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Solutions to homework 2

Problem 1 :
Let C be a set of finite $\mathcal{L}$-structures. Let $T = \{\varphi : \varphi$ is a sentence and for all $M \in C$, $M \vDash \varphi\}$.

1. Give a necessary and sufficient condition for T to have an infinite model.

   Solution: T has an infinite model if and only if for all $n \in \mathbb{Z}_{\geqslant 0}$, there exists $M \in C$ such that $|M| \geqslant n$.

   Indeed, let us assume that for all $n \in \mathbb{Z}_{\geqslant 0}$, there exists $M \in C$ such that $|M| \geqslant n$. Consider $\varphi_n := \exists x_1 \dots \exists x_n \bigwedge_{i \neq j} \neg x_i = x_j$. Then $M \vDash \varphi_n$ if and only if $|M| \geqslant n$. Infinite model of T are exactly the models of $T' := T \cup \{\varphi_n : n \in \mathbb{Z}_{>0}\}$. By compactness, it suffices to show that T' is finitely consistent. Let $T_0 \subseteq T$ be finite. Then there exists $m \in \mathbb{Z}_{>0}$ such that $T_0 \subseteq T \cup \{\varphi_n : 0 < n \leqslant m\}$. By hypothesis, there exists $M \in C$ such that $|M| \geqslant m$, then $M \vDash T_0$.

   Conversely, if there exists m such that every $M \in C$ has cardinality smaller than m, then $\neg\varphi_m \in T$ and every model of T has cardinality smaller than m.

2. Assume that T has infinite models, give a theory T' such that the models of T' are exactly the infinite models of T.

   Solution: The infinite models of T are exactly the models of $T' := T \cup \{\varphi_n : n \in \mathbb{Z}_{>0}\}$.

3. Show that $T' \vDash \varphi$ if and only if there exists some $n \in \mathbb{N}$ such that for all $M \in C$ of cardinality greater than n, $M \vDash \varphi$.

   Solution: By compactness, $T' \vDash \varphi$ if and only if there exists $m \in \mathbb{Z}_{>0}$ such that $T \cup \varphi_m \vDash \varphi$. It follows that for every $M \in C$ such that $|M| \geqslant m$, $M \vDash \varphi$.

   Conversely if every $M \in C$ such that $|M| \geqslant m$ verify $M \vDash \varphi$, then $\varphi_n \to \varphi \in T$. Since every model of T' is a model of φ_n and a model of T, it follows that $T \vDash \varphi$.

Problem 2 :
A sentence φ is said to be universal if it is of the form $\forall x_1 \dots \forall x_n\, \psi(x_1, \dots, x_n)$ where ψ is quantifier free.

1. Let φ be universal, M and N $\mathcal{L}$-structures and $f : N \to M$ and embedding. Show that if $M \vDash \varphi$ then $N \vDash \varphi$.

   Solution: If $M \vDash \exists \forall \overline{x}\, \psi(\overline{x})$ then for all $\overline{a} \in M^{\overline{x}}$, $M \vDash \psi(\overline{a})$. In particular, for all $\overline{a} \in N^{\overline{x}}$, $M \vDash \psi(f(\overline{a}))$. But since f is an embedding and ψ is quantifier free, this implies that $N \vDash \psi(\overline{a})$.

2. Let T be an $\mathcal{L}$-theory and $\overline{c}$ a tuple of new constants (i.e. that do not appear in $\mathcal{L}$). Let $\varphi(\overline{x})$ be an $\mathcal{L}$-formula, such that $\overline{x}$ is sorted as $\overline{c}$. Show that if $T \vDash \varphi(\overline{c})$ then $T \vDash \forall \overline{x}\, \varphi(\overline{x})$.

   Solution: Let $M \vDash T$ and $\overline{a} \in M^{\overline{x}}$. Let $M^\star$ the $\mathcal{L} \cup \{\overline{c}\}$-enrichment of M such that $\overline{c}^{M^\star} = \overline{a}$. Then, since $\overline{c}$ does not appear in T and $M \vDash T$, we have $M^\star \vDash T$ and hence $M^\star \vDash \varphi(\overline{c})$, i.e. $M \vDash \varphi(\overline{a})$. Since $\overline{a}$ and M were arbitrary, we do get $T \vDash \forall \overline{x}\, \varphi(\overline{x})$.

3. Let $M \models T_\forall := \{\varphi \text{ universal } \mathcal{L}\text{-sentence} : T \models \varphi\}$. Show that the $\mathcal{L}(M)$-theory $\Delta_M(M) \cup T$ is consistent.

