exactly infinite $\mathcal{L}$-structures. So this class is elementary. This class is not finitely axiomatisable because if T_0 is a finite $\mathcal{L}$-theory whose models are all infinite, then $T_\infty \vDash \bigcap_{\varphi \in T_0} \varphi$. By compactness, it follows that for some $n \in \mathbb{Z}_{>0}$, $\varphi_n \vdash \bigcap_{\varphi \in T_0} \varphi$. Let M be any $\mathcal{L}$-structure with cardinality n (we can always find such a structure by taking all constants equal, all functions equal and constant and all predicates empty). But then $M \vDash T_0$ is finite, a contradiction.

2. Finite sets in some language $\mathcal{L}$ with one sort.

   Solution: Essentially for the same reason as above, finite $\mathcal{L}$-structures are not an elementary class. Indeed assume that T is an $\mathcal{L}$-theory whose models are all finite $\mathcal{L}$-structures. Then $T \cup T_\infty$ is satisfiable. Indeed a finite subset of $T \cup T_\infty$ is contained in $T \cup \{\varphi_n : n \leqslant m\}$ and any $\mathcal{L}$-structure of cardinal m is a model of that theory.

But $M \vDash T \cup T_\infty$ is an infinite model of T, a contradiction. If you think the two last proofs are similar, it is because it is essentially the same proof. You can actually prove that if a class C is elementary and its complement is elementary then both are finitely axiomatisable.

3. Fields in $\mathcal{L}_{\mathrm{rg}} := \{\mathbf{K}; 0 : \mathbf{K}, 1 : \mathbf{K}, - : \mathbf{K} \to \mathbf{K}, + : \mathbf{K}^2 \to \mathbf{K}, \cdot : \mathbf{K}^2 \to \mathbf{K}\}$.

   Solution: This class is finitely axiomatisable (by T_{field}) and you can find the axioms in any textbook on abstract algebra.

4. Characteristic p fields (for some prime p) in $\mathcal{L}_{\mathrm{rg}}$.

   Solution: This class is finitely axiomatized by $T_{field} \cup \{\sum_{i=1}^{p} 1 = 0\}$.

5. Characteristic 0 fields in $\mathcal{L}_{\mathrm{rg}}$.

   Solution: Let $T_{field,0} = T_{field} \cup \{\neg \sum_{i=1}^{p} 1 = 0 : p \in \mathbb{Z}_{>0}\}$. Then models of $T_{field,0}$ are exactly characteristic zero fields. This class is not finitely axiomatisable because if it where, by some theory T', then we would have $T_{field,0} \vdash \bigcap_{\varphi \in T'} \varphi$ and by compactness, $T_{field} \cup \{\sum_{i=1}^{p} 1 = 0 : 0 < p < m\} \vdash \bigcap_{\varphi \in T'} \varphi$. But any field of characteristic larger than m is a model of the theory on the left, but not of the theory on the right, a contradiction.

6. Algebraically closed fields in $\mathcal{L}_{\mathrm{rg}}$ (in that case, some of the proofs require a certain amount of algebra).

   Solution: Let $\{\psi_n := \forall \overline{x}\, \neg x_n = 0 \to (\exists y \sum_{i=0}^{n} x_i y^i = 0)\}$ and ACF $:= T_{field} \cup \{\psi_n : n \in \mathbb{Z}_{>0}\}$. Models of ACF are exactly algebraically closed fields. This class is not finitely axiomatisable because if it where, by the same arguments as above, there ould be a formula θ whose models are all algebraically closed fields such that $T_{field} \cup \{\psi_n : 0 < n < m\}$. To obtain a contradiction, we just have to find a field K over which every polynomial of degree at most m has a root but which is not algebraically closed.

   This can be done in the following way. Let p be a prime integer larger than m and K be the union of all the extensions of $\mathbb{Q}$ (inside some fixed algebraic closure $\overline{\mathbb{Q}}^{\mathrm{a}}$) whose degree is prime to p. Let $P \in \mathbb{Q}[X]$ be irreducible of degree p (for example $X^p - 2$). If P has a root in $a \in K$, then $\mathbb{Q}(a) \leqslant L \leqslant K$ with $[L : \mathbb{Q}]$ prime to p (by definition of K). But by multiplicativity of the degree in towers, since $[\mathbb{Q} : \mathbb{Q}(a)]$, we must have $p | [L : \mathbb{Q}]$, a contradiction. So K is not algebraically closed. Now if P is a polynomial of degree smaller than m and $a \in \overline{\mathbb{Q}}^{\mathrm{a}}$ is a root of P, then $[\mathbb{Q}(a) : \mathbb{Q}] \leqslant m$ and is therefore prime to p, it follows that $\mathbb{Q}(q) \leqslant K$ and P has a root in K.