Silvain Rideau silvain.rideau@berkeley.edu
1091 Evans www.normalesup.org/~srideau/en/teaching

# Solutions to homework 3

**Problem 1** (Vaught's test) **:**
We say that a theory $T$ is $\kappa$-categorical if all models of $T$ of cardinality $\kappa$ are isomorphic. Show that if $T$ only has infinite models and is $\kappa$-categorical for some $\kappa \geqslant |\mathcal{L}|$, then $T$ is complete.

***Solution:*** Let $M$, $N \vDash T$. Assume $|M| \leqslant \kappa$. By upwards Löwenheim-Skolem, there exists $M^\star \succcurlyeq M$ such that $|M| = \kappa$. If $|M| \geqslant \kappa$, By downwards Löwenheim-Skolem, there exists $M^\star \preccurlyeq M$ such that $|M| = \kappa$. Note that $M \equiv M^\star$. Simialrly, we find $N^\star$ such that $|N^\star| = \kappa$ and $N \equiv N^\star$. By $\kappa$-categoricity, $N^\star \simeq M^\star$. It follows that $M \equiv M^\star \equiv N^\star \equiv N$ and $T$ is complete.

**Problem 2** (Vector spaces) **:**
Let $K$ be an field (if you prefer, take $K = \mathbb{R}$). Let $\mathcal{L}$ be the language with one sort $\mathbf{V}$ a constant 0, a function $- : \mathbf{V} \to \mathbf{V}$, a function $+ : \mathbf{V}^2 \to \mathbf{V}$ and for all $x \in K$, a function $\lambda_x : \mathbf{V} \to \mathbf{V}$. Any $K$-vector space can naturally be made into an $\mathcal{L}$-structure by interpreting $+$ as the addition, 0 the additive identity, $-$ as the additive inverse and $\lambda_x$ as scalar multiplication be $x$

1. Show there exist a theory $T$ whose models are all the infinite $K$-vector spaces.

   ***Solution:*** Let $T$ contain:

$$
\begin{array}{cl}
\forall x \forall y \forall z\, (x+y)+z = x+(y+z) & \\
\forall x\, x+0 = x & \\
\forall x\, x+(-x) = 0 & \\
\forall x \forall y\, x+y = y+x & \\
\forall x \forall y\, \lambda_k(x+y) = \lambda_k(x)+\lambda_k(y) & \text{for all } k \in K \\
\forall x \forall y\, \lambda_{k+l}(x) = \lambda_k(x)+\lambda_l(x) & \text{for all } k, \in K \\
\forall x\, \lambda_{kl}(x) = \lambda_k(\lambda_l(x)) & \text{for all } k, \in K \\
\forall x\, \lambda_1(x) = x & \\
\exists x_1 \ldots \exists x_n \bigwedge_{i \neq j} x_i \neq x_j & \text{for all } n \in \mathbb{Z}_{>0}
\end{array}
$$

   Then the models of $T$ are exactly the infinite $K$-vector spaces.

2. Let $V_i \leqslant U_i$, for $i = 1, 2$ be $K$-vector spaces such that $\dim(U_1) = \dim(U_2) > \dim(V_1) = \dim(V_2)$. Show that there exists an isomorphism $f : U_1 \to U_2$ extending any given isomorphism $g : V_1 \to V_2$.

   ***Solution:*** Let $W_i$ be such that $U_i = V_i \oplus W_i$. I claim that $\dim(W_1) = \dim(W_2)$. Indeed, $\dim(U_i) = \dim(V_i) + \dim(W_i)$. If $\dim(U_i)$ is finite, then $\dim(W_i) = \dim(U_i) - \dim(V_i)$. If $\dim(U_i)$ is infinite, then $\dim(W_i) = \dim(U_i)$. By sending a basis a basis of $W_1$ to a basis of $W_2$, we get the required extension.

3. Show that $T$ is $\kappa$-categorical for all $\kappa > |\mathcal{L}|$. Conclude that $T$ is complete.

   ***Solution:*** We have $\kappa > |\mathcal{L}| = |K| + \aleph_0 \geqslant |K|$. Let $U_1$ and $U_2$ be two vector spaces of cardinality $\kappa$. Every element of $U_i$ is a finite sum of $K$-linear combinations of elements of the basis. It follows that if $\dim(U_i)$ is finite, then $|U_i| = |K|^n = |K|$, a contradiction. So $|U_i|$ is infinite and $|U_i| = |K|\dim(U_i)$. Since $|U_i| > |K|$, it follows that $\dim(U_i) = |U_i| = \kappa$. So $U_1$ and $U_2$ have the same dimension and they are therefore isomorphic.

We now conclude that $T$ is complete by Vaught's criterion (first problem).

4. Let $V \leqslant U$ be two infinite $K$-vector spaces such that $\dim(V) < \dim(U)$ and $\dim(U) > |\mathcal{L}|$, then $V \preccurlyeq U$.

   ***Solution:*** By upward Löwenheim-Skolem, we can find $V^\star \succcurlyeq V$ such that $|V^\star| = |U|$. As in the previous question, we have that $\dim(V^\star) = |V^\star| = |U| = \dim(U)$. By question 2, we can find an isomorphism $f : V^\star \to U$ fixing $V$. Let $\varphi(\overline{x})$ be an $\mathcal{L}$-formula and pick $\overline{v} \in V^{\overline{x}}$. We have $V \vDash \varphi(\overline{v})$ if and only if $V^\star \vDash \varphi(\overline{v})$ if and only if $U \vDash \varphi(f(\overline{v}))$ if and only if $U \vDash \varphi(\overline{v})$.