   Solution: By compactness, if this theory is not consistent, there exists a quantifier free $\mathcal{L}$-formula $\varphi(\overline{x})$ such that $\varphi(\overline{m}) \in \Delta(M)$ and $T \cup \{\varphi(\overline{m})\}$ is inconsistent. To be precise, there are finitely many $\varphi_i(m_i)$ but we can take their conjunction. If $T \cup \{\varphi(\overline{m})\}$, it follows that $T \models \neg\varphi(\overline{m})$ and, by the previous question, $T \models \forall\overline{x}\,\neg\varphi(\overline{x})$. So $\forall\overline{x}\,\neg\varphi(\overline{x}) \in T_\forall$ and therefore $M \models \forall\overline{x}\,\varphi(\overline{x})$. In particular, $M \models \neg\varphi(\overline{m})$, contradicting the fact that $\varphi(\overline{m}) \in \Delta(M)$.

4. Let T and T' be two theories, show that the following are equivalent:

   a) $T_\forall \subseteq T'_\forall$;

   b) Every model of T' can be embedded in a model of T.

   Solution: Let us first assume that $T_\forall \subseteq T'_\forall$. If $M \models T'$, then $M \models T'_\forall$ and in particular $M \models T_\forall$. By the previous question, We can find $N^\star \models \Delta(M) \cup T$. Let $N := N^\star|_{\mathcal{L}}$, then $N \models T$ and there is en embedding $M \to N$.

   Conversely, pick $\varphi \in T_\forall$ and let $M \models T'$. By hypothesis, we find $N \models T$ and an embedding $M \to N$. By definition of $T_\forall$, $N \models \varphi$ and, by the first question, $M \models \varphi$. It follows that $T' \models \varphi$ and that $\varphi \in T'_\forall$.

5. Show that T is stable under substructure (i.e. if $N \models T$ and $f : M \to N$ is an embedding, then $M \models T$) if and only if T is equivalent to $T_\forall$.

   Solution: Every model of T is a models of $T_\forall$. It follows that we only have to show that every model of $T_\forall$ is a model of T. Let $M \models T_\forall$. By question 3, we can find $N \models T$ and and embedding $M \to N$. Since T is stable under substructure, we also have $M \models T$.

6. Let T be the $\mathcal{L}_{\mathrm{rg}}$-theory of algebraically closed fields (where $\{\mathbf{K}; 0 : \mathbf{K}, 1 : \mathbf{K}, - : \mathbf{K} \to \mathbf{K}, + : \mathbf{K}^2 \to \mathbf{K}, \cdot : \mathbf{K}^2 \to \mathbf{K}\}$). Which are the models of $T_\forall$.

   Solution: Let T' be the $\mathcal{L}_{\mathrm{rg}}$-theory of integral domains. It contains only universal statements: because we have 0, 1 and $-$ in the language, the axioms of rings are universal and integrality can be expressed as $\forall x \forall y\, x \cdot y = 0 \to (x = 0 \vee y = 0)$. So T' is equivalent to $T'_\forall$.

   Moreover, every every integral domain embeds in a field (its faction field) which in turn embeds in an algebraically closed field (its algebraic closure). So $T_\forall \subseteq T'_\forall = T'$. Finally, every algebraically closed field is n integral domain so $T \models T' = T'_\forall$ and hence $T' \subseteq T_\forall$. So $T_\forall$ is exactly the theory of integral domains.

7. Let $\mathcal{L} = \{X; < : X^2\}$ and T, T' be two $\mathcal{L}$-theories containing the theory of infinite total orders. Show that $T_\forall = T'_\forall$.

   Solution: Let us show that, given any two infinite total orders X and Y, we can embed X in a elementary extension of Y. This will conclude the proof as it will imply that any model of T, here X, can be embedded in a model of T', here the elementary extension of Y. So $T'_\forall \subseteq T_\forall$. Symmetrically, $T_\forall \subseteq T'_\forall$.

   Let us prove that $\mathcal{D}^{\mathrm{el}}(Y) \cup \Delta(X)$ is consistent (here we are very careful to take distinct new constants to represent X and Y). A finite subset of that theory is included in $\mathcal{D}^{\mathrm{el}}(Y) \cup \Delta(X_0)$ for some finite $X_0 \subseteq X$. Since Y is infinite, we can find a finite chain in Y of length $|X_0|$. The map sending X_0 to that chain (in the right order) is an embedding. It follows that, interpreting the elements of X_0 as given by the embedding (and the other elements of X arbitrarily) we can enrich Y into a model of $\mathcal{D}^{\mathrm{el}}(Y) \cup \Delta(X_0)$. By compactness, $\mathcal{D}^{\mathrm{el}}(Y) \cup \Delta(X)$ is consistent.