5. Show that this remains true without the dimension hypothesis.

   ***Solution:*** By upwards Löwenheim-Skolem, we can find $U^\star \succcurlyeq U$ such that $\dim(U) > |\mathcal{L}| + \dim(V)$. By the previous question $V \preccurlyeq U^\star$. Let $\varphi(\overline{x})$ be an $\mathcal{L}$-formula and pick $\overline{v} \in V^{\overline{x}}$. We have $V \vDash \varphi(\overline{v})$ if and only if $U^\star \vDash \varphi(\overline{v})$ if and only if $U \vDash \varphi(\overline{v})$.

6. Let $U \vDash T$, $V \leqslant U$ and $\pi(x) = \{\neg x = c : c \in V\}$. Let $a$, $b \in U$ be two realizations of $\pi$. Show that $\mathrm{tp}^U(a/V) = \mathrm{tp}^U(b/V)$.

   ***Solution:*** First of all, since $a\ b \notin V$, we have an isomorphism $V + Ka \to V + Kb$ fixing $V$ and sending $a$ to $b$. By question 2, we can find an automorphism of $U^\star$ fixing $V$ and sending $a$ to $b$. It follows that $\mathrm{tp}^{U^\star}(a/V) = \mathrm{tp}^{U^\star}(b/V)$. Since $U^\star \succcurlyeq U$, we also have $\mathrm{tp}^U(a/V) = \mathrm{tp}^U(b/V)$.

7. Let $x$ be a single variable, show that $\mathcal{S}^U_x(V) = \{\mathrm{tp}^U(a/V) : a \in V\} \cup \{p_\infty\}$ where $p_\infty \supseteq \pi$ (in particular, you have to show that there is a unique complete type over $V$ extending $\pi$).

   ***Solution:*** Let $p \in \mathcal{S}^U_x(V)$. If $p$ contains a formula $x = a$ for some $a \in V$, then the only realisation of $p$ in any $U^\star \preccurlyeq U$ is $a$ and therefore $p = \mathrm{tp}^U(a/V)$. Indeed, let $\varphi(x)$ be an $\mathcal{L}$-formula and $c \in U^\star \succcurlyeq U$ be a realisation of $p$. Since the formula $x = a$ is in $p$, we must have $c = a$ nd hence $\varphi \in p$ if and only if $U^\star \vDash \varphi(a)$ if and only if $U \vDash \varphi(a)$.

   Let us now assume that $p$ and $q$ contains $\pi$ and let $a \in U^\star \succcurlyeq U$ realize $p$ and $b \in U^\star \succcurlyeq U$ realize $q$. By the previous question $p = \mathrm{tp}^{U^\star}(a/V) = \mathrm{tp}^\star(b/V) = q$. So any $p \in \mathcal{S}^U_x(V)$ is either $\mathrm{tp}^U(a/V)$ or the unique type $p_\infty$ containing $\pi$.

8. Pick $p \in \mathcal{S}^U_x(V) \smallsetminus \{p_\infty\}$. Show that $\{p\}$ is open.

   ***Solution:*** Let $a \in V$ be such that $p = \mathrm{tp}(a/V)$, then the open set $\{q \in \mathcal{S}^U_x(V) : q$ contains $x = a\}$ is the singleton $\{p\}$.

9. Show that $X \subseteq \mathcal{S}^U_x(V)$ containing $p_\infty$ is open if and only if $X$ is cofinite.

   ***Solution:*** Let $X \subseteq \mathcal{S}^U_x(V)$ be open containing $p_\infty$, then $\mathcal{S}^U_x(V) = X \cup \bigcup_{p \notin X}\{p\}$ is an open covering of $\mathcal{S}^U_x(V)$ which is compact. It follows that $\mathcal{S}^U_x(V) = X \cup \bigcup_{p \in Y_0}\{p\}$ where $Y_0$ is finite and $X \cap Y_0 = \varnothing$. But then $Y_0$ is exactly the complement of $X$ which is therefore cofinite.

   Conversely since every point is closed (we are in an Hausdorff space), finite sets are closed and hence cofinite sets are open.

10. Let $\varphi(x)$ be an $\mathcal{L}$-formula, show that $\varphi(V)$ is finite or cofinite.

    ***Solution:*** Consider the clopen set $\langle\varphi\rangle \subseteq \mathcal{S}_x^U(V)$. If $p\infty \in \langle\varphi\rangle$, then $\langle\varphi\rangle$ is cofinite and $\langle\neg\varphi\rangle$ is finite. We have $V \vDash \neg\varphi(a)$ if and only if $\mathrm{tp}(a/V) \in \langle\neg\varphi\rangle$ and since two points of $V$ have distinct types over $V$, it follows that $\neg\varphi(V)$ is finite and hence $\varphi(V)$ is cofinite. If $p\infty \notin \langle\varphi\rangle$, then $p\infty \in \langle\neg\varphi\rangle$ and be the same proof has above, $\varphi(V)$ is finite.

If you don't like linear algebra, you can replace $T$ by the theory of infinite sets in the language with one sort and no symbol (except equality) and dimension by cardinal. All of the (obvious adaptations of the) questions above remain